Two-Dimensional Models for Soft Theorems

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Based on—

* 2011.11412 and
* 2107.06660.

Main Results

1. Soft theorems in QED and gravity equivalent to Ward identities of dual two-dimensional (read holographic) models.

2. Global symmetries of the model imply gauge invariance and equivalence principle.

3. QED and gravity models related by an incarnation of the double copy.

4. Asymptotic symmetries find nice interpretations.

Introduction

* Route to flat space holography—massless scattering.

* Due to massless kinematics—

$$p = \omega(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}).$$

* Achieved by analysis of amplitudes on null infinity.

* Much easier in the *soft sector*.

Soft Theorems on $\mathbb{CP}^1$

* Analytic structure of soft radiation encoded by so-called soft factor in QED

$$S^{(0)}_{\text{QED}} = \sum_{i=1}^{n} \frac{e_i}{z - z_i}$$

and gravity

$$S^{(0)}_{\text{grav}} = \sum_{i=1}^{n} \kappa\omega_i \frac{\bar{z} - \bar{z}_i}{z - z_i}.$$

* Two observations -
  1. Dynamically dependent on positions on $\mathbb{CP}^1$.
  2. Singularities suggest existence of two-dimensional dynamical duals.

Learning from Ward Identities

- Recall that soft theorems in QED and gravity can be obtained from the statement that the charges

$$Q_\varepsilon^{\mathrm{QED}} = \int \varepsilon(z,\bar{z}) \left(\underbrace{\partial_{\bar{z}} a_+(z,\bar{z},u) + \mathrm{c.c}}_{\mathrm{soft}} + \underbrace{j_u(z,\bar{z},u)}_{\mathrm{hard}} \right) du d^2 z$$

$$Q_\sigma^{\mathrm{grav}} = \int \sigma(z,\bar{z}) \left(\underbrace{\partial_{\bar{z}}^2 g_+(z,\bar{z},u) + \mathrm{c.c}}_{\mathrm{soft}} + \underbrace{T_{uu}(z,\bar{z},u)}_{\mathrm{hard}} \right) du d^2 z$$

commute with the S-matrix. Soft theorems equivalent to specific choices of ε and σ.

- Our strategy leverages the global symmetries due to the soft-hard split.

Global Symmetries from Soft Charges

* Expectation: two-dimensional duals share the symmetries of the asymptotic charges. Simplest starting point is via the soft pieces.

* In QED, soft charge left invariant by shifts $\varepsilon \longrightarrow \varepsilon + f$ such that

$$\partial_z f(z, \bar{z}) = \partial_{\bar{z}} f(z, \bar{z}) = 0$$

which are solved by $f(z, \bar{z}) = a$ for constant a.

* Look for action that has the same symmetry.

The 2D Dual for Soft QED

* Unique choice is the free boson—

$$I_{\mathrm{QED}} = -\int dz \wedge d\overline{z}[\partial_z \varepsilon(z,\overline{z})\partial_{\overline{z}}\varepsilon(z,\overline{z})].$$

* Result 1: This dynamically captures leading soft sector in QED

* To do this, we perform a dressing quite analogous to that of Faddeev and Kulish, using instead the two-dimensional field ε.

The 2D for Soft QED

* Assumption: external states created by operators $\mathcal{O}_i(\omega_i, z_i, \bar{z}_i)$. We dress them according to

$$\mathcal{O}_i(\omega_i, z_i, \bar{z}_i) \longrightarrow \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i) = \exp(ie_i\varepsilon(z_i, \bar{z}_i))\mathcal{O}_i(\omega_i, z_i, \bar{z}_i).$$

* Soft theorem now equivalent to Ward identity of the current $\partial_z\varepsilon(z, \bar{z})$ (and the conjugate). Indeed,

$$\partial_z\varepsilon(z, \bar{z}) \prod_{i=1}^{n} \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i) \sim \frac{1}{\pi} \left(\sum_{i=1}^{n} \frac{ie_i}{z - z_i} \right) \prod_{i=1}^{n} \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i)$$

which is the positive helicity soft photon theorem. Thus, global shift Ward identity in 2D equivalent to 4D soft theorem.

The 2D Dual for Soft Gravity

* Similar story in gravity. Soft charges symmetric under shifts that obey

$$\partial_z^2 f(z,\bar{z}) = \partial_{\bar{z}}^2 f(z,\bar{z}) = 0.$$

Solutions given by $f(z,\bar{z}) = a_1 + a_2 z + a_3 \bar{z} + a_4 z\bar{z}$ for constant a_i.

* The simplest action invariant under these global symmetries is simply

$$I_{\text{grav}} = \int dz \wedge d\bar{z}[\partial_z^2 \sigma(z,\bar{z}) \partial_{\bar{z}}^2 \sigma(z,\bar{z})]$$

with the corresponding Noether charges given by $\partial_z^2 \sigma(z,\bar{z})$ and $\partial_{\bar{z}}^2 \sigma(z,\bar{z})$

The 2D Dual for Soft Gravity

- Indeed, we can now generalize the dressing to

$$\mathcal{O}_i(\omega_i, z_i, \bar{z}_i) \longrightarrow \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i) = \exp(i\kappa\omega_i\sigma(z_i, \bar{z}_i))\mathcal{O}_i(\omega_i, z_i, \bar{z}_i).$$

- By virtue of the fact that we have

$$\partial^2\sigma(z, \bar{z})\sigma(z', \bar{z}') \sim \frac{1}{\pi}\frac{\bar{z} - \bar{z}'}{z - z'}$$

we can recast the soft graviton theorem as the following operator product expansion

$$\partial_z^2\sigma(z, \bar{z})\prod_{i=1}^{n}\widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i) \sim \frac{1}{\pi}\left(\sum_{i=1}^{n} i\kappa\omega_i\frac{\bar{z} - \bar{z}_i}{z - z_i}\right)\prod_{i=1}^{n}\widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i).$$

Implications of Global Symmetries

- Upshot: consistency conditions from symmetries.

- In QED: shift by constant a gives a phase—

$$\prod_{i=1}^{n} \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i) \longrightarrow \exp\left(i(e_1 + \cdots + e_n)\right) \prod_{i=1}^{n} \widetilde{\mathcal{O}}_i(\omega_i, z_i, \bar{z}_i).$$

- Single-valuedness expectation value $\longrightarrow$ phase vanishes, equivalent to charge conservation.

Implications of Global Symmetries

* Similar upshot for gravity.

* Shift $\sigma(z,\bar{z}) \longrightarrow \sigma(z,\bar{z}) + a_1 + a_2 z + a_3 \bar{z} + a_4 z\bar{z}$ yields phase—

$$\delta = \kappa \sum_{i=1}^{n} \left(a_1\omega_i + a_2\omega_i z_i + a_3\omega_i\bar{z}_i + a_4\omega_i z_i\bar{z}_i \right).$$

* Vanishing of phase $\longrightarrow$ momentum conservation (or equivalence principle).

Double Copy Structures

- Double copy - $(\text{gravity}) = (\text{gauge})^2$ for observables. Usual method is replacing color by kinematics.

- Similar structure in the soft sector; gravity side obtained from QED by the replacements

$$(\partial_z, \partial_{\bar{z}}) \longrightarrow (\partial_z^2, \partial_{\bar{z}}^2)$$

and

$$e_i \longrightarrow \kappa\omega_i.$$

Relation to Asymptotic Symmetries

* Asymptotic symmetries also easily manifest. Exponential dressings $\exp\left(ie_i\varepsilon(z,\bar{z})\right)$ essentially large gauge transformations

* Since these functions are manifestly angle-dependent (thinking in terms of a stereographic projection), we see that for each classical configuration, the dressing corresponds to a specific large gauge transformation.

Relation to Asymptotic Symmetries

* Gravity case understood more easily by Mellin transforming energies -

$$\mathcal{O}(\omega_i, z_i, \bar{z}_i) \longrightarrow \int_0^\infty e^{iu_i\omega_i}\omega_i^\Delta \mathcal{O}(\omega_i, z_i, \bar{z}_i)\, d\omega_i.$$

In this basis, the dressing has the effect of performing the local translation

$$u_i \longrightarrow u_i + \sigma(z_i, \bar{z}_i).$$

By solving the biharmonic equation, we can see that any classical configuration for σ has the effect of generating supertranslations spanned by $\{z_i^n, \bar{z}_i z_i^n, \bar{z}_i^n, z_i \bar{z}_i^n\}$.

Further Questions at Higher Orders

* Informed by the Ward identities which are equivalent to higher order soft theorems, similar two-dimensional models with higher derivative kinetic terms can be shown to produce soft theorems beyond leading order.
* These higher derivative models have larger global symmetry groups, which would result in angle-dependent analogues of the conservation of charge and momentum.
* We have so far dealt with scattering amplitudes involving massless particles. Extending these results to include the effects of massive particles would be interesting.
* Generalizing this framework to describe soft theorems coming from loop corrections might be instructive as well.

Thank you!