Black-Hole Scattering

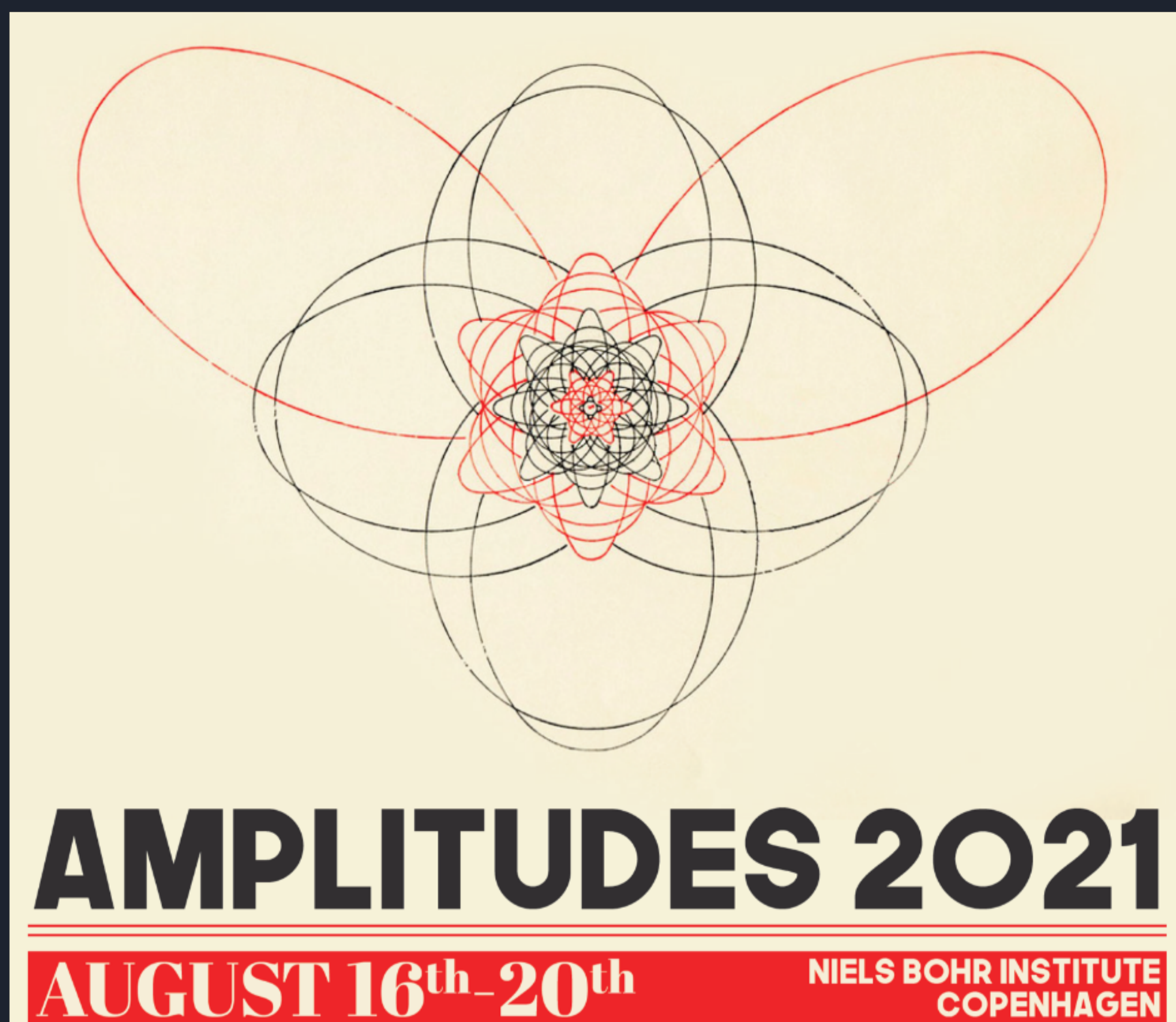
Eikonal Resummation

Gravitational Waves

NORDITA

Uppsala University

Carlo Heissenberg



BASED ON:

2008.12743 2101.05772 2104.03256

IN COLLABORATION WITH

P. Di Vecchia, R. Russo,

G. Veneziano

AND ON:

2105.04594

+ in progress...

Outline

- **Classical Gravity from Scattering Amplitudes**

  EIKONAL EXPONENTIATION

- **$\mathcal{N}=8$ as a Theoretical Laboratory**

  POTENTIAL + RADIATION REACTION @ 3PM

- **Radiative Effects in General Relativity**

  3-PARTICLE CUT & RADIATION

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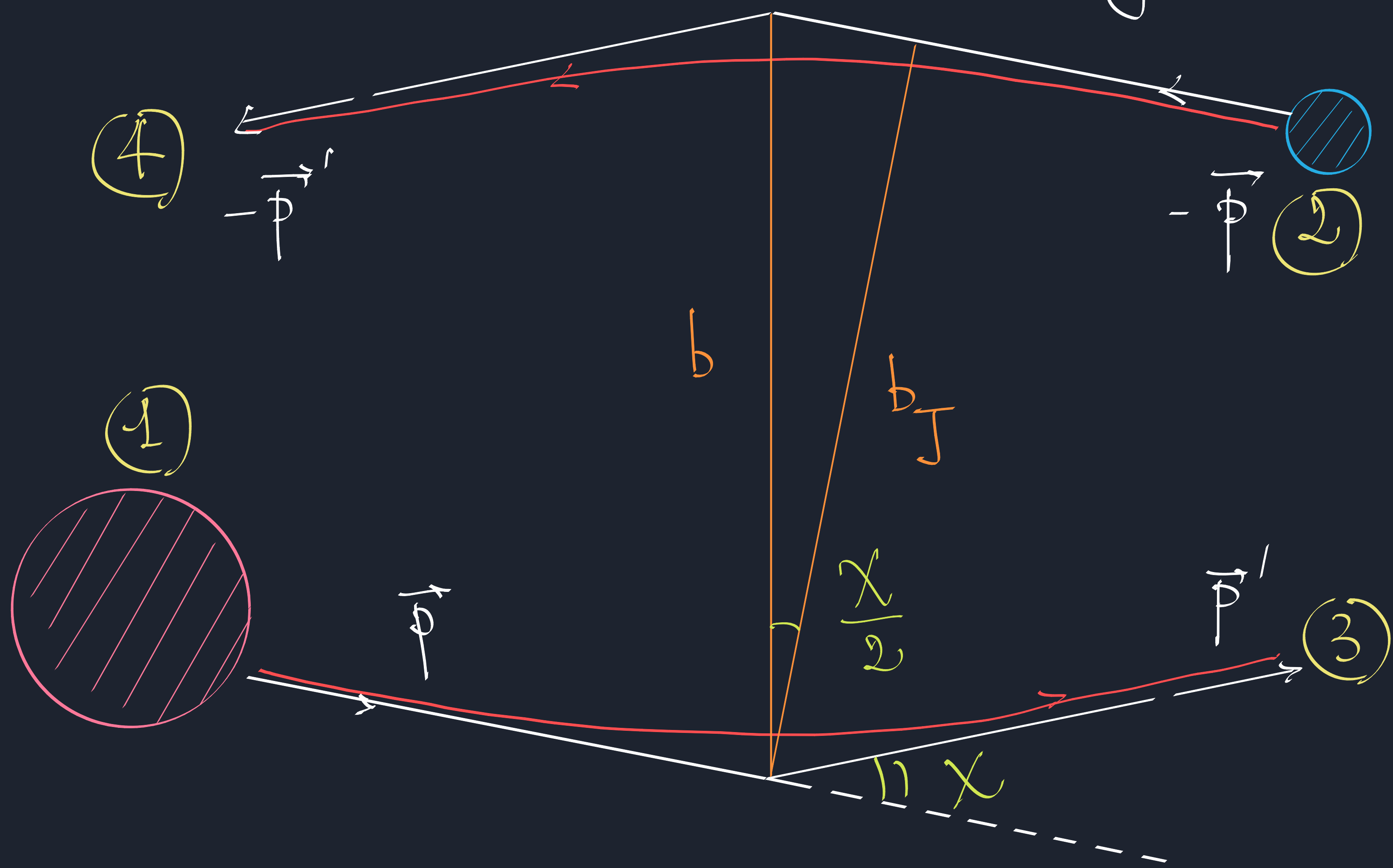
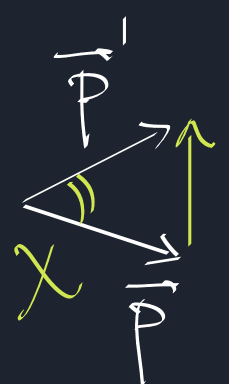




$\vec{Q}$ CLASSICAL MOMENTUM TRANSFER $\quad |\vec{Q}| = 2|\vec{p}|\sin\frac{\chi}{2}$

$b_J |\vec{p}| = J$ ANGULAR MOMENTUM $\quad b_J = b\cos\frac{\chi}{2}$

- Hierarchy of length scales:

Quantum wavelength $\frac{\hbar}{m}$ or $\frac{\hbar}{\sqrt{s}}$

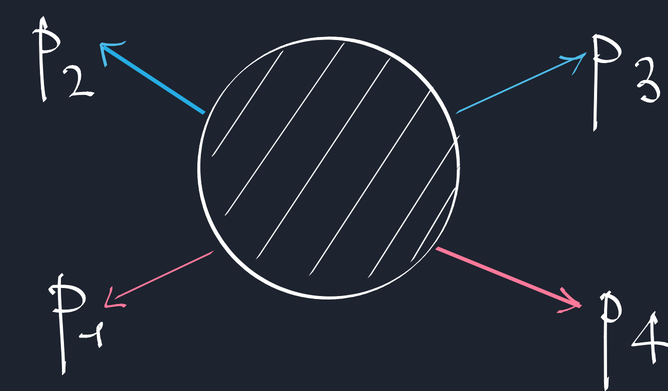
Schwarzschild radius Gm or $G\sqrt{s}$

Impact parameter

$$\lambda \ll R \ll b$$

$\boxed{\text{CLASSICAL LIMIT}}$ $\boxed{\text{PM REGIME}}$

- Amplitude:



$= \mathcal{A}(s,q)$

PERTURBATIVE MOMENTUM TRANSFER $\quad q = p_4 - p_1$

$$s = E^2 = -(p_1+p_2)^2 = m_1^2 + 2m_1 m_2\sigma + m_2^2$$

Large-b / Small-q Expansion

- Impact-parameter space representation:

$$D = 4 - 2\epsilon$$

$$\boxed{\tilde{A}(s,b) = \frac{1}{4m_1 m_2\sqrt{\sigma^2-1}} \int e^{ib\cdot q} A(s,q) \frac{d^{2-2\epsilon} q}{(2\pi)^{2-2\epsilon}}}$$

$$\begin{cases} \sigma = \dfrac{1}{\sqrt{1-v^2}} \\ E|\vec{p}| = m_1 m_2 \sqrt{\sigma^2-1} \end{cases}$$

* Large b $\Longleftrightarrow$ Small q^2

* Long-range in b $\Longleftrightarrow$ Non-Analytic in q^2

- Loop-expansion: $A = A_0 + A_1 + A_2 + \dots$

$$\tilde{A}_0 = \boxed{c_0\left(\frac{R}{\lambda}\right)}$$

$$\tilde{A}_1 = c_{1,0}\left(\frac{R}{\lambda}\right)^2 + \boxed{c_{1,1}\left(\frac{R}{\lambda}\right)\left(\frac{R}{b}\right)} + \boxed{c_{1,2}\left(\frac{R}{b}\right)^2\left[1+\mathcal{O}\left(\frac{\lambda}{b}\right)\right]}$$

$$\tilde{A}_2 = c_{2,0}\left(\frac{R}{\lambda}\right)^3 + c_{2,1}\left(\frac{R}{\lambda}\right)^2\left(\frac{R}{b}\right) + \boxed{c_{2,2}\left(\frac{R}{\lambda}\right)\left(\frac{R}{b}\right)^2} + \boxed{c_{2,3}\left(\frac{R}{b}\right)^3\left[1+\mathcal{O}\left(\frac{\lambda}{b}\right)\right]}$$

$$\boxed{\begin{aligned} &R = Gm\,(b^{2\epsilon}) \\ &\lambda = \frac{\hbar}{m} \\ &\frac{R}{\lambda} \gg 1 \ \text{(CLASSICAL LIMIT)}, \quad \frac{R}{b} \ll 1 \ \text{(PM REGIME)} \end{aligned}}$$

Eikonal Exponentiation

cf. Bern et al. - 2101.07254
Damgaard, Planté, Vanhove - 2107.12891,
Brandhuber, Chen, Travaglini, Wen - 2108.04216

- $$\boxed{1 + i\tilde{A}(s,b) = e^{2i\delta(s,b)}\left[1 + 2i\Delta(s,b)\right]}$$

(Classical) Eikonal $\left[\sim e^{\frac{i}{\hbar}S_{\text{classical}}}\right]$

$$\boxed{\delta = \delta_0 + \delta_1 + \delta_2 + \dots}$$

$$\delta_{n-1} \sim \left(\frac{R}{\lambda}\right)\left(\frac{R}{b}\right)^{n-1}$$

Quantum remainder

$$\boxed{\Delta = \Delta_1 + \Delta_2 + \dots}$$

$$\Delta_{n-1} \sim \left(\frac{R}{b}\right)^{n}\left[1 + O\left(\frac{\lambda}{b}\right)\right]$$

$$R = Gm\left(b^{2\epsilon}\right)$$

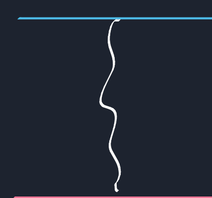
$$\lambda = \frac{\hbar}{m}$$

$$\frac{R}{\lambda} \gg 1\,, \quad \frac{R}{b} \ll 1$$

CLASSICAL LIMIT | PM REGIME

- For "small G":

$$i\tilde{A}_0(s,b) = \boxed{2i\delta_0}$$

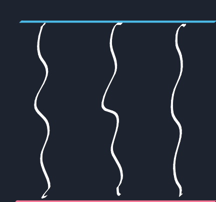


$$i\tilde{A}_1(s,b) = \boxed{\frac{(2i\delta_0)^2}{2!}}$$

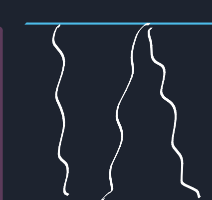
$$+ \boxed{2i\delta_1}$$

$$+ \boxed{2i\Delta_1}$$

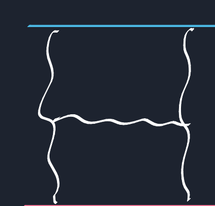
$$i\tilde{A}_2(s,b) = \boxed{\frac{(2i\delta_0)^3}{3!}}$$

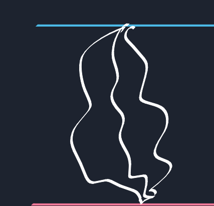


$$+ \boxed{2i\delta_0\, 2i\delta_1}$$



$$+ \boxed{2i\delta_2 + 2i\delta_0\, 2i\Delta_1} + \boxed{2i\Delta_2}$$





Kabat, Ortiz - hepth/9112020
Akhoury, Saotome, Sterman - 1308.5204
Bjerrum-Bohr, Damgaard, Festuccia, Vanhove - 1806.04920

Eikonal & Deflection Angle

- STATIONARY PHASE APPROXIMATION: after exponentiation

$$\boxed{\mathcal{A}(s,\vec{Q}) \sim \int e^{-i\vec{b}\cdot\vec{Q}+2i\delta}\, d^{2-2\epsilon}\vec{b}} \quad \underset{\text{CLASSICAL LIMIT}}{\Longrightarrow} \quad \boxed{\vec{Q} = \frac{\partial}{\partial \vec{b}} \mathrm{Re}(2\delta)}$$

$$|\vec{Q}| = 2|\vec{p}| \sin\frac{\chi}{2} \simeq \left(\chi - \frac{\chi^3}{24}\right)$$

- $\delta = \delta_0 + \delta_1 + \delta_2 + \dots$, schematically

$$2\delta_0 \propto G m_1 m_2 \frac{b^{2\epsilon}}{2\epsilon} \Longrightarrow \chi_{1PM} \propto \frac{G m_1 m_2}{|\vec{p}|\, b} \sim \frac{R}{b}$$

$$2\delta_1 \propto G m_1 m_2 \frac{G(m_1+m_2)}{b^{1-4\epsilon}} \Longrightarrow \chi_{2PM} \propto \frac{G^2 m_1 m_2 (m_1+m_2)}{|\vec{p}|\, b^2} \sim \left(\frac{R}{b}\right)^2$$

$$2\delta_2 \propto G m_1 m_2 \frac{G^2 m_1 m_2}{b^{2-6\epsilon}} \left[(\dots) + \frac{i}{\epsilon}(\dots) \right] \Longrightarrow \chi_{3PM} \propto \frac{G^3 (m_1 m_2)^2}{|\vec{p}|\, b^3} \sim \left(\frac{R}{b}\right)^3$$

IR finite REAL PART

IR divergent IMAGINARY PART

due to $\left| \text{TREE} \right|^2$

(Weinberg '65)

Nontrivial part:

precise dependence on $\boxed{\sigma}$,

$\sigma \to 1$ STATIC LIMIT

$\sigma \to \infty$ ULTRAREL. LIMIT

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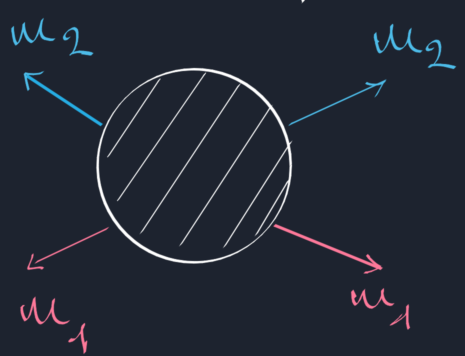
$\mathcal{N}=8$ SUGRA

- Useful toy model: * simpler amplitudes compared to GR
  * same technical and conceptual challenges

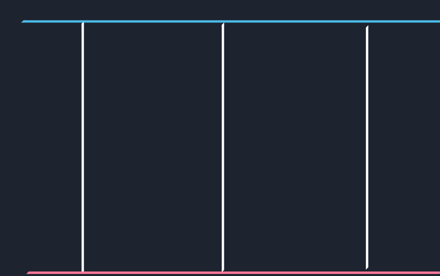
- s-u symmetric amplitude for two massive scalars

Caron-Huot, Zahraee - 1810.04694
Parra-Martinez, Ruf, Zeng - 2005.04236



$$A_2(s,q) = \frac{(8\pi G)^3}{2}\left[\left(s-m_1^2-m_2^2\right)^4+\left(u-m_1^2-m_2^2\right)^4-\left(q^2\right)^4\right]$$

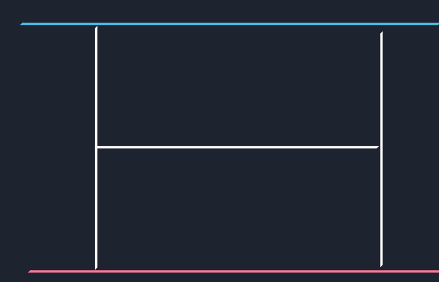
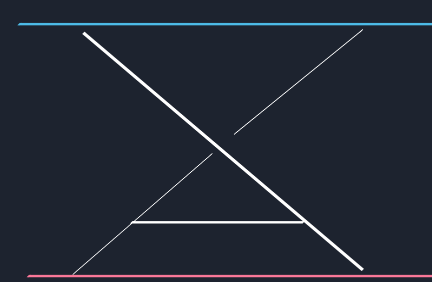
$$\times\Big[\left(s-m_1^2-m_2^2\right)^2\Big($$

 $+$  $+$  $\Big)$

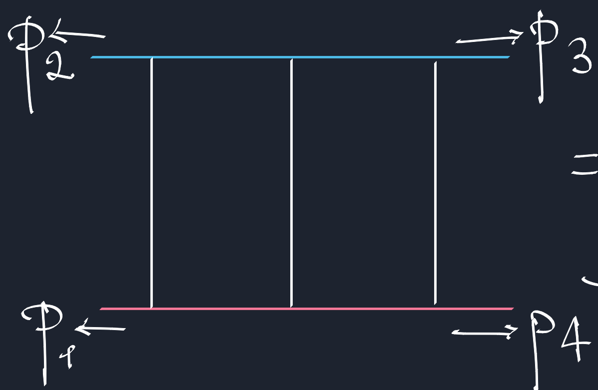
$$+\left(u-m_1^2-m_2^2\right)^2\Big($$

 $+$  $+$  $\Big)$

$$+\left(q^2\right)^2\Big($$

 $+$  $+ \ldots \Big)\Big]$

Loop Integrals for Small q^2



$$= \iint \frac{d^D l_1 \; d^D l_2}{\left[-2\bar{p}_1\cdot l_1 - i0 + (l_1^2 - l_1\cdot q)\right]\left[-2\bar{p}_2\cdot l_1 - i0 + (l_1^2 - l_1\cdot q)\right]\left[+2\bar{p}_1\cdot l_2 - i0 + (l_2^2 - l_2\cdot q)\right]\left[+2\bar{p}_2\cdot l_2 - i0 + (l_2^2 - l_2\cdot q)\right]}$$

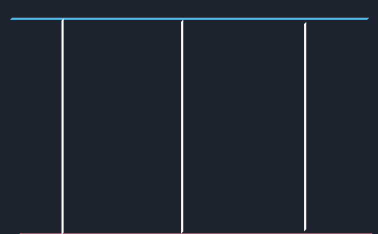
$$\times \frac{}{(l_1^2 - i0)(l_2^2 - i0)\left[(l_1 + l_2 - q)^2 - i0\right]}$$

$$\begin{cases} p_1 = -\bar{p}_1 + \frac{1}{2} q \\ p_2 = -\bar{p}_2 - \frac{1}{2} q \\ p_3 = \bar{p}_2 - \frac{1}{2} q \\ p_4 = \bar{p}_4 + \frac{1}{2} q \end{cases}$$

As $q^2 \to 0$, METHOD OF REGIONS

Beneke, Smirnov
- hep-ph/9781391

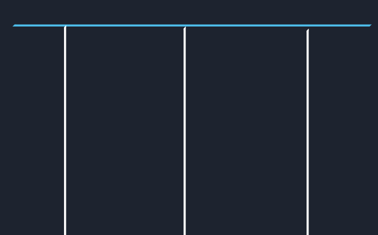
- HARD REGION: $\boxed{l_{1,2} \sim \mathcal{O}(m)}$



$$\overset{(h)}{=} c_0 (q^2)^0 + c_1 (q^2)^1 + c_2 (q^2)^2 + \ldots$$

analytic in q^2
$\Rightarrow$ irrelevant for the long-range behavior

- SOFT REGION: $\boxed{l_{1,2} \sim \mathcal{O}(q)}$

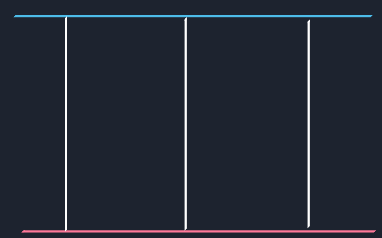


$$\overset{(s)}{=} \frac{c_{ssl}}{(q^2)^{1+2\epsilon}} + \frac{c_{sl}}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{c_l}{(q^2)^{2\epsilon}} + \ldots$$

relevant non-analytic terms

IBP + ODE Toolkit

- Integration-by-Parts (IBP) identities reduce the calculation to a few SOFT master integrals [LiteRed, FIRE6]

 $\overset{(s)}{=} \sum_{j=1}^{14} c_j(\epsilon;x)\, f_j(\epsilon;x)$

$$\begin{cases} y \simeq \sigma + \mathcal{O}(q^2) \\ x = y - \sqrt{y^2-1} \end{cases}$$

- The master integrals can be determined by solving ODE's in canonical form [epsilon]

$$d\vec{f} = \epsilon \left[A_0 \, d\log x + A_{+1} \, d\log(x+1) + A_{-1} \, d\log(x-1) \right] \vec{f}$$

Factorized ϵ - dependence

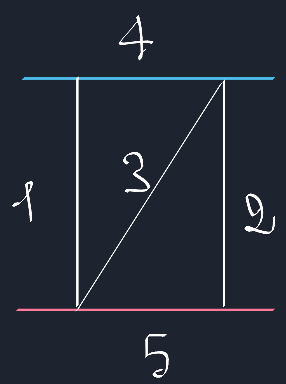
Constant Matrices

Parra-Martinez, Ruf, Zeng - 2005.04236

- One still needs the Boundary Conditions ...

Soft Boundary Conditions

- Boundary conditions in the static limit $\sigma \to 1^+$
- EXAMPLE



$$\overset{(S)}{=} \frac{1}{m_1 m_2} \int_{\mathbb{R}^3_+} \frac{dt_1\, dt_2\, dt_3}{T^{1-\epsilon}}\, e^{-\frac{t_1 t_2 t_3}{T} q^2} \left[\int_{\mathbb{R}^2_+} dx_4\, dx_5\, e^{-\left(t_{23} x_5^2 + 2 t_3 (-\sigma - i0) x_4 x_5 + t_{13} x_4^2\right)} \right]$$

$T = t_1 t_2 + t_2 t_3 + t_3 t_1$ $\qquad t_{23} = t_2 + t_3$ $\qquad t_{13} = t_1 + t_3$

As $\sigma \to 1^+$, METHOD OF REGIONS $\left(\tau \equiv \sqrt{\sigma^2 - 1}\right)$

* "ORDINARY" region : $\boxed{t_{1,2,3} \sim \mathcal{O}(\tau^0)}$

$$[\ldots] \overset{(\sigma)}{=} \int_{\mathbb{R}^2_+} dx_4\, dx_5\, e^{-\left(t_{23} x_5^2 - 2 t_3 x_4 x_5 + t_{13} x_4^2\right)} = \text{Real} \quad (\& \text{ Finite})$$

* "SINGULAR" region : $\boxed{t_{1,2} \sim \mathcal{O}(\tau^0) \quad t_3 \sim \mathcal{O}(\tau^{-2})}$ $\qquad (\alpha = x_5 + x_4,\ \beta = x_5 - x_4)$

$$[\ldots] \overset{(S)}{=} \frac{1}{2} \int_{-\infty}^{+\infty} d\beta\ e^{-t_3 \beta^2} \int_0^{\infty} d\alpha\ e^{-\frac{\alpha^2}{4}\left(t_{12} + t_3(-\tau^2 - i0)\right)} = \frac{i\pi}{2\sqrt{t_3}\sqrt{t_3 \tau^2 + (-t_{12} + i0)}}$$

$2\delta_2$ from the Soft-Region

Potential Contribution

Parra-Martinez, Ruf, Zeng - 2005.04236

Radiation-Reaction

DVHRV - 2008.12743

- $$\mathrm{Re}(2\delta_2) = \frac{16 G^3 m_1^2 m_2^2 \sigma^4}{b^2(\sigma^2-1)}\left[-\operatorname{arccosh}\sigma + \frac{1}{\sigma^2-1}\left(\sigma^2 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\operatorname{arccosh}\sigma\right)\right]$$

✓ IR finite

✓ UR limit $\mathrm{Re}(2\delta_2) \underset{\sigma\to\infty}{\sim} \dfrac{16 G^3 (m_1 m_2 \sigma)^2}{b^2}$

(UNIVERSAL)

Amati, Ciafaloni, Veneziano '90

Linked by ANALYTICITY and CROSSING SYMMETRY

DVHRV - 2104.03256

IR divergence due to soft particles in the 3-particle cut

- $$\mathrm{Im}(2\delta_2) = -\frac{16 G^3 m_1^2 m_2^2}{\pi b^2}\frac{\sigma^4}{(\sigma^2-1)^2}\Bigg\{\left[\frac{1}{\epsilon} - \log\left(4(\sigma^2-1)\right) + 3\log\left(\pi b^2 e^{\gamma_E}\right)\right]\left(\sigma^2 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\operatorname{arccosh}\sigma\right)$$
$$+ (\sigma^2-1)\left[1 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\right](\operatorname{arccosh}\sigma)^2 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\mathrm{Li}_2(1-z^2) + 2\sigma^2\Bigg\} + \mathcal{O}(\epsilon)$$

$$z \equiv \sigma - \sqrt{\sigma^2-1}$$

Towards GR

- $\underline{\text{IDEA}}$: add RR effects $\mathrm{Re}(2\delta_2^{RR})$ to the (known) POTENTIAL PART
- By ANALYTICITY

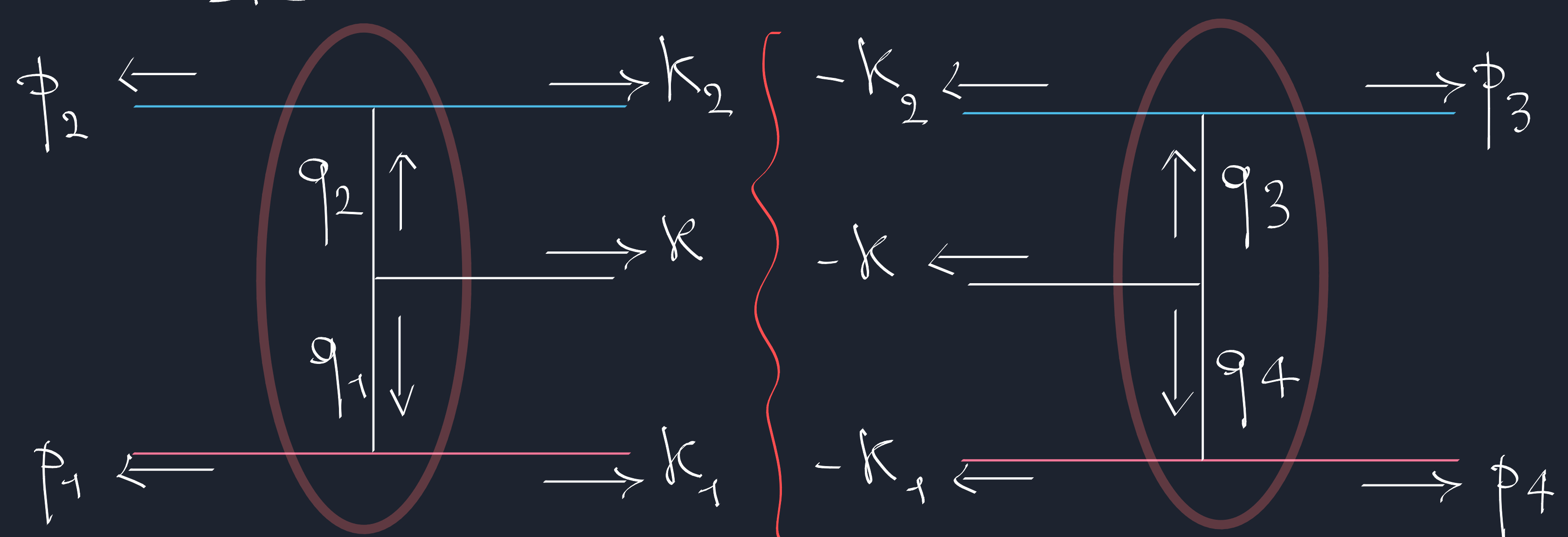
DVHRV - 2101.05772, 2104.03256

$$\boxed{\log(\sigma^2-1)} = \mathrm{Im}\left[i \log(-(\sigma^2-1)-i0)\right] \Rightarrow \mathrm{Re}\left[i \log(-(\sigma^2-1)-i0)\right] = \boxed{+\pi}$$

- $\log(\sigma^2-1)$ term of $\mathrm{Im}(2\delta_2) \Rightarrow \mathrm{Re}(2\delta_2^{RR})$
- By UNITARITY,

$$\delta_+(t) \equiv 2\pi\Theta(t)\delta(t)$$

$$\left[\mathrm{Im}(2\mathcal{A}_2)\right]_{3PC} = \int d^D k \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\,\delta^{(D)}(p_1+p_2+k_1+k_2+k_3)\,\delta_+(k^2)\,\delta_+(k_1^2+m_1^2)\,\delta_+(k_2^2+m_2^2)\left|\mathcal{A}_{5pt}\right|^2$$



In b-space

$$\boxed{\left[\mathrm{Im}\,\tilde{\mathcal{A}}_2\right]_{3PC} = \mathrm{Im}\,2\delta_2}$$

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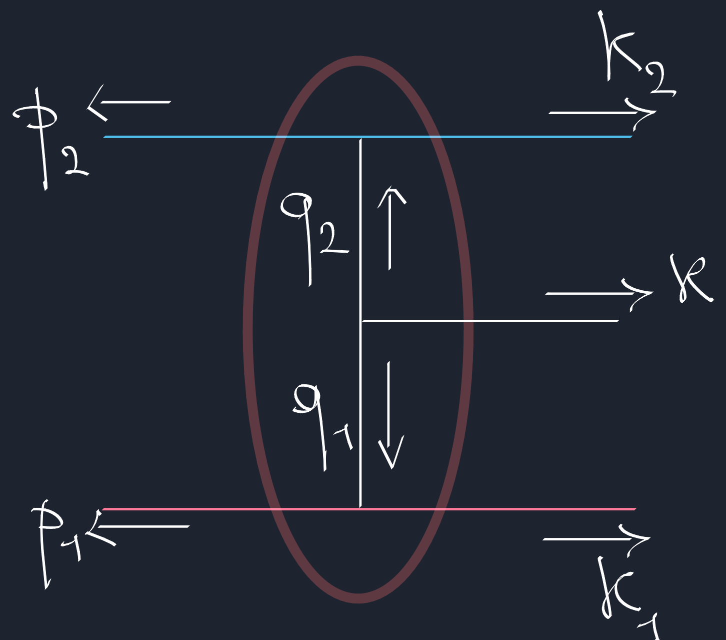
5-pt Amplitudes in the Classical Limit $\quad q_i \to 0$

Goldberger, Ridgway; Luna, Nicholson, O' Connell, White; Mogull, Plefka, Steinhoff



$$A_{5pt}(P_1,P_2,K_1,K_2,k) = (8\pi G)^{\frac{3}{2}}\Big\{ \frac{8(P_1\cdot k\, P_2^M - P_2\cdot k\, P_1^M)(P_1\cdot k\, P_2^N - P_2\cdot k\, P_1^N)}{q_1^2 q_2^2}$$

$$+ 8 P_1\cdot P_2 \Big[\Big(\frac{P_1^M P_2^N \frac{k\cdot P_2}{k\cdot P_1} - P_1^{(M}P_2^{N)}}{q_2^2} + 1\leftrightarrow 2\Big) - 2\,\frac{P_1\cdot k\, P_2^{(M} q_1^{N)} - P_2\cdot k\, P_1^{(M} q_1^{N)}}{q_1^2 q_2^2}\Big]$$

$$+ \beta\Big[-\Big(\frac{P_1^M P_1^N\, k\cdot q_1}{(P_1\cdot k)^2\, q_2^2} + 1\leftrightarrow 2\Big) + 2\Big(\frac{P_1^{(M} q_1^{N)}}{P_1\cdot k\, q_2^2} - \frac{P_2^{(M} q_1^{N)}}{P_2\cdot k\, q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2}\Big)\Big]\Big\}$$

Leading Weinberg term $(K_i \ll q)$

$P_1^2 = 0 = P_2^2$,

$$\boxed{|A_{5pt}|^2 \equiv A^{MN}_{5pt}(P_1,P_2,K_1,K_2,K)\,\Pi_{MN,RS}\,A^{RS}_{5pt}(P_4,P_3,-K_1,-K_2,-k)}$$

$\boxed{\mathcal{N}=8}$

$P_1 = (p_1;\, 0,0,0,0,0,m_1)$

$P_2 = (p_2;\, 0,0,0,0,m_2,0)$

$\beta = 4m_1^2 m_2^2 \sigma^2$, $\quad \Pi_{MN,RS} = \eta_{MR}\eta_{NS}$

$\boxed{GR}$

$P_1 = (p_1;\, 0,0,0,0,0,0)$

$P_2 = (p_2;\, 0,0,0,0,0,0)$

$\beta = 4m_1^2 m_2^2\left(\sigma^2 - \frac{1}{D-2}\right)$, $\quad \Pi_{MN,RS} = \eta_{MR}\eta_{NS} - \frac{\eta_{MN}\eta_{RS}}{D-2}$

Evaluation of the Cut Integrals

DVHRV - 2104.03256

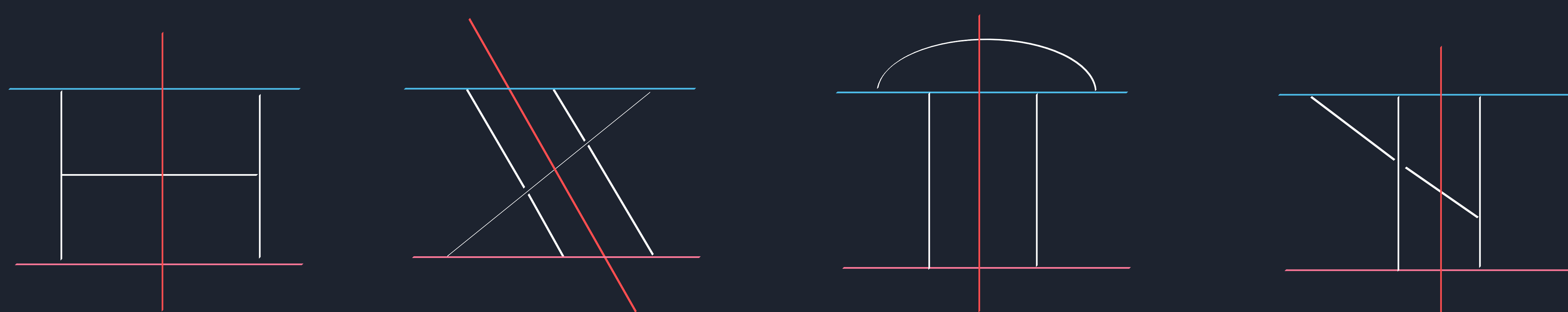
- $$|\mathcal{A}_{5pt}|^2 = \frac{N(u_1\cdot\ell_2,\, u_2\cdot\ell_2,\, \ell_1\cdot q,\, \ell_2\cdot q,\, u_1\cdot\ell_1,\, u_2\cdot\ell_1,\, \ell_1^2,\, \ell_1\cdot\ell_2,\, \ell_2^2)}{D_1^{\alpha_1} D_2^{\alpha_2} D_3 D_4 D_5 D_6 \cancel{D_7} \cancel{D_8} \cancel{D_9}}$$

$$\begin{cases} q_1 = -\ell_1 \\ q_2 = \ell_1 + \ell_2 \\ k = -\ell_2 \end{cases}$$

$D_1 = u_1\cdot\ell_2$, $D_2 = u_2\cdot\ell_2$, $D_3 = \ell_1^2$, $D_4 = (\ell_1+\ell_2)^2$, $D_5 = (\ell_1+q)^2$, $D_6 = (\ell_1+\ell_2+q)^2$

$D_7 = \ell_2^2$, $D_8 = u_1\cdot\ell_1$, $D_9 = u_2\cdot(\ell_1+\ell_2)$

- This can be reduced to



- These integrals can be decomposed into the same MASTER INTEGRALS discussed previously up to the choice of (cut) BOUNDARY CONDITIONS

[REVERSE UNITARITY]

Anastasiou, Melnikov - 0207004
Herrmann, Parra-Martinez, Ruf, Zeng - 2101.07255, 2104.03957

3PM Eikonal in GR

POTENTIAL

Bern, Cheung, Shen, Solon, Ruf, Zeng - 1901.04424

$$\mathrm{Re}\,2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left\{ \frac{s(12\sigma^4 - 10\sigma^2 + 1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1}\,\mathrm{arccosh}\,\sigma \right.$$

PROBE LIMIT

RADIATION REACTION

ANALYTICITY & CROSSING

DVHRV - 2101.05772, 2104.03256
(Agrees with Damour - 2010.01641)

$$\left. + \frac{(2\sigma^2-1)^2}{2(\sigma^2-1)^2}\left[\frac{8-5\sigma^2}{3} + \frac{\sigma(2\sigma^2-3)}{\sqrt{\sigma^2-1}}\,\mathrm{arccosh}\,\sigma\right]\right\}$$

$$\mathrm{Im}\,2\delta_2 = \frac{2G^3 m_1^2 m_2^2}{\pi b^2}\frac{(2\sigma^2-1)^2}{(\sigma^2-1)^2}\left\{\left[-\frac{1}{\epsilon} + \log(4(\sigma^2-1)) - 3\log(\pi b^2 e^{\gamma_E})\right]\left[\frac{8-5\sigma^2}{3} + \frac{\sigma(2\sigma^2-3)}{\sqrt{\sigma^2-1}}\,\mathrm{arccosh}\,\sigma\right]\right.$$

$$+ (\mathrm{arccosh}\,\sigma)^2\left[\frac{\sigma(3-2\sigma^2)}{\sqrt{\sigma^2-1}} - 2\frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{(2\sigma^2-1)^2}\right]$$

$$+ \sigma\,\mathrm{arccosh}\,\sigma\,\frac{88\sigma^6-240\sigma^4+240\sigma^2-97}{3(2\sigma^2-1)^2\sqrt{\sigma^2-1}}$$

$$\left. + \frac{\sigma(3-2\sigma^2)}{\sqrt{\sigma^2-1}}\mathrm{Li}_2(1-z^2) + \frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9(2\sigma^2-1)^2}\right\}$$

The $-\frac{1}{\epsilon}$ always accompanies $\log(\sigma^2-1) \Rightarrow$ the DIVERGENT PART governs RR effects

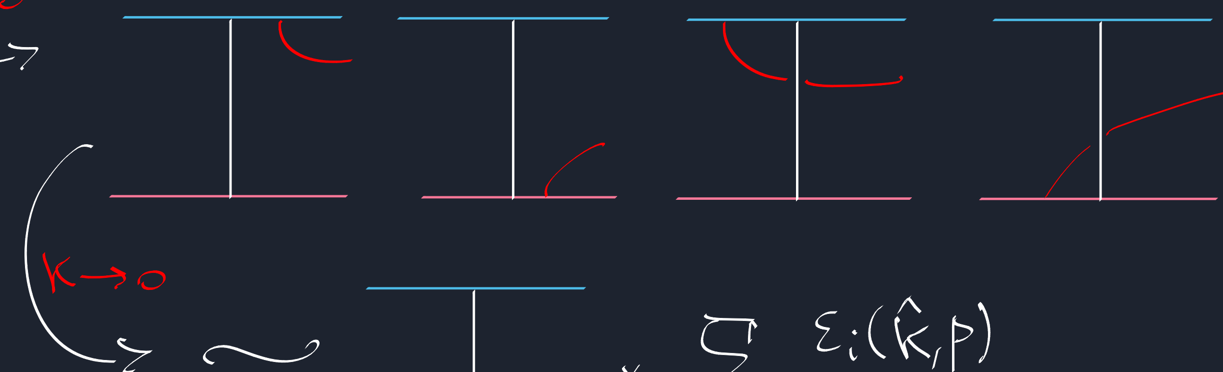
Explicitly checked in Bjerrum-Bohr, Damgaard, Planté, Vanhove - 2105.05218

The $\frac{1}{\epsilon}$ Part: Unitarity + Soft Theorems

DVHRV - 2101.05772

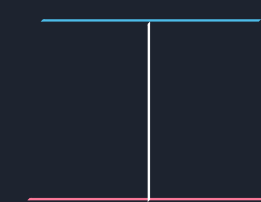
- $$\left[\mathrm{Im}(2A_2)\right]_{3PC} = \int d^D k \, \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \, \delta^{(D)}(p_1+p_2+k_1+k_2+k_3)\, \delta_+(k^2)\, \delta_+(k_1^2+m_1^2)\, \delta_+(k_2^2+m_2^2) \left|A_{5pt}\right|^2$$

- Weinberg limit $\quad k \ll q \ll m$



- $$\mathrm{Im}\, 2\delta_2 = \left[\mathrm{Im}(2\tilde{A}_2)\right]_{3PC}$$

$$\sim \int \frac{d^{D-1}\vec{k}}{|\vec{k}|} \left(\frac{1}{|\vec{k}|} \hat{A}(\hat{k}, b) \right) \left(\frac{1}{|\vec{k}|} \hat{A}(\hat{k}, b) \right)^*$$

$\sim$  $\times \sum_{ext} \frac{\varepsilon_i(\hat{k}, p)}{p \cdot k}$

$$\sim \underbrace{\int_0^{\omega_{MAX}} \frac{d\omega}{\omega^{1+2\epsilon}}} \int d\Omega_{2-2\epsilon}(\hat{k}) \left|\hat{A}(\hat{k}, b)\right|^2$$

[Going to b-space BEFORE integrating over $\vec{k}$

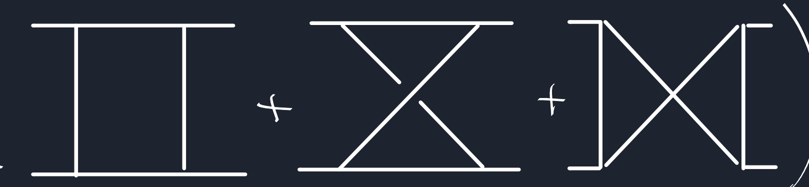
$$-\frac{1}{2\epsilon}\frac{1}{\omega_{MAX}^{2\epsilon}} = -\frac{1}{2\epsilon} + \log \omega_{MAX} \underset{\text{small velocity}}{\sim} \boxed{-\frac{1}{2\epsilon} + \log v} \xrightarrow{\text{generic velocity}} \boxed{\frac{1}{2}\left[-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right]}$$

The $\frac{1}{\epsilon}$ Part: Exponentiation of IR Divergences

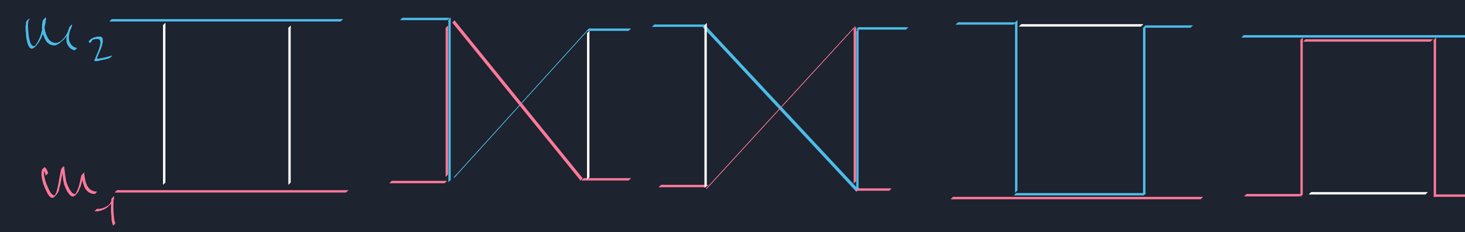
H - 2105.04594

- $\boxed{\mathcal{N}=8}$

$$A_1 = \frac{1}{2}\frac{(8\pi G)^2}{(4\pi)^{2-\epsilon}}\left[(s-m_1^2-m_2^2)^4+(u-m_1^2-m_2^2)^4-t^4\right]\Big(\ \ +\ \ +\ \ \Big)$$



$$\boxed{A = A_0\, e^{\hat{A}_1}\left(1+\hat{A}_2+\dots\right)}$$



+ crossing

$\hat{A}_1 \equiv A_1/A_0$

IR FINITE

IR – DIVERGENT PART $\Longrightarrow$ $\boxed{A_2 \underset{\epsilon\to 0}{\sim} \frac{A_1^2}{2A_0}}$ $\Longrightarrow$ $2\delta_2 \underset{\epsilon\to 0}{\sim} -\frac{i}{\pi\epsilon}\frac{16G^3 m_1^2 m_2^2 \sigma^5}{b^2(\sigma^2-1)^2}\left[\sigma+(\sigma^2-2)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}\right]$

- $\boxed{\mathrm{GR}}$

(Weinberg '65)

$$\boxed{A = e^{W}\left(A_0+A_1^0+A_2^0+\dots\right)}$$

IR FINITE

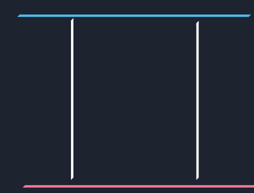
$$W = \frac{G}{2\pi\epsilon}\sum_{n,m} m_n m_m\left(\sigma_{nm}^2-\frac{1}{2}\right)\frac{\pm\operatorname{arccosh}\sigma_{nm}-i\pi}{\sqrt{\sigma_{nm}^2-1}}$$

$\sigma_{nm} = -\frac{p_n\cdot p_m}{m_n m_m}$

out/out, in/in $\}$ +
in/out $\}$ −

only if $n\neq m$, both in (out)

$\boxed{A_2 \underset{\epsilon\to 0}{\sim} W A_1 + \text{ANALYTIC in } q^2}$



+ CROSSING

$$\Longrightarrow 2\delta_2 \underset{\epsilon\to 0}{\sim} -\frac{i}{\pi\epsilon}\frac{(2G)^3 m_1^2 m_2^2\left(\sigma^2-\frac{1}{2}\right)^2}{b^2(\sigma^2-1)^2}\left[\frac{8-5\sigma^2}{3}+\sigma(2\sigma^2-3)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}\right]$$

Summary of RR from Analiticity + Crossing

- **Unitarity + Soft Th.**
  DVHRV - 2101.05772

$$\Rightarrow \mathrm{Im}\, 2\delta_2 \underset{\epsilon \to 0}{\sim} -\frac{1}{\pi\epsilon}\left[\frac{1}{2} - \epsilon \log v\right](\dots)_{RR}$$

small v

- **Full Unitarity:**
  DVHRV - 2104.03256

$$\Rightarrow \mathrm{Im}\, 2\delta_2 = -\frac{1}{\pi\epsilon}\left[\frac{1}{2} - \epsilon \log(\sigma^2 - 1)\right](\dots)_{RR} + \epsilon^0 \left(\text{analytic as } \sigma \to 1\right) + \mathcal{O}(\epsilon)$$

- **IR exponentiation:** $\mathcal{A} = e^{\text{1-loop div}} \mathcal{A}_{\text{FINITE}} \Rightarrow 2\delta_2 \underset{\epsilon \to 0}{\sim} -\frac{i}{2\pi\epsilon}(\dots)_{RR}$
  H - 2105.04594

Then by Analyticity + Crossing:

$$\boxed{\left(\mathrm{Re}\, 2\delta_2^{RR}\right)_{D=4} = \lim_{\epsilon \to 0}\left[-\pi\epsilon\, \mathrm{Im}\, 2\delta_2\right]}$$

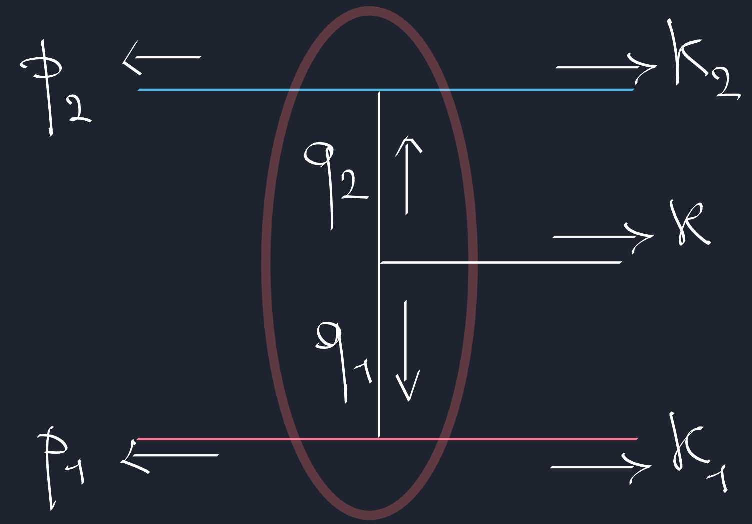
Bonus: Zero-frequency limit of the emission spectrum

$$\boxed{\left.\frac{dE^{GW}}{d\omega}\right|_{\substack{\omega = 0 \\ D=4}} = \lim_{\epsilon \to 0} -4\epsilon\, \mathrm{Im}\, 2\delta_2}$$

Waveforms

- Going to b-space BEFORE integrating over $\vec{k}$:

$$\begin{cases} \vec{q}_1 + \vec{q}_2 = 0 \\ \vec{q}_1 - \vec{q}_2 = \frac{1}{2}\vec{\Delta} \end{cases}$$



$$\tilde{\mathcal{A}}_{5pt,\,i} = \int \frac{d^{2-2\epsilon}\underline{\Delta}}{(2\pi)^{2-2\epsilon}} \frac{e^{i\underline{b}\cdot\underline{\Delta}}}{4m_1 m_2\sqrt{\sigma^2-1}} \underset{\downarrow}{\epsilon_{i,\mu\nu}} \mathcal{A}^{\mu\nu}_{5pt}$$

$\boxed{\text{POLARIZATIONS} \quad \times, +}$

- Waveform in the far zone: $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa H_{\mu\nu} \quad (\kappa = \sqrt{8\pi G})$

$\Box H^{\mu\nu} = \tau^{\mu\nu}$, with $\hat{\tau}^{\mu\nu} = \frac{1}{\kappa}\mathcal{A}^{\mu\nu}_{5pt}$

$$\underset{D=4}{\Longrightarrow} \quad \boxed{H^{\mu\nu}_{ret} = \frac{\kappa}{4\pi R}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{-i\omega u}\, \mathcal{A}^{\mu\nu}_{5pt}(\vec{k} = \omega\hat{x})}$$

detector's distance $\qquad$ retarded time $\qquad$ observation direction

Waveforms: $\times$

cf. Kovacs, Thorne (1978)
Jakobsen, Mogull, Plefka, Steinhoff - 2101.10256,
Mougiakakos, Riva, Vernizzi - 2102.08339

We work in the CM frame:

$$\tilde{\mathcal{A}}_{5pt,\times} = \frac{\kappa^3}{4pE}\,\frac{i\,\hat{b}\cdot\vec{e}_\phi}{2\pi}\Bigg\{\beta\Bigg[-\frac{m_2\, s(\theta)}{\sqrt{s}}\, e^{i\frac{\underline{b}\cdot\underline{k}}{2}} K_2\big(bc_2|\underline{k}|\big) - \frac{m_1\, s(\theta)}{\sqrt{s}}\, e^{-i\frac{\underline{b}\cdot\underline{k}}{2}} K_1\big(bc_1|\underline{k}|\big)$$

$$+\, i\int_0^1 dx\; e^{i\frac{\underline{b}\cdot\underline{k}}{2}(1-2x)}\Big(\hat{b}\cdot\vec{e}_\theta\; b\sqrt{f}\,K_1\big(b|\underline{k}|f\big) + i\big(x-\tfrac{1}{2}\big)\,\underline{k}\cdot\vec{e}_\theta\; b\,K_0\big(b|\underline{k}|\sqrt{f}\big)\Big)\Bigg]$$

$$+\left(4\,p_1\cdot p_2\,pE\,\omega\, s(\theta) + \beta\, s(\theta)\,\frac{q_1^L - q_2^L}{2}\right)\int_0^1 dx\; e^{i\frac{\underline{b}\cdot\underline{k}}{2}(1-2x)}\, b\,K_0\big(b|\underline{k}|\sqrt{f}\big)\Bigg\}$$

$$\begin{pmatrix} s(\theta)\equiv\sin\theta \\ c(\theta)\equiv\cos\theta \end{pmatrix}$$

$$f = x(1-x) + c_1^2 x + c_2^2(1-x)$$

$$|\underline{k}|\,c_1 = \frac{p_2\cdot k}{m_2\sqrt{\sigma^2-1}}$$

$$|\underline{k}|\,c_2 = \frac{p_1\cdot k}{m_1\sqrt{\sigma^2-1}}$$

$$\vec{e}_\phi = \big(-s(\phi),\; c(\phi),\; 0\big)$$

$$\vec{e}_\theta = \big(c(\theta)\,c(\phi),\; c(\theta)\,s(\phi),\; -s(\theta)\big)$$

$$k^\mu = \omega\big(1,\; s(\theta)c(\phi),\; s(\theta)s(\phi),\; c(\theta)\big)$$

$$\underline{k} = \omega\, s(\theta)\big(c(\phi),\; s(\phi)\big)$$

WARNING, PRELIMINARY

Waveforms: +

u-space expressions of X,+ agree with soft theorems dictating the large-u behavior (Sahoo, Sen - 2105.08739 and refs. therein)

$$\tilde{\mathcal{A}}_{5pt,+} = \frac{\kappa^3}{4E_p}\frac{1}{4\pi}\Bigg\{ 8\left(p_1\cdot k\, p_2\cdot\vec{e}_\theta - p_2\cdot k\, p_1\cdot\vec{e}_\theta\right)^2 \int_0^1 dx\, e^{i\frac{b\cdot k}{2}(1-2x)}\, \frac{b}{2}\,\frac{K_1(b|\underline{k}|\sqrt{f})}{|\underline{k}|\sqrt{f}}$$

$$+\,\beta\Bigg[\frac{(p_1\cdot\vec{e}_\theta)^2}{(p_1\cdot k)^2}\, e^{i\frac{k\cdot b}{2}}\,\underline{k}^2\left(d_2 K_0(bc_2|\underline{k}|) + \frac{i\,\underline{k}\cdot b}{b|\underline{k}|}\, c_2 K_1(bc_2|\underline{k}|)\right) + \frac{(p_2\cdot\vec{e}_\theta)^2}{(p_2\cdot k)^2}\, e^{-i\frac{k\cdot b}{2}}\,\underline{k}^2\left(d_1 K_0(bc_1|\underline{k}|) - \frac{i\,\underline{k}\cdot b}{b|\underline{k}|}\, c_1 K_1(bc_1|\underline{k}|)\right)$$

$$-\frac{p_1\cdot\vec{e}_\theta}{p_1\cdot k}\, e^{i\frac{b\cdot k}{2}}\left(\vec{e}_\theta\cdot\underline{k} + s(\theta)(q_1^L-q_2^L)K_0(bc_2|\underline{k}|) + 2i\,\vec{e}_\theta\cdot\hat{b}\; c_2|\underline{k}|K_1(bc_2|\underline{k}|)\right) + \frac{s(\theta)^2}{2}(q_1^L-q_2^L)^2\int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}\,\frac{b}{2}\,\frac{K_1(b|\underline{k}|\sqrt{f})}{|\underline{k}|\sqrt{f}}$$

$$-\frac{p_2\cdot\vec{e}_\theta}{p_2\cdot k}\, e^{-i\frac{b\cdot k}{2}}\left(\vec{e}_\theta\cdot\underline{k} - s(\theta)(q_1^L-q_2^L)K_0(bc_1|\underline{k}|) - 2i\,\vec{e}_\theta\cdot\hat{b}\; c_1|\underline{k}|K_1(bc_1|\underline{k}|)\right) + \frac{1}{2}\int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}(\vec{e}_\theta\cdot\underline{k})^2(1-2x)^2\,\frac{b}{2}\,\frac{K_1(b|\underline{k}|\sqrt{f})}{|\underline{k}|\sqrt{f}}$$

$$-s(\theta)(q_1^L-q_2^L)\int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}\left(-\vec{e}_\theta\cdot\underline{k}(1-2x)\frac{b}{2}\frac{K_1(b|\underline{k}|\sqrt{f})}{|\underline{k}|\sqrt{f}} - i\vec{e}_\theta\cdot\hat{b}\; bK_0(b|\underline{k}|\sqrt{f})\right) + \int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}\left(-s(\theta)^2K_0(b|\underline{k}|\sqrt{f}) + \left[(\vec{e}_\phi\cdot\hat{b})^2 - (\vec{e}_\theta\cdot\hat{b})^2\right]b|\underline{k}|\sqrt{f}K_1(b|\underline{k}|\sqrt{f})\right)\Bigg]$$

$$+\,8\,p_1\cdot p_2\Bigg[\left((p_1\cdot\vec{e}_\theta)^2\frac{k\cdot p_2}{k\cdot p_1} - p_1\cdot\vec{e}_\theta\, p_2\cdot\vec{e}_\theta\right)e^{i\frac{k\cdot b}{2}}K_0(bc_2|\underline{k}|) + \left((p_2\cdot\vec{e}_\theta)^2\frac{k\cdot p_1}{k\cdot p_2} - p_1\cdot\vec{e}_\theta\, p_2\cdot\vec{e}_\theta\right)e^{-i\frac{b\cdot k}{2}}K_0(bc_1|\underline{k}|)$$

$$+\left(p_1\cdot k\, p_2\cdot\vec{e}_\theta - p_2\cdot k\, p_1\cdot\vec{e}_\theta\right)\int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}\left(i\, e_\theta\cdot\hat{b}\; bK_0(b|\underline{k}|\sqrt{f}) + s(\theta)(q_1^L-q_2^L)\,\frac{b}{2}\,\frac{K_1(b|\underline{k}|\sqrt{f})}{|\underline{k}|\sqrt{f}}\right)\Bigg]$$

$$+\,i\int_0^1 dx\, e^{i\frac{k\cdot b}{2}(1-2x)}\,\vec{e}_\theta\cdot\underline{k}\;\, e_\theta\cdot\hat{b}\,(1-2x)\, bK_0(b|\underline{k}|\sqrt{f})\Bigg\}$$

cf. Kovacs, Thorne (1978)
Jakobsen, Mogull, Plefka, Steinhoff - 2101.10256,
Mougiakakos, Riva, Vernizzi - 2102.08339

WARNING/
PRELIMINARY

Towards an Eikonal with Real Radiation

- $1 + i\underset{\uparrow \atop \text{ELASTIC } 2\to 2}{\tilde{A}}(s,b) \underset{\text{CLASSICAL}}{\sim} e^{2i\delta(s,b)}$ fails to be "unitary" because $\boxed{\text{Im}\, 2\delta_2 \neq 0}$ (in fact $\sim \frac{1}{\epsilon}$)

- $\text{Im}\, 2\delta_2$ cancels against $[\text{Im}\, \tilde{A}_2]_{3PC} = \int d\Gamma_{int}$

INELASTIC $2 \to 3$

Ciafaloni, Colferai, Veneziano - 1812.08137

- The appropriate way to restore unitarity is to combine $2 \to 2$ and $2 \to 3$ via a Hermitian eikonal OPERATOR

  - The 1st step exponentiation of the $2 \to 3$ amplitude

    The $\frac{1}{\epsilon}$, superclassical term in $A^{\text{1-loop}}_{5pt,\, \mathcal{N}=8 \text{ massless}}$ is compatible with $2i\delta_0 \cdot A^{\text{tree}}_{5pt,\, \mathcal{N}=8 \text{ massless}}$

WARNING: PRELIMINARY

Conclusions & Outlook

- 3PM eikonal: $\mathrm{Re}\, 2\delta_2$ (Potential & Radiation Reaction) ✓
  ($\mathcal{N}=8$ & GR) $\mathrm{Im}\, 2\delta_2$ ✓
- Radiation-Reaction from:
  ANALYTICITY + CROSSING + UNITARITY ✓
  ANALYTICITY + CROSSING + IR EXPONENTIATION ✓
- Leading-Order Waveforms $\mathcal{O}(G^2)$ ✓
- Inelastic Eikonal ...
- Eikonal Operator @ 4PM