Elliptic Feynman integrals and modular transformations

Stefan Weinzierl

Institut für Physik, Universität Mainz

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Section 1

Motivation

An analogy: The choice of an gauge

- *Consider a scattering amplitude in Yang-Mills theory.*
- *It can be expressed as a sum of Feynman diagrams.*
- *In computing Feynman diagrams we make a gauge choice.*
- *The scattering amplitude is independent of the gauge choice.*

Remark: This can be non-trivial. Consider for example the calculation of the Altarelli-Parisi splitting functions in Feynman gauge and in the axial gauge.

This talk: The choice of a pair of periods for an elliptic curve

- *Consider a (scalar, elliptic) Feynman integral.*
- *It can be expressed as a linear combination of master integrals.*
- *In computing elliptic master integrals we make a choice for a pair of periods.*
- *The original Feynman integral is independent of this choice.*

Modular transformations

Let's assume we choose as periods (ψ_2, ψ_1), while somebody else made the choice (ψ_2', ψ_1').

The two choices are related by a (2×2)-matrix γ:

$$\left(\begin{array}{c} \psi_2' \\ \psi_1' \end{array} \right) = \gamma \left(\begin{array}{c} \psi_2 \\ \psi_1 \end{array} \right)$$

This is called a **modular transformation**.

Questions

Suppose our elliptic Feynman integral can be expressed for a particular choice of periods as **iterated integrals of modular forms** for a **congruence subgroup Γ**.

1. What happens for a modular transformation $\gamma \notin \Gamma$?

2. Even worse, for $\gamma \in \Gamma$ we can show that in general iterated integrals of modular forms for Γ do not transform into iterated integrals of modular forms for Γ. What is going on here?

Both questions have a nice answer.

Section 2

Background from physics

Feynman integrals

Notation:

$$I_{\nu_1\nu_2\dots\nu_n}(\varepsilon, x) = e^{l\varepsilon\gamma_\mathrm{E}} \left(\mu^2\right)^{\nu-\frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{n_\mathrm{int}} \frac{1}{\left(-q_j^2+m_j^2\right)^{\nu_j}}$$

Kinematic variables $x_1, \dots, x_{N_B+1}$:

$$\frac{-p_i \cdot p_j}{\mu^2}, \qquad \frac{m_i^2}{\mu^2}.$$

As μ^2 is arbitrary, we may set one kinematic variable to one.

Integration-by-parts allows us to express any $I_{\nu_1\nu_2\dots\nu_n}$ as a linear combination of master integrals.

Notation

$\mathbf{N_F} = N_{\mathrm{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

$\mathbf{N_B} = N_{\mathrm{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_{N_B})$.

The method of differential equations

We want to calculate

$$I = (I_1, ..., I_{N_F})$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integrals.*

$$[d + A(\varepsilon, x)] I = 0.$$

2. *Transform the differential equation into a simple form.*

$$[d + \varepsilon A(x)] I = 0.$$

3. *Solve the latter differential equation with appropriate boundary conditions.*

Transformations

- **Change the basis of the master integrals**

$$I' = UI,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

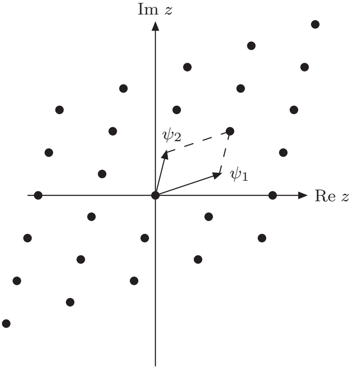
The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Section 3

Background from mathematics

Representing an elliptic curve as $\mathbb{C}/\Lambda$



Points inside fundamental parallelogram $\Leftrightarrow$ Points on elliptic curve

An elliptic curve together with a choice of periods (ψ_2, ψ_1) is called a framed elliptic curve.

Notation

Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$, where

$$\tau = \frac{\psi_2}{\psi_1}$$

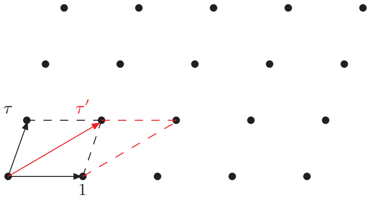
and require $\text{Im}(\tau) > 0$.

Definition (The complex upper half-plane)

$$\mathbb{H} = \{\tau \in \mathbb{C} | \text{Im}(\tau) > 0\}$$

Modular transformations

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a **modular form** of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

1. f transforms under modular transformations as
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$
2. f is holomorphic on $\mathbb{H}$,
3. f is holomorphic at $i\infty$.

Define the $|_k\gamma$ operator by
$$(f|_k\gamma)(\tau) \;=\; (c\tau+d)^{-k} \cdot f(\gamma(\tau)).$$

Then item 1 can be written as
$$f|_k\gamma \;=\; f.$$

Congruence subgroups

Apart from $\mathrm{SL}_2(\mathbb{Z})$ we may also look at congruence **subgroups**, for example

$$\Gamma_0(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d \equiv 1 \bmod N, \; b, c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "**nice**" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Modular forms

For a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ denote by $\mathcal{M}_k(\Gamma)$ the **space of modular forms of weight k**.

We have the inclusions

$$\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq \mathcal{M}_k(\Gamma_0(N)) \subseteq \mathcal{M}_k(\Gamma_1(N)) \subseteq \mathcal{M}_k(\Gamma(N))$$

$\mathcal{M}_k(\Gamma(N))$ is the largest space.

Fourier expansions

Let N' be the smallest positive integer such that

$$\left(\begin{array}{cc} 1 & N' \\ 0 & 1 \end{array} \right) \in \Gamma.$$

It follows that modular forms for Γ have a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n \bar{q}_{N'}^n, \qquad \bar{q}_{N'} = \exp\left(\frac{2\pi i \tau}{N'}\right).$$

Iterated integrals of modular forms

Let $f_1, \dots, f_n$ be modular forms for the congruence subgroup Γ.

$$I\left(f_1, f_2, ..., f_n; \tau\right) = \left(2\pi i\right)^n \int\limits_{i\infty}^{\tau} d\tau_1 f_1\left(\tau_1\right) \int\limits_{i\infty}^{\tau_1} d\tau_2 f_2\left(\tau_2\right) ... \int\limits_{i\infty}^{\tau_{n-1}} d\tau_n f_n\left(\tau_n\right)$$

An integral over a modular form is in general **not** a modular form.

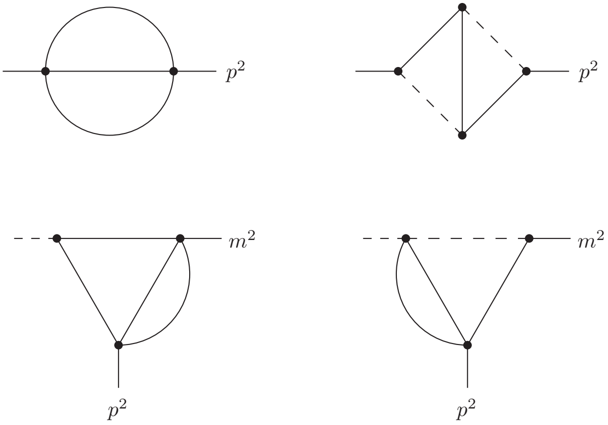
Analogy: An integral over a rational function is in general not a rational function.

Section 4

One kinematic variable

Elliptic Feynman integrals

Examples of elliptic Feynman integrals depending on one kinematic variable $x = \frac{-p^2}{m^2}$ (solid internal lines of mass m, dashed lines massless):



Transformation to an ε-form

We may put the differential equation for these Feynman integrals into an ε-form with only simple poles by

- making a choice for the two periods (ψ_2, ψ_1),
- performing a fibre transformation $J = UI$.
- performing a base transformation $x \to \tau$.

Example: The equal mass sunrise integral

$$(d+A)J = 0, \qquad A = 2\pi i\,\varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & f_2(\tau) & f_0(\tau) \\ f_3(\tau) & f_4(\tau) & f_2(\tau) \end{pmatrix} d\tau.$$

For this particular example, the $f_k(\tau)$'s are modular forms of modular weight k for $\Gamma_1(6)$.

Solution given as **iterated integrals of modular forms for $\Gamma_1(6)$**:

$$J_2 = \left[3\,\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) + 4I(f_0, f_3; \tau)\right]\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right).$$

Modular transformations

Let us now consider the base transformation

$$\tau' = \gamma(\tau) \quad \textit{with} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Modular transformations

- For $f \in \mathcal{M}_k(\Gamma)$ of level N and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ we have in general

$$f|_k\gamma \quad \notin \quad \mathcal{M}_k(\Gamma).$$

- However if $f \in \mathcal{M}_k(\Gamma)$ we also have $f \in \mathcal{M}_k(\Gamma(N))$.
  As the principal congruence subgroup $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ we have for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$f|_k\gamma \quad \in \quad \mathcal{M}_k(\Gamma(N)).$$

This **solves question** ❶ for the case of one kinematic variable.

Consider now the coordinate transformation

$$\tau \;=\; \gamma^{-1}\left(\tau'\right) \;=\; \frac{a\tau'+b}{c\tau'+d}, \qquad \gamma^{-1} \in \Gamma.$$

We have

$$\begin{aligned} I\left(f;\tau\right) &= 2\pi i \int\limits_{i\infty}^{\tau} f\left(\tilde{\tau}\right) d\tilde{\tau} \\ &= 2\pi i \int\limits_{\gamma(i\infty)}^{\gamma(\tau)} \left(c\tilde{\tau}'+d\right)^{k-2} (f|_k\gamma^{-1})(\tilde{\tau}')\, d\tilde{\tau}'. \end{aligned}$$

For $k \neq 2$ we pick up a power of the automorphic factor $(c\tilde{\tau}'+d)$ and **leave the space of iterated integrals of modular forms**!

Transformations of multiple polylogarithms

In the case of multiple polylogarithms

$$G(z_1, \dots, z_k; y)$$

the transformations

$$y' = 1 - y, \quad y' = \frac{1}{y}, \quad y' = \frac{1}{1-y}, \quad y' = \frac{1-y}{1+y},$$

don't leave the space of multiple polylogarithms.

Solution to question 2:

In order not to leave the space of iterated integrals of modular forms a base transformation

$$\tau' = \gamma(\tau)$$

needs to be accompanied by a fibre transformation

$$J' = UJ.$$

Example: The equal mass sunrise integral

For

$$\gamma(\tau) \;=\; \frac{a\tau+b}{c\tau+d}, \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z})$$

we have to consider the combined transformation

$$\begin{aligned} J' &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (c\tau+d)^{-1} & 0 \\ 0 & \frac{c}{2\pi i \varepsilon f_0} & (c\tau+d) \end{pmatrix} J, \\ \tau' &= \frac{a\tau+b}{c\tau+d}. \end{aligned}$$

Example: The equal mass sunrise integral

Working out the transformed differential equation we obtain

$$\left(d + A'\right) J' \quad = \quad 0$$

with

$$A' \quad = \quad 2\pi i\,\varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & (f_2|_2\gamma^{-1})(\tau') & (f_0|_0\gamma^{-1})(\tau') \\ (f_3|_3\gamma^{-1})(\tau') & (f_4|_4\gamma^{-1})(\tau') & (f_2|_2\gamma^{-1})(\tau') \end{pmatrix} d\tau'.$$

We have

$$f_k|_k\gamma^{-1} \quad \in \quad \mathcal{M}_k(\Gamma(6))$$

and J' can be expressed as **iterated integrals of modular forms for $\Gamma(6)$**.

Example: The equal mass sunrise integral

The fact that we need to redefine the master integrals is not too surprising.

$$J' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (c\tau+d)^{-1} & 0 \\ 0 & \frac{c}{2\pi i \varepsilon f_0} & (c\tau+d) \end{pmatrix} J$$

We originally defined J_2 by

$$\mathbf{J_2} = \varepsilon^2 \frac{\pi}{\psi_1} \mathbf{I_{111}}(\mathbf{2} - \mathbf{2}\varepsilon, \mathbf{x}).$$

The automorphic factor $(c\tau + d)$ is the ratio of two periods

$$c\tau + d = \frac{\psi_1'}{\psi_1}.$$

We find that J_2' is given by

$$\mathbf{J_2'} = \varepsilon^2 \frac{\pi}{\psi_1'} \mathbf{I_{111}}(\mathbf{2} - \mathbf{2}\varepsilon, \mathbf{x}).$$

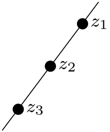
Section 5

Several kinematic variables

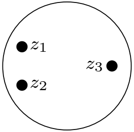
Moduli spaces

$\mathcal{M}_{g,n}$: Space of **isomorphism classes of** smooth (complex, algebraic) **curves of genus g with n marked points**.

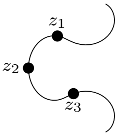
complex curve



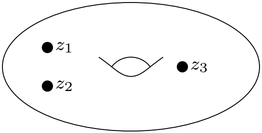
$\Leftrightarrow$



real surface



$\Leftrightarrow$



Coordinates

Genus 0: $\dim \mathcal{M}_{0,n} = n-3$.
Sphere has a unique shape
Use Möbius transformation to fix $z_{n-2} = 1$, $z_{n-1} = \infty$, $z_n = 0$
Coordinates are $(\mathbf{z_1}, \dots, \mathbf{z_{n-3}})$

Genus 1: $\dim \mathcal{M}_{1,n} = n$.
One coordinate describes the shape of the torus
Use translation to fix $z_n = 0$
Coordinates are $(\tau, \mathbf{z_1}, \dots, \mathbf{z_{n-1}})$

Iterated integrals on $\mathcal{M}_{0,n}$

We are interested in differential one-forms, which have only simple poles:

$$\omega^{\mathrm{mpl}}(z_j) = \frac{dy}{y - z_j}.$$

Multiple polylogarithms:

$$G(z_1, \ldots, z_k; y) = \int\limits_0^y \frac{dy_1}{y_1 - z_1} \int\limits_0^{y_1} \frac{dy_2}{y_2 - z_2} \ldots \int\limits_0^{y_{k-1}} \frac{dy_k}{y_k - z_k}, \quad z_k \neq 0$$

The Kronecker function

Define the **first Jacobi theta function** $\theta_1(z, \bar{q})$ by

$$\theta_1(z, \bar{q}) = -i \sum_{n=-\infty}^{\infty} (-1)^n \bar{q}^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} e^{i\pi(2n+1)z}.$$

The **Kronecker function** $F(z, \alpha, \tau)$:

$$F(z, \alpha, \tau) = \theta_1'(0, \bar{q}) \frac{\theta_1(z+\alpha, \bar{q})}{\theta_1(z, \bar{q})\, \theta_1(\alpha, \bar{q})} = \frac{1}{\alpha} \sum_{k=0}^{\infty} \mathbf{g}^{(\mathbf{k})}(\mathbf{z}, \tau) \alpha^k$$

We are interested in the coefficients $g^{(k)}(z, \tau)$ of the Kronecker function.

The coefficients $g^{(k)}(z,\tau)$ of the Kronecker function

Properties of $g^{(k)}(z,\tau)$:

1. **only simple poles** as a function of z
2. **quasi-periodic** as a function of z: Periodic by 1, quasi-periodic by τ.

$$
\begin{aligned}
g^{(k)}(z+1,\tau) &= g^{(k)}(z,\tau), \\
g^{(k)}(z+\tau,\tau) &= \sum_{j=0}^{k} \frac{(-2\pi i)^j}{j!} g^{(k-j)}(z,\tau)
\end{aligned}
$$

3. **quasi-modular**:

$$
g^{(k)}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \sum_{j=0}^{k} \frac{(2\pi i)^j}{j!} \left(\frac{cz}{c\tau+d}\right)^j g^{(k-j)}(z,\tau)
$$

Differential one-forms on $\mathcal{M}_{1,n}$

1. *From modular forms:*

$$\omega_k^{\mathrm{modular}} = 2\pi i \, f_k(\tau) d\tau$$

2. *From the Kronecker function:*

$$\omega_k^{\mathrm{Kronecker}} = (2\pi i)^{2-k} \left[g^{(k-1)}\left(L(z),\tau\right) dL(z) + (k-1)\, g^{(k)}\left(L(z),\tau\right) \frac{d\tau}{2\pi i} \right],$$

$$L(z) = \sum_{j=1}^{n-1} \alpha_j z_j + \beta.$$

Modular transformations

A modular transformation acts as

$$\tau' = \frac{a\tau+b}{c\tau+d}, \quad z_j' = \frac{z_j}{c\tau+d}, \quad \beta' = \frac{\beta}{c\tau+d}.$$

We may view β as being a further marked point in a higher dimensional space $\mathcal{M}_{1,n'}$ with $n' > n$.

We find

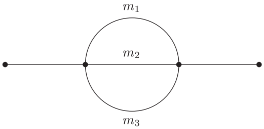
$$\omega_k^{\text{Kronecker}}\left(L'\left(z'\right),\tau'\right) = \left(c\tau+d\right)^{k-2} \sum_{j=0}^{k} \frac{1}{j!} \left(\frac{cL\left(z\right)}{c\tau+d}\right)^j \omega_{k-j}^{\text{Kronecker}}\left(L\left(z\right),\tau\right).$$

Example: The unequal mass sunrise

- 7 master integrals

- 3 kinematic variable

$$x_1 = \frac{-p^2}{m_3^2}, \quad x_2 = \frac{m_1^2}{m_3^2}, \quad x_3 = \frac{m_2^2}{m_3^2}.$$



Example: The unequal mass sunrise

We may put the differential equation into an ε-form with only simple poles by

- making a choice for the two periods (ψ_2, ψ_1),
- performing a fibre transformation $J = UI$.
- performing a base transformation $(x_1, x_2, x_3) \rightarrow (\tau, z_1, z_2)$.

$$(d + A) J = 0, \qquad A = \varepsilon \sum_{j=1}^{N_L} C_j \, \omega_j$$

and the ω_j's are either of the form $\omega^{\mathrm{modular}}$ or $\omega^{\mathrm{Kronecker}}$.

Example: The unequal mass sunrise

A modular transformation on the base

$$(\tau, z_1, z_2, \beta) \quad \rightarrow \quad (\tau', z_1', z_2', \beta')$$

is accompanied by the fibre transformation

$$J' \quad = U \, J,$$

with U given by

$$U \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c\tau+d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{6ic(z_1+z_2)}{c\tau+d} & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{2ic(z_1-z_2)}{c\tau+d} & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{c}{2\pi i \varepsilon} + \frac{c^2\left(z_1^2+z_1 z_2+z_2^2\right)}{c\tau+d} & -\frac{ic(z_1+z_2)}{4} & -\frac{ic(z_1-z_2)}{4} & c\tau+d \end{pmatrix}.$$

Section 6

Comments

$\bar{q}$-expansions

Why do we bother about modular transformations?

- *Iterated integrals in the elliptic case are evaluated with the help of their $\bar{q}$-expansions.*
- *The $\bar{q}$-series converge for $|\bar{q}| < 1$.*
- *By a modular transformation we may map τ to the fundamental domain, resulting in*

$$|\bar{q}| \leq e^{-\pi\sqrt{3}} \approx 0.0043,$$

resulting in a fast converging series.

Uniqueness

Are master integrals of uniform weight unique?

- *Assume that J is a set of master integrals of uniform weight and U an* *x-independent* *invertible $(N_F \times N_F)$-matrix. Then*

$$J' = UJ$$

  is also of uniform weight.
- *Assume that J is of uniform weight and contains elliptic Feynman integrals. A modular transformation induces a* **x-dependent** *transformation U, such that J' is again of uniform weight.*

Maybe we shouldn't use the word "canonical" in this context.

Bootstrap

Modularity puts *(in addition to integrability)* **constraints** *on the matrix A:*

Example

Consider a system consisting of one master integral J depending on two variables (z, τ) with differential equation

$$(d+A)J = 0.$$

The matrix

$$A = \varepsilon \left[\omega_2^{\mathrm{Kronecker}}(z,\tau) - 2\omega_2^{\mathrm{Kronecker}}(z,2\tau) \right]$$

is modular, while the apparent simpler choice

$$A = \varepsilon \omega_2^{\mathrm{Kronecker}}(z,\tau)$$

is not.

Conclusions

- For elliptic Feynman integrals we expect that the choice of periods does not matter. This implies that the system should be modular.
- By a modular transformation we can always achieve $|\bar{q}| \leq 0.0043$.
- Unfortunately, as a modular transformation is always accompanied by a fibre transformation, there is no black-box numerical evaluation algorithm just for iterated integrals of $\omega^{\mathrm{modular}}$ and $\omega^{\mathrm{Kronecker}}$.
- There are x-dependent fibre transformations, which transform master integrals of uniform weight into master integrals of uniform weight.
- Modularity puts constraints on the matrix A.