Navier-Stokes to Maxwell via Einstein

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August 18, 2021

arXiv:2005.04242 with T. Manton and N. Monga; forthcoming with N. Monga

Overview

Outline

- From Einstein to Maxwell: the classical double copy via Weyl
- From Navier-Stokes to Einstein: fluid-gravity duality via a cutoff
- Algebraic Speciality in Fluids
  - Type D Fluids: constant vorticity
  - Type N Fluids: potential flows
- Towards a general fluid?

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From Einstein to Maxwell: The Classical Double Copy

Yang-Mills amplitudes $\mathcal{A}^{\text{YM}}$ (properly gauged) ‘square’ to gravity amplitudes $\mathcal{M}^{\text{grav}}$:

$$\mathcal{A}^{\text{YM}} \sim \sum_k \frac{n_k c_k}{props} \quad \longrightarrow \quad \mathcal{M}^{\text{grav}} \sim \sum_k \frac{n_k n_k}{props}$$

Also scalar theory with amplitudes $\mathcal{A}^{\text{s}} \sim \sum_k c_k \tilde{c}_k / props$

For review see ch 10 of Bern, Carrasco, Chiodaroli, Johansson, Roiban 1909.01358

Kerr-Schild Map (Monteiro, O’Connell, White 1410.0239)

Pick metric in Kerr-Schild coordinates (with $k^2 = 0$):

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi k_\mu k_\nu \quad \longrightarrow \quad G_{\mu\nu} = 0$$

$$A_\mu = \phi k_\mu \quad \longrightarrow \quad \nabla_\nu F^{\mu\nu} = 0$$

$$\phi \quad \longrightarrow \quad \nabla^2 \phi = 0$$

Note our color factors will always be trivial, so we are restricting to the $U(1)$ sector.

From Einstein to Maxwell: The Weyl Classical Double Copy

For Type D/N spacetimes with principal null vectors aligning in pairs/all four align

- Rewrite Weyl tensor in spinor notation: $C_{ABCD} = \frac{1}{4} W_{\mu\nu\lambda\gamma}\sigma^{\mu\nu}_{AB}\sigma^{\lambda\gamma}_{CD}$
- Decompose in principle spinors $C_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
- $C^D_{ABCD} \sim \alpha_{(A}\alpha_B\beta_C\beta_{D)}$, $C^N_{ABCD} \sim \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$

For these special spacetimes, can 'square root' the Weyl tensor:

$$C_{ABCD} = \frac{1}{S} f_{(AB}f_{CD)}, \qquad \text{with e.g. } f_{(AB)} = \alpha_{(A}\beta_{B)}$$

and $\nabla_0^2 S = 0$. Spinor $f_{AB} \to F_{\mu\nu}$ which satisfies $\nabla_0^\mu F_{\mu\nu} = 0$.

Luna, Monteiro, Nicholson, O'Connell 1810.08183;

Godazgar^2, Monteiro, Veiga, Pope 2010.02925

Why Fluid-Gravity Duality?

Questions from the Classical Double Copy

- Why is there a spacetime (not momentum space) double copy?
  ∃ linearized derivation via twistors and the Penrose transform
  White 2012.02479; Chacon, Nagy, White 2103.16441
- Can we extend the classical copy to Petrov type II or type I solutions? 2012.02479: type III example, 2103.16441: (linearized) whenever ∃ a Penrose transform
  Fluid-gravity gives physically interesting type II solutions to test
- Can we go beyond the linearized level?
  Cutoff fluid-gravity duality is non-linear but still perturbative. Good forum to ask.
- Can we build a Weyl double copy in higher dimensions? Yes, for Schwarzschild Monteiro, Nicholson, O'Connell 1809.03906
  Fluid-grav duals generalize to higher d; 3+1d fluid=5d grav.

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History of Fluid/Gravity Duality

Membrane Paradigm

- Began with prescient thesis of Damour in 1978
- Fluctuations of a black hole horizon act like a viscous fluid
- Fluid viscosity is computed to be $\eta = 1/16\pi G$
- Dividing by the entropy density $s = 1/4G$ gives $\eta/s = 1/4\pi$
- Always considers fluctuations at the black hole horizon $r = r_h$ itself; produces Damour-Navier Stokes equation

AdS/CFT Method

- Policastro, Son, Starinets hep-th/0205052 considered the hydrodynamics of $\mathcal{N} = 4\, SU(N)$ SYM via AdS/CFT
- Again find $\eta/s = 1/4\pi$
- Performed at AdS spatial infinity $r = \infty$
- Requires string theory, SUSY gauge theory, and AdS/CFT

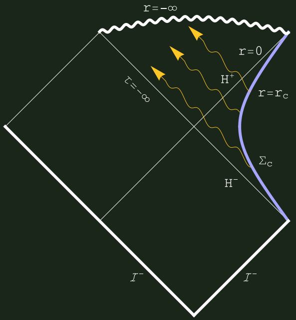
A 'Wilsonian' Approach

Fluid-gravity duality in the cutoff approach relates solutions of the incompressible Navier-Stokes equation

$$\partial^i v_i = 0, \quad \partial_\tau v_i - \bar{\eta}\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$$

to solutions of the Einstein equation:

$$G_{\mu\nu} = 0$$



Fixing cutoff surface $r = r_c$, then perturbing:

- induced metric at $r = r_c$ is Ricci flat
- waves are infalling at $r = r_h$
- extrinsic curvature at $r = r_c$ becomes fluid stress tensor . . .
- in a hydrodynamic limit

Bredberg, CAK, Lysov, Strominger, 1006.1902 and 1103.2355

Satisfying the Einstein Constraints

The Nonlinear Metric in the Hydrodynamic Limit

$$
\begin{aligned}
ds^2 = &- r d\tau^2 + 2 d\tau dr + dx_i dx^i \\
&- 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
&+ \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
&- \frac{\left(r^2 - r_c^2\right)}{r_c} \partial^2 v_i dx^i d\tau + \ldots \mathcal{O}(\epsilon^3)
\end{aligned}
$$

with $v_i \sim \mathcal{O}(\epsilon), \quad P \sim \mathcal{O}(\epsilon^2), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2)$.

- Induced metric at $r = r_c$ cutoff is flat
- constraint eqns at $\mathcal{O}(\epsilon^2)$ are $\partial^i v_i = 0$
- constraint eqns at $\mathcal{O}(\epsilon^3)$ are $\partial_\tau v_i - r_c \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0$, Navier-Stokes with viscosity $\bar{\eta} = r_c$
- $G_{ra}, G_{ab}, G_{rr} = \mathcal{O}(\epsilon^4)$

Cutoff Approach

Highlights

- Does not require AdS, but is connectible to the AdS approach
  (Brattan, Camps, Loganayagam, Rangamani 1106.2577)
  Toy to test how double copy relates to AdS/CFT?
- Extendible to higher orders
  (Compere, McFadden, Skenderis, Taylor, 1103.3022; Pinzani-Fokeeva, Taylor 1401.5975)
- Hydrodynamic limit can be recast as near horizon limit
- Spacetime is algebraically special!
  Generic 4d fluid-duals are Petrov type II through $\mathcal{O}(\epsilon^{14})$
  (Bredberg, Keeler, Lysov, Strominger 1101.2451
- More restricted fluids are more special!

Petrov type II: $C^{II}_{ABCD} \sim \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$

Type D: $C^{D}_{ABCD} \sim \alpha_{(A}\alpha_B\beta_C\beta_{D)}$ Type N: $C^{N}_{ABCD} \sim \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$

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Algebraic Speciality in Fluids

Can prove algebraic speciality by writing

$$C_{ABCD} = \Psi_0 \iota_A \iota_B \iota_C \iota_D - 4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)} - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$

If only Ψ_2 is nonzero, then the spacetime is type D.
If only Ψ_4 is nonzero, then the spacetime is type N.
For general fluid-dual spacetimes, $\Psi_0, \Psi_1, \Psi_3 = 0 + \mathcal{O}(\epsilon^3)$,

$$\Psi_2 = -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right)/4r_c + \mathcal{O}(\epsilon^3)$$
$$\Psi_4 = -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right)/2r + \mathcal{O}(\epsilon^3).$$

Algebraically special fluid-dual spacetimes (τ-independent)

- Type D fluids have constant vorticity
- Type N fluids are potential flows

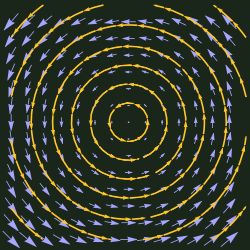
Type D fluids: Constant Vorticity

Only nonzero Ψ_I is

$$\Psi_2 = -i\epsilon^2\omega/2r_c + \mathcal{O}(\epsilon^3)$$

with natural background

$$ds^2_{(0)} = -r d\tau^2 + 2 dr d\tau + dx^2 + dy^2$$



Single and Zeroth Copies

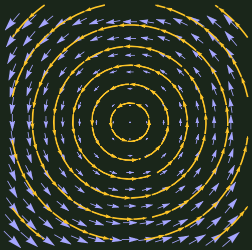
$$S = i\omega r_c e^{2i\theta}, \quad f_{AB} = e^{i\theta}\omega \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to \begin{cases} F^{\tau r} = -\omega\cos\theta \\ F^{xy} = -\omega\sin\theta \end{cases}$$

- all other $F^{\mu\nu}$ components are zero
- S is constant so trivially solves $\nabla^2_{(0)} S = 0$
- $\nabla^{(0)}_\nu F^{\mu\nu} = 0, \quad \nabla^{(0)}_{[\mu} F_{\rho\sigma]} = 0$

Type D fluid single copy: A giant solenoid

Choosing $\theta = 3\pi/2$ we have

$$v_x = -\omega y, \quad v_y = \omega x$$
$$F^{\tau r} = 0, \quad F^{xy} = \omega$$
$$E_\mu = 0, \quad B_\mu = \omega \delta^r_\mu$$



Type D Fluid Double Copy Summary

- Fluid is solution inside of slowly rotating cylinder with no-slip conditions at the wall
- Magnetic field $\vec{B} = \omega \hat{r}$ is uniform field inside a big solenoid with current proportional to ω
- zeroth copy field S is constant and thus plays a passive role
- Fluid only in hydro regime for $x, y \sim \epsilon^{-1}$; can fix by going to near-horizon expansion instead

Type N fluids: Potential flow: The Double Copy Story

The potential ϕ resides in the zeroth copy scalar S .
We have, using $z = x + iy$,

$$v_x = \partial_x \phi, \quad v_y = \partial_y \phi \text{ with } \phi = f(z) + \bar{f}(\bar{z})$$

The zeroth and single copy fields become

$$S = -\frac{e^{2i\theta}}{2\partial_{\bar{z}}^2 \bar{f}(\bar{z})}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Type N Fluid Double Copy Summary

- $\nabla^2_{(0)} S = 0$ nontrivially; because $\phi = f(z) + \bar{f}(\bar{z})$
- 'Background' single copy field is $\vec{E} = -\hat{x}$, $\vec{B} = \hat{y}$
- Poynting vector of single copy is $\vec{S} = -\hat{r}$.
- Gauge field is single copy necessary to build up any fluid with a potential component.

Algebraically Special Fluid Double Copy Summary

Type D Fluid Double Copy Summary

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Type N Fluid Double Copy Summary

- $\nabla^2_{(0)} S = 0$ nontrivially; because $\phi = f(z) + \bar{f}(\bar{z})$
- 'background' single copy field is still $\vec{E} = -\hat{x},\ \vec{B} = \hat{y}$
- Poynting vector of single copy is $\vec{S} = -\hat{r}$.
- Gauge field is single copy necessary to build up any fluid with a potential component.

Can we generalize to all Fluid Dual spacetimes?

Generic fluid-dual spacetime is type II, so its Weyl spinor is

$$C_{ABCD} = 6\Psi_2 o_{(A}\iota_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D = \frac{1}{S} f^{(1)}_{(AB} f^{(2)}_{CD)} \text{ for}$$

$$\begin{aligned} f^{(1)}_{AB} =& \beta \left(i\sqrt{6\Psi_2} o_{(A}\iota_{B)} + \sqrt{\Psi_4} o_A o_B \right) \\ f^{(2)}_{AB} =& \frac{S}{\beta} \left(-i\sqrt{6\Psi_2} o_{(A}\iota_{B)} + \sqrt{\Psi_4} o_A o_B \right) \end{aligned}$$

but we cannot pick a single β and S for which both gauge fields solve Maxwell and the scalar solves Klein-Gordon for all fluids.

Possible Solutions

- Extension: $C = \frac{1}{s_1} f^{(1,1)} f^{(1,2)} + \frac{1}{s_2} f^{(2,1)} f^{(2,2)}$
- Twistor method? cf. White 2012.02479; Chacon, Nagy, White 2103.16441

Details on General Type II Fluid-Dual spacetimes

The Weyl Spinor

In two dimensions for incompressible fluids: $v_i = \epsilon_{ij}\partial_j\chi$, where χ is the stream function and $z = x + iy$:

$$C_{ABCD} = 6\Psi_2 o_{(A}\iota_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$
$$\Psi_2 = i\partial_z\partial_{\bar{z}}\chi, \quad \Psi_4 = 2i\partial_{\bar{z}}\partial_{\bar{z}}\chi.$$

(In near horizon, λ expansion)

Factorizing $C_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$:

$$\alpha_A = \frac{P}{i\sqrt{6\Psi_2}} o_A$$
$$\beta_A = i\sqrt{6\Psi_2}\iota_A + \sqrt{\Psi_4} o_A$$
$$\gamma_A = -iC\sqrt{6\Psi_2}\iota_A + C\sqrt{\Psi_4} o_A$$

Setting $S = -CP^2/6\Psi_2$, we can factorize either as $f_1 = \alpha_{(A}\beta_{B)}$, $f_2 = \alpha_{(A}\gamma_{B)}$ or as $f_1 = \alpha_{(A}\alpha_{B)}$, $f_2 = \beta_{(A}\gamma_{B)}$.

Details on General Type II Fluid-Dual spacetimes, Pt 2

The Weyl Spinor

$$C_{ABCD} = 6\Psi_2 o_{(A}\iota_B o_C \iota_{D)} + \Psi_4 o_A o_B o_C o_D$$
$$\Psi_2 = i\partial_z\partial_{\bar{z}}\chi, \quad \Psi_4 = 2i\partial_{\bar{z}}\partial_{\bar{z}}\chi.$$

Cases with $\Psi_2 \neq 0$ and $\Psi_4 \neq 0$:

- $\chi_{\text{Couette}} = A(z-\bar{z})^3 + B(z-\bar{z})^2 + C(z-\bar{z})$
- $\chi_{\text{Oseen-Lamb}} = \mathrm{Ei}\left[-z\bar{z}/4\eta t\right]/4\pi$

- Recover type D constant vorticity and type N potential flow
- For $\alpha\beta$, $\alpha\gamma$ factorization, Maxwell's give $\partial_{\bar{z}}\left[\Psi_4/\Psi_2\right] = 0$. But (e.g. for Couette flow) can't always get $\nabla^2 S \neq 0$ (instead have $\nabla^2(1/S) = 0$)
- For $\alpha\alpha$, $\beta\gamma$ factorization: Maxwell's: $\partial_{\bar{z}}\left[\Psi_4/\Psi_2\right] = 0$, again Couette flow gives $\nabla^2(1/S) = 0$).
- Oseen-Lamb vortex doesn't work under either factorization

Future Directions

Future Questions

- Solving for Type II fluids (e.g. Couette or Oseen-Lamb):
  - Consider extension to sum of terms:

$$C = \frac{1}{s_1} f^{(1,1)} f^{(1,2)} + \frac{1}{s_2} f^{(2,1)} f^{(2,2)}$$

  - Use Penrose transform/twistor story?
  - Perturbative but nonlinear in Navier-Stokes
- higher orders in ϵ or λ?
- relate to other fluid-gravity dualities
  - large D and near horizon physics
  - AdS/CFT: study fluid modes?
- Larger dimensions: 5d gravity= 3+1 d fluid

  forthcoming: (S. Chawla+C.K.) on general separable spacetimes as a double copy

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\begin{aligned}
\Psi_2 &= -i\epsilon^2 \left(\partial_x v_y - \partial_y v_x\right) / 4r_c + \mathcal{O}(\epsilon^3) \\
\Psi_4 &= -\epsilon^2 \left(\partial_x v_x - \partial_y v_y + i(\partial_x v_y + \partial_y v_x)\right) / 2r + \mathcal{O}(\epsilon^3).
\end{aligned}
$$

Algebraically special fluid-dual spacetimes

- Type D fluids have constant vorticity
- Type N fluids are potential flows

Type N fluids: Planar Extensional Flow

The simplest Type N fluid has $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

The zeroth and single copy fields become

$$S = \frac{e^{2i\theta}}{\alpha}, \quad f_{AB} = \frac{e^{i\theta}}{\sqrt{r}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Again choosing $\theta = 3\pi/2$ the nonzero components of F become

$$F^{rx} = 1,\ F^{\tau x} = \frac{2}{r} \quad \longrightarrow \quad \vec{E} = -\hat{x},\ \vec{B} = \hat{y}.$$

On the background $ds^2_{(0)} = -r d\tau^2 + 2 dr d\tau + dx^2 + dy^2$ again both Klein-Gordon and Maxwell's are solved.

Poynting vector is

$$\vec{S} = -\hat{r}.$$

Gauge field is single copy necessary to build up any fluid with a potential component.

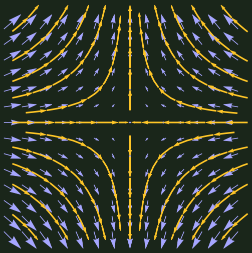
What if we consider a different potential ϕ?

Type N fluids: Potential flow: The Double Copy Story

We already studied extensional flow: $\phi = \frac{\alpha}{2}(y^2 - x^2)$, so

$$v_x = \partial_x \phi = -\alpha x,\ v_y = \partial_y \phi = \alpha y$$

$$\vec{E} = -\hat{x}, \vec{B} = \hat{y}$$



But there are many other potential flow fluids!

	Potential ϕ	v_x	v_y
Ext. flow	$-\frac{\alpha}{2}(x^2 - y^2)$	$-\alpha x$	αy
Source/Sink	$\ln(x^2 + y^2)$	$2x/(x^2 + y^2)$	$2y/(x^2 + y^2)$
Dipole	$x/(x^2 + y^2)$	$(y^2 - x^2)/(x^2 + y^2)^2$	$-2xy/(x^2 + y^2)^2$
Line Vortex	$\arctan(y/x)$	$-y/(x^2 + y^2)$	$x/(x^2 + y^2)$

If $F_{\mu\nu}$ is just a 'support' single copy, then what distinguishes these fluids from each other? S!