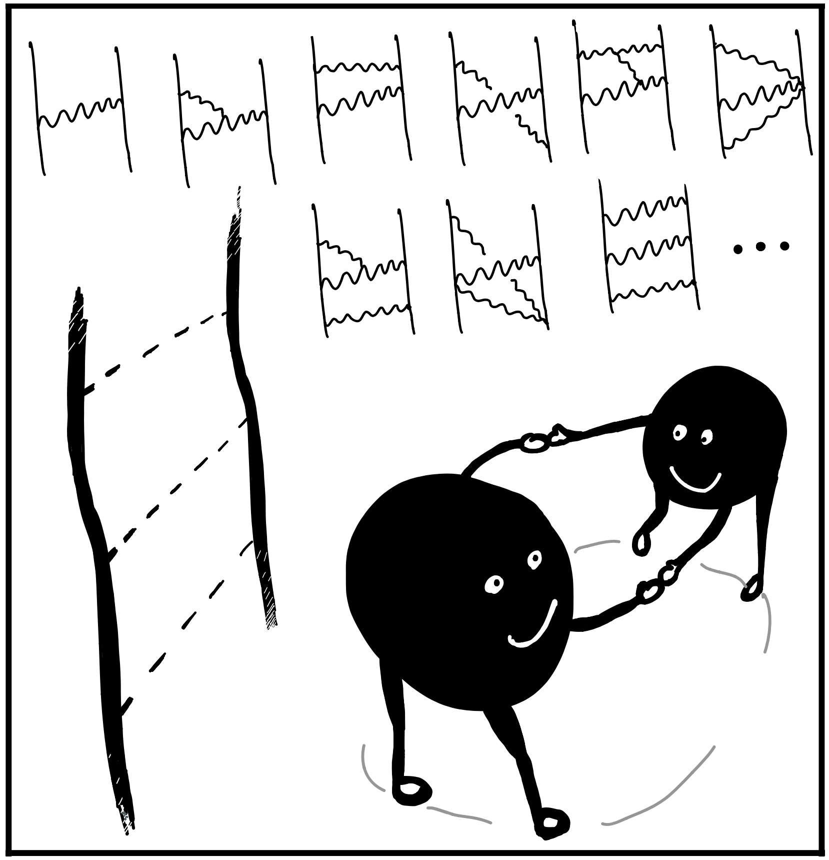



# Scattering Amplitudes and Conservative Binary Dynamics at $G^4$

Mikhail Solon

Bhaumik Institute for Theoretical Physics, UCLA

Quantum Field Theory

$$\Big\downarrow$$

Gravitational Wave Science

**Cheung**, **Rothstein**, MS (PRL 18)

**Bern**, Cheung, **Roiban**, **Shen**, MS, **Zeng** (PRL 19, JHEP 19)

Cheung, MS (JHEP 20, PRL 20)

Cheung, **Shah**, MS (PRD 20)

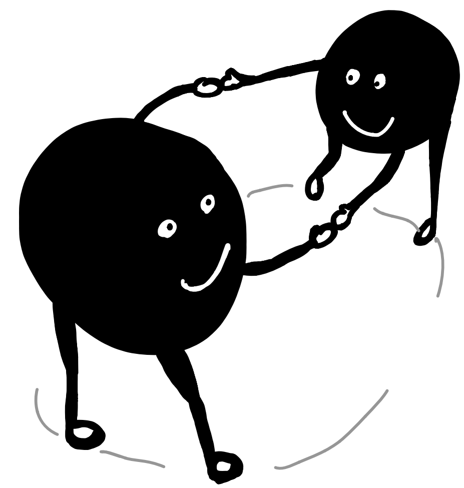
Bern, **Parra-Martinez**, Roiban, **Ruf**, Shen, MS, Zeng (PRL 21)

…

# Gravitational Wave Science

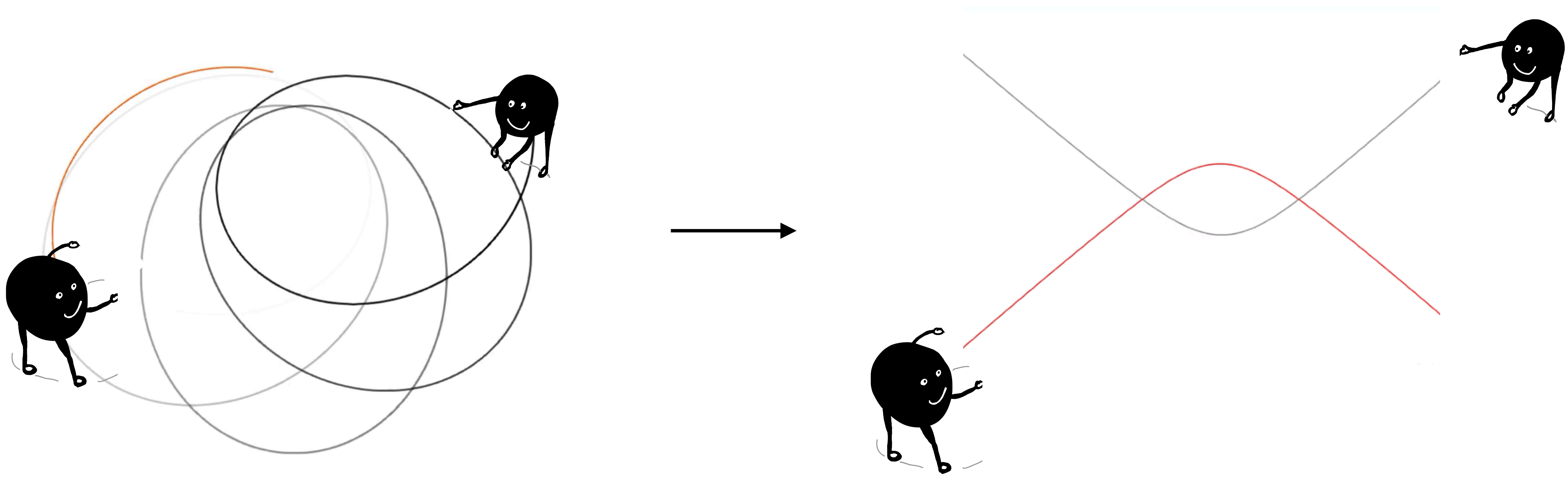
| 2015 | 2020 | 2025 | 2030+ |
|---|---|---|---|
| LIGO | LIGO/Virgo | + KAGRA, INDIGO, … | LISA<br>Einstein Telescope<br>Cosmic Explorer<br>… |

# Gravitational Wave Science

$$G_{ab} = 8\pi T_{ab} \longrightarrow$$ 

Decades of heroic effort

# Turn it into an easier problem



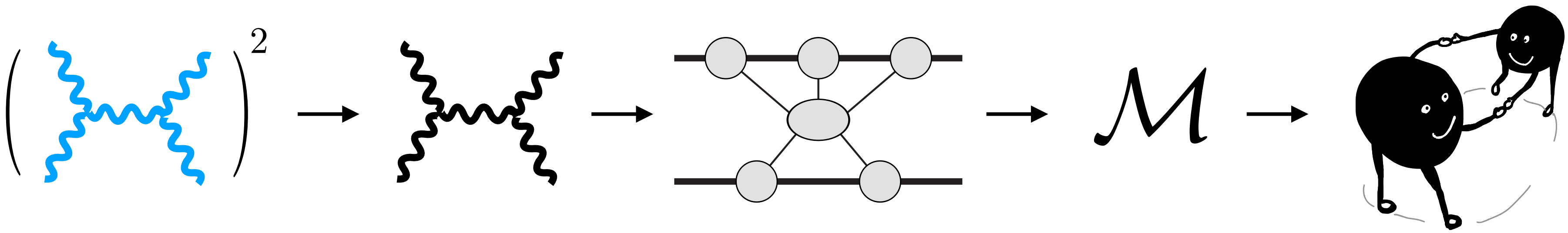
Relativity

Tools of Theoretical High Energy Physics

onshell methods,  advanced multiloop integration,  effective field theory

Underlying dynamics are universal

# Scalable pipeline using tools from QFT:



# Conservative Two-Body Potential (no radiation)

$$
\begin{array}{r}
G(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^2(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^3(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^4(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^5(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^6(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots) \\
G^7(1 + v^2 + v^4 + v^6 + v^8 + v^{10} + v^{12} + \cdots)
\end{array}
$$

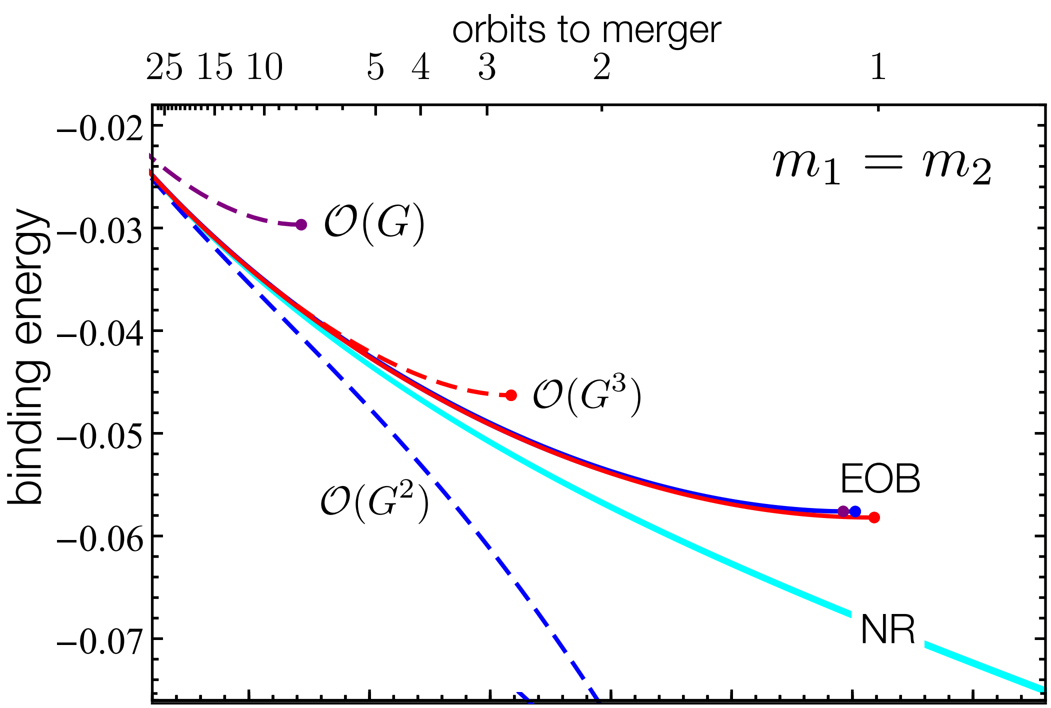
Cheung, Rothstein, MS PRL18

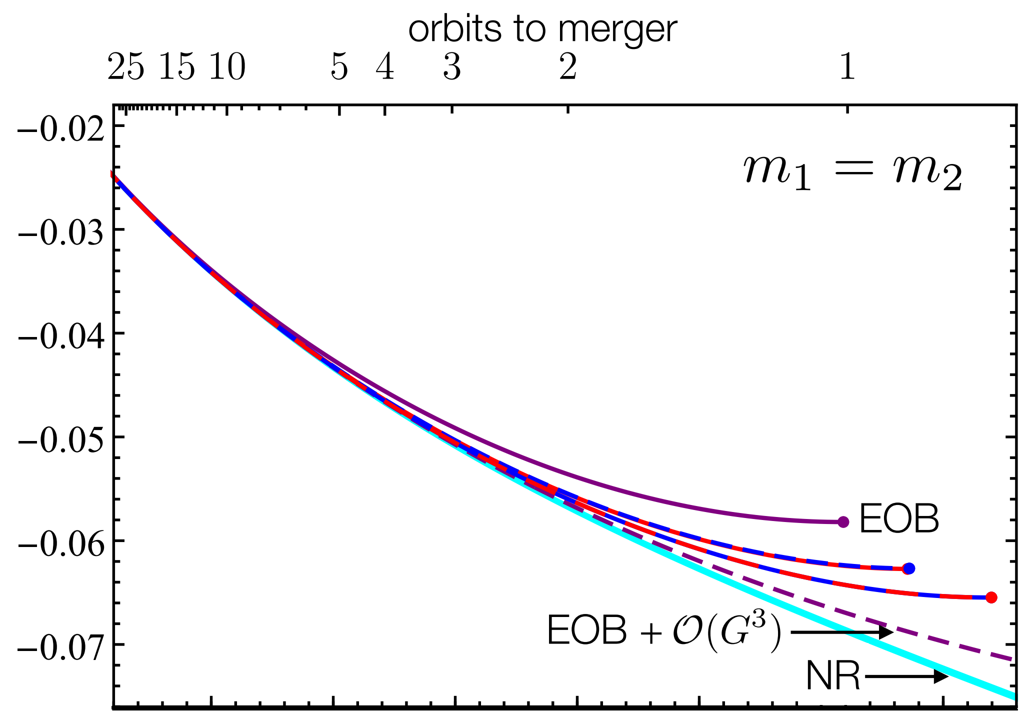
Bern, Cheung, Roiban, Shen, MS, Zeng PRL19

Bern, Parra-Martinez, Roiban, Ruf, Shen, MS, Zeng PRL 21

# Theorists at LIGO are interested

Antonelli, Buonanno, Steinhoff, van de Meent, Vines 2019



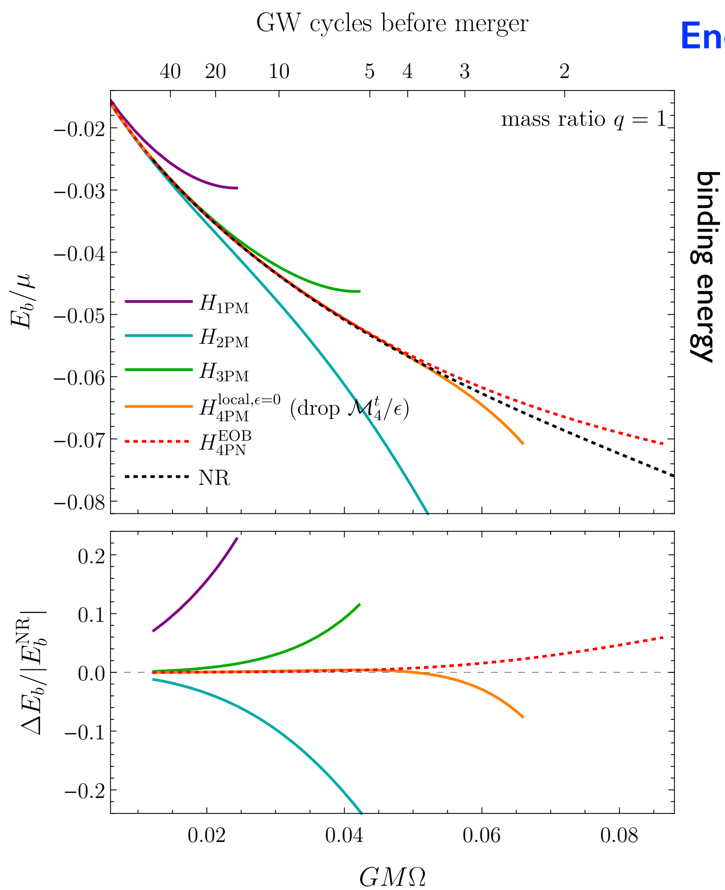
# NEW! from Alessandra Buonanno’s recent talk at GGI

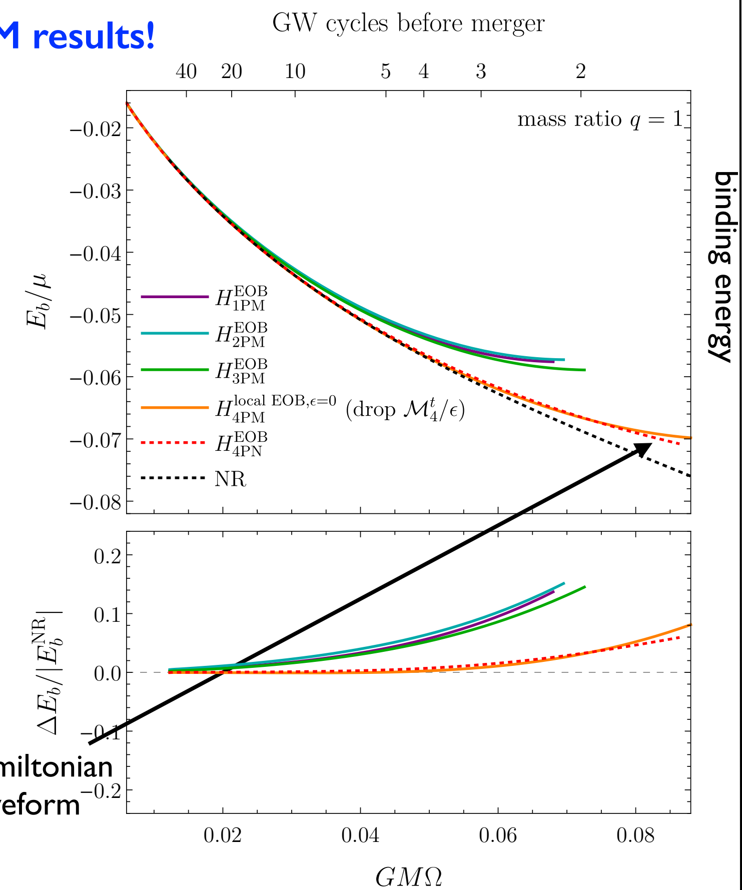
**Crucial** to **push PM** calculations at **higher order,** and **resume them** in EOB formalism.

*(Damour 19, Antonelli, AB, Steinhoff, van de Meent & Vines 19, Khalil, AB, Steinhoff & Vines in prep 21)*

**Encouraging** (local-in-time) **4PM results!**

*(Khalil, AB, Steinhoff & Vines in prep 21)*



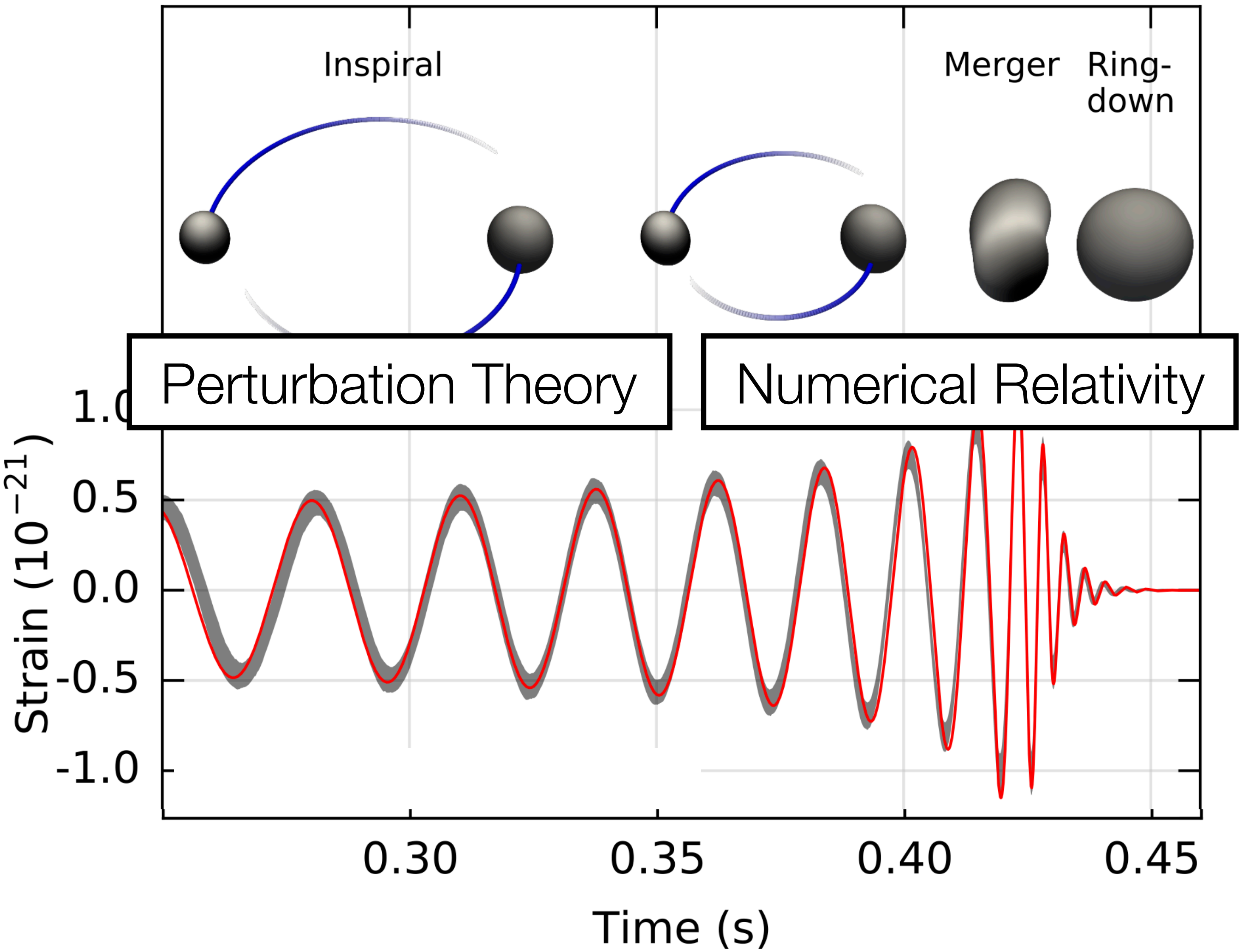




current (uncalibrated) Hamiltonian used to build EOBNR waveform models for LIGO/Virgo

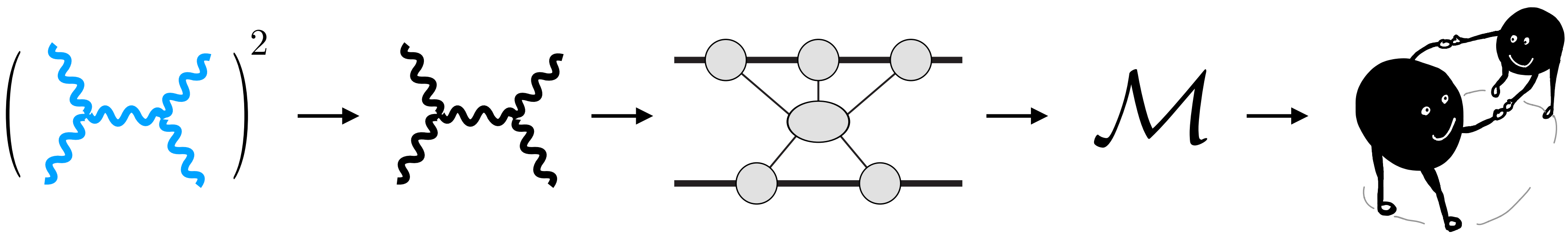
## Fruitful exchanges between GR and HEP

Bini, Damour, Geralico 2019, 2020, … Damour 2019 Damour 2020

# Not yet in the pipeline

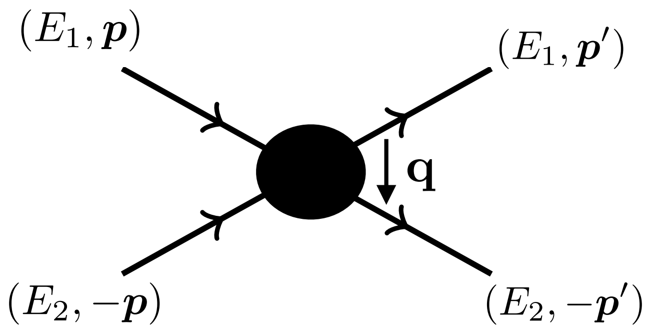


# Scalable pipeline using tools from QFT:




Classical limit is taken at the earliest stages.

# Classical Limit




$$\left(\frac{R_s}{r}\right)^{\mathcal{C}} \sim \left(\frac{m^2}{m_{\rm pl}^2}\frac{q}{m}\right)^{\mathcal{C}}$$

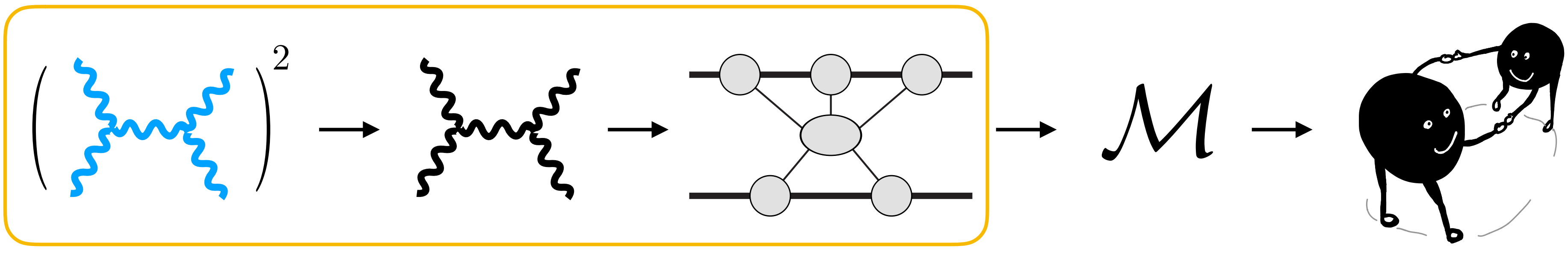
$$\mathcal{M}_{\rm GR} \sim \left(\frac{m^2}{m_{\rm pl}^2}\frac{q}{m}\right)^{\mathcal{C}} \left(\frac{q}{m}\right)^{\mathcal{Q}} \qquad M_{\odot} \sim 10^{38} m_{\rm pl}$$

"In reality classical effects are smaller than quantum." - Aneesh Manohar

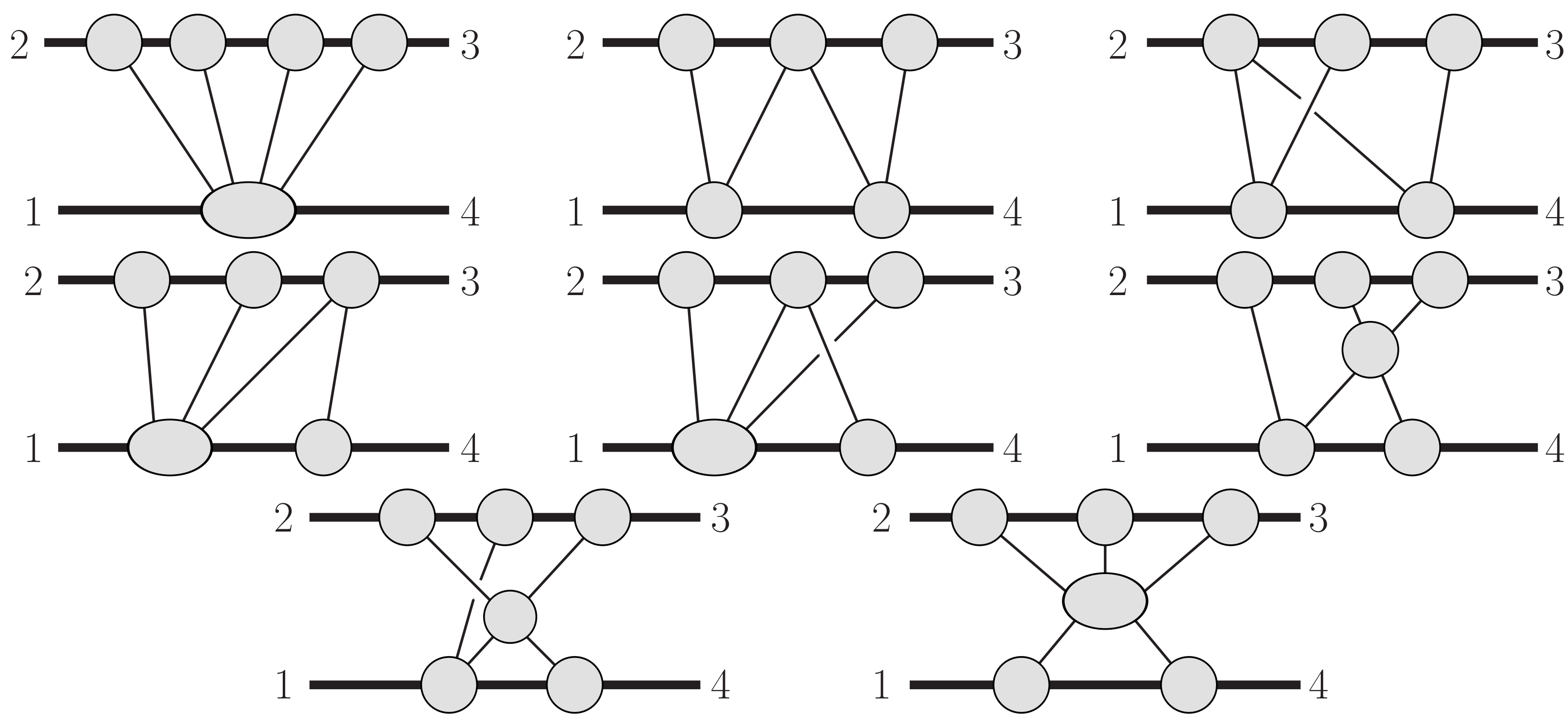
$$\mathcal{M}_{\rm QED} \sim \left(\frac{Q^2}{Q_{\rm pl}^2}\frac{q}{m}\right)^{\mathcal{C}} \left(\frac{q}{m}\right)^{\mathcal{Q}} \qquad e \sim 10^{-1} Q_{\rm pl}$$

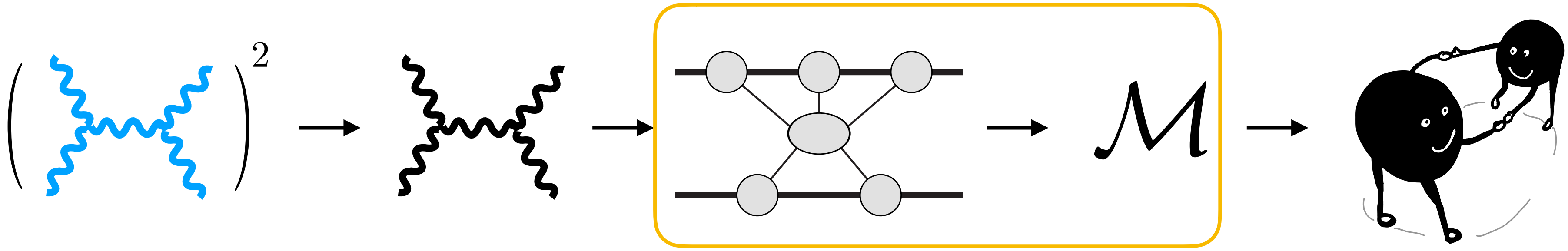
$$\text{soft}: \quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|, |\boldsymbol{q}|) \longrightarrow \begin{array}{ll} \text{potential}: & (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|) \\ \text{radiation}: & (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}||\boldsymbol{v}|) \end{array}$$

NRGR Goldberger, Rothstein




There are eight classical cuts:





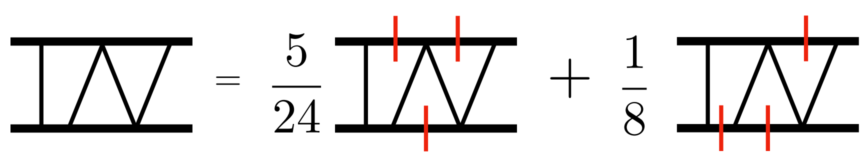

$$\mathcal{I} \sim 1 + v^2 + v^4 + v^6 + \cdots$$

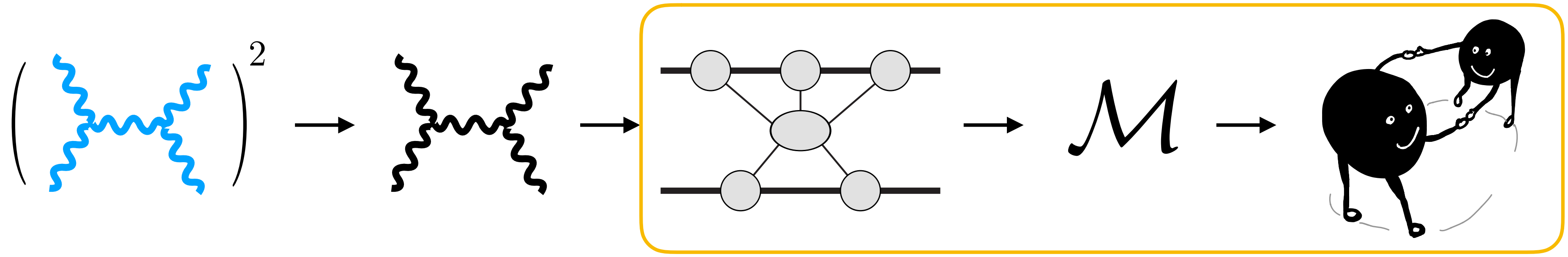
Obtain from differential equations

Parra-Martinez, Ruf, Zeng

Boundary condition from NR integration

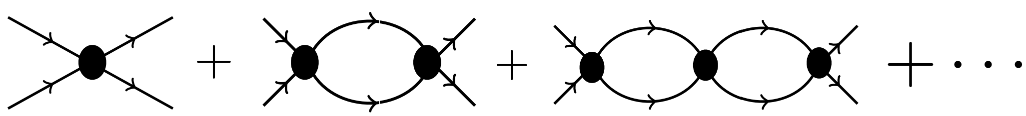
$$\int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{d\omega_3}{2\pi}\mathcal{I}(\omega_1,\omega_2,\omega_3) = \sum S_i \, \mathrm{Res}_i \mathcal{I}(\omega_1,\omega_2,\omega_3)$$






$$\mathcal{L} = \mathcal{L}_{\rm kin} - \int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}')A(\boldsymbol{k})B^\dagger(-\boldsymbol{k}')B(-\boldsymbol{k})$$

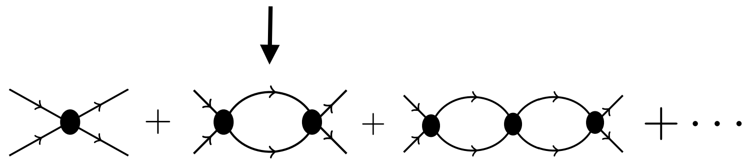
$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i$$



$$= \mathcal{M} + \int \mathcal{M}\frac{1}{Y}\mathcal{M} + \int \mathcal{M}\frac{1}{Y}\mathcal{M}\frac{1}{Y}\mathcal{M} + \cdots$$

$$= \mathcal{M}' + \int \mathcal{M}'\frac{1}{Z}\mathcal{M}' + \int \mathcal{M}'\frac{1}{Z}\mathcal{M}'\frac{1}{Z}\mathcal{M}' + \cdots$$

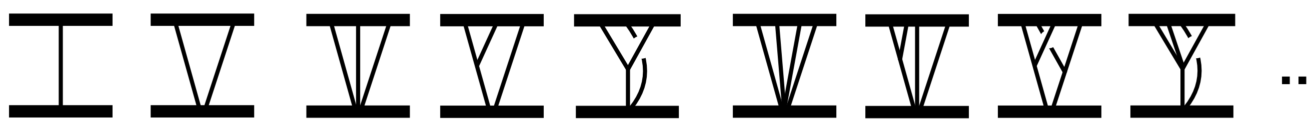
$$E - \sqrt{(\boldsymbol{p}+\boldsymbol{l})^2 + m^2} = E - \sqrt{E + Y}$$



$$= \mathcal{M} + \int \mathcal{M}\frac{1}{Y}\mathcal{M} + \int \mathcal{M}\frac{1}{Y}\mathcal{M}\frac{1}{Y}\mathcal{M} + \cdots$$

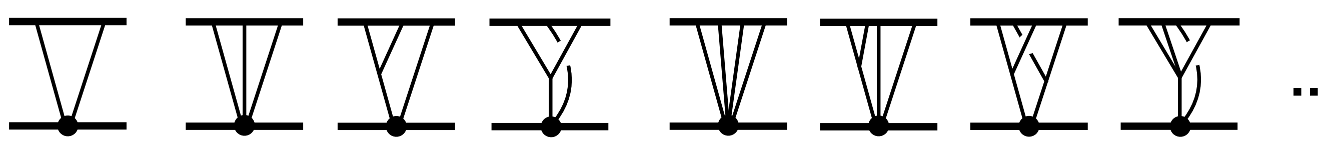
$$\mathcal{M} = \frac{p(r)^2 - p^2}{2E}$$

Test-particle in Schwarzschild:



$$= \frac{1}{2E}\left[m^2(1 - g_{rr}^{\mathrm{iso}}(r)) - E^2\left(1 + \frac{g_{rr}^{\mathrm{iso}}(r)}{g_{tt}^{\mathrm{iso}}(r)}\right)\right]$$

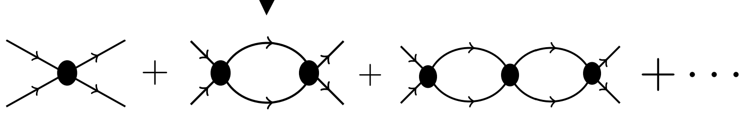
Test-particle with tidal distortion:



$$= \frac{64}{5005Em^4(\rho-1)^5\rho^6(\rho+1)^8R^4}\Big[E^4(\rho+1)^4\big(525525\rho^9 - 53625\rho^8 + 226512\rho^7 + 16952\rho^6 - 26598\rho^5 - 4090\rho^4 + 3240\rho^3 + 464\rho^2 - 231\rho - 21\big)$$
$$+ m^4\big(165165\rho^8 + 12870\rho^7 + 20592\rho^6 + 20098\rho^5 + 13390\rho^4 + 6050\rho^3 + 1760\rho^2 + 294\rho + 21\big)(\rho-1)^5 - 2E^2m^2(\rho+1)^2\big(225225\rho^9 - 102960\rho^8$$
$$+ 65637\rho^7 + 16809\rho^6 + 3367\rho^5 + 2865\rho^4 + 775\rho^3 - 501\rho^2 - 252\rho - 21\big)(\rho-1)^2\Big]$$

$$\rho = 4r/R$$

$$E - \sqrt{(\boldsymbol{p}+\boldsymbol{l})^2 + m^2} = 2\boldsymbol{p}' \cdot \boldsymbol{l} + \cdots$$



$$= \mathcal{M}' + \int \mathcal{M}' \frac{1}{Z} \mathcal{M}' + \int \mathcal{M}' \frac{1}{Z} \mathcal{M}' \frac{1}{Z} \mathcal{M}' + \cdots$$

$$\mathcal{M}' = I_r$$

The classical part of the amplitude is the radial action.

Mass dependence $\quad \mathcal{M}_N \sim G^N \left[ m_1^{N+1} m_2^2 + m_1^N m_2^3 + \cdots + \{1 \leftrightarrow 2\} \right]$

Bini, Damour, Geralico

$$\begin{array}{rl} \mathcal{M}_1 \sim G & \left[ m_1^2 m_2^2 \right] \\ \mathcal{M}_2 \sim G^2 & \left[ m_1^3 m_2^2 + \{1 \leftrightarrow 2\} \right] \\ \mathcal{M}_3 \sim G^3 & \left[ m_1^4 m_2^2 + m_1^3 m_2^3 + \{1 \leftrightarrow 2\} \right] \\ \mathcal{M}_4 \sim G^4 & \left[ m_1^5 m_2^2 + m_1^4 m_2^3 + \{1 \leftrightarrow 2\} \right] \\ \mathcal{M}_5 \sim G^5 & \left[ m_1^6 m_2^2 + m_1^5 m_2^3 + m_1^4 m_2^4 + \{1 \leftrightarrow 2\} \right] \\ \mathcal{M}_6 \sim G^6 & \left[ m_1^7 m_2^2 + m_1^6 m_2^3 + m_1^5 m_2^4 + \{1 \leftrightarrow 2\} \right] \\ \vdots & \end{array}$$

0SF 1SF 2SF

# Amplitudes

$$\mathcal{M}_1 = 16\pi G\nu^2 M^4 |\boldsymbol{q}|^{-2}(2\sigma^2 - 1)$$

$$\mathcal{M}_2 = 6\pi^2 G^2\nu^2 M^5 |\boldsymbol{q}|^{-1}(5\sigma^2 - 1)$$

$$\mathcal{M}_3 = 2\pi G^3\nu^2 M^6 \log \boldsymbol{q}^2 \Bigg[ \frac{(1+2\nu(\sigma-1))(5-60\sigma^2+120\sigma^4-64\sigma^2)}{3(\sigma^2-1)^2} + 8\nu\left(\frac{\sigma(14\sigma^2+25)}{3} + (4\sigma^4 - 12\sigma^2 - 3)\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\Bigg]$$

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2} \qquad M = m_1 + m_2 \qquad \sigma = \frac{p_1 \cdot p_2}{m_1 m_2}$$

$$\mathcal{M}_4(\boldsymbol{q}) = G^4 M^7 \nu^2 |\boldsymbol{q}| \left(\frac{\boldsymbol{q}^2}{4^{\frac{1}{3}}\tilde{\mu}^2}\right)^{-3\epsilon} \pi^2 \left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}}\right)\right]$$

$$\mathcal{M}_4^{\mathrm{p}} = -\frac{35\left(1-18\sigma^2+33\sigma^4\right)}{8\left(\sigma^2-1\right)} \qquad \mathcal{M}_4^{\mathrm{t}} = h_1 + h_2 \log\left(\tfrac{\sigma+1}{2}\right) + h_3 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$

$$\begin{aligned}\mathcal{M}_4^{\mathrm{f}} &= h_4 + h_5 \log\left(\tfrac{\sigma+1}{2}\right) + h_6 \frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} + h_7 \log(\sigma) - h_2 \frac{2\pi^2}{3} + h_8 \frac{\operatorname{arccosh}^2(\sigma)}{\sigma^2-1} + h_9 \left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) + \frac{1}{2}\log^2\left(\tfrac{\sigma+1}{2}\right)\right] \\ &+ h_{10}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{2}\right) - \frac{\pi^2}{6}\right] + h_{11}\left[\mathrm{Li}_2\left(\tfrac{1-\sigma}{1+\sigma}\right) - \mathrm{Li}_2\left(\tfrac{\sigma-1}{\sigma+1}\right) + \frac{\pi^2}{3}\right] + h_2 \frac{2\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\left[\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - \mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right)\right] \\ &+ \frac{2h_3}{\sqrt{\sigma^2-1}}\left[\mathrm{Li}_2\left(1-\sigma-\sqrt{\sigma^2-1}\right) - \mathrm{Li}_2\left(1-\sigma+\sqrt{\sigma^2-1}\right) + 5\mathrm{Li}_2\left(\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) - 5\mathrm{Li}_2\left(-\sqrt{\tfrac{\sigma-1}{\sigma+1}}\right) + 2\log\left(\tfrac{\sigma+1}{2}\right)\operatorname{arccosh}(\sigma)\right] \\ &+ h_{12}\mathrm{K}^2\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{13}\mathrm{K}\left(\tfrac{\sigma-1}{\sigma+1}\right)\mathrm{E}\left(\tfrac{\sigma-1}{\sigma+1}\right) + h_{14}\mathrm{E}^2\left(\tfrac{\sigma-1}{\sigma+1}\right),\end{aligned}$$

High-energy limit: $\mathcal{M} \sim G^4 \boldsymbol{p}^8 |\boldsymbol{q}| \dfrac{(m_1+m_2)}{m_1 m_2}$

$$h_1 = \frac{1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6}{12(\sigma^2 - 1)}$$

$$h_2 = \frac{1}{2}\left(5 - 76\sigma + 150\sigma^2 - 60\sigma^3 - 35\sigma^4\right)$$

$$h_3 = \sigma\frac{\left(-3 + 2\sigma^2\right)}{4(\sigma^2 - 1)}\left(11 - 30\sigma^2 + 35\sigma^4\right)$$

$$h_4 = \frac{1}{144\left(\sigma^2 - 1\right)^2\sigma^7}\big(-45 + 207\sigma^2 - 1471\sigma^4 + 13349\sigma^6 - 37566\sigma^7 + 104753\sigma^8 - 12312\sigma^9 - 102759\sigma^{10} - 105498\sigma^{11} + 134745\sigma^{12} + 83844\sigma^{13} - 101979\sigma^{14} + 13644\sigma^{15} + 10800\sigma^{16}\big)$$

$$h_5 = \frac{1}{4(\sigma^2 - 1)}\big(1759 - 4768\sigma + 3407\sigma^2 - 1316\sigma^3 + 957\sigma^4 - 672\sigma^5 + 341\sigma^6 + 100\sigma^7\big)$$

$$h_6 = \frac{1}{24(\sigma^2 - 1)^2}\big(1237 + 7959\sigma - 25183\sigma^2 + 12915\sigma^3 + 18102\sigma^4 - 12105\sigma^5 - 9572\sigma^6 + 2973\sigma^7 + 5816\sigma^8 - 2046\sigma^9\big)$$

$$h_7 = 2\sigma\frac{\left(-852 - 283\sigma^2 - 140\sigma^4 + 75\sigma^6\right)}{3(\sigma^2 - 1)}$$

$$h_8 = \frac{\sigma}{8(\sigma^2 - 1)^2}\big(-304 - 99\sigma + 672\sigma^2 + 402\sigma^3 - 192\sigma^4 - 719\sigma^5 - 416\sigma^6 + 540\sigma^7 + 240\sigma^8 - 140\sigma^9\big)$$

$$h_9 = \frac{1}{2}\left(52 - 532\sigma + 351\sigma^2 - 420\sigma^3 + 30\sigma^4 - 25\sigma^6\right)$$

$$h_{10} = 2\left(27 + 90\sigma^2 + 35\sigma^4\right)$$

$$h_{11} = 20 + 111\sigma^2 + 30\sigma^4 - 25\sigma^6$$

$$h_{12} = \frac{834 + 2095\sigma + 1200\sigma^2}{2(\sigma^2 - 1)}$$

$$h_{13} = -\frac{1183 + 2929\sigma + 2660\sigma^2 + 1200\sigma^3}{2(\sigma^2 - 1)}$$

$$h_{14} = \frac{7\left(169 + 380\sigma^2\right)}{4(\sigma - 1)}$$

# Hamiltonian

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2+m_1^2} + \sqrt{\boldsymbol{p}^2+m_2^2} + \sum_{i=1}^{\mathbf{4}} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right) \qquad \xi = E_1E_2/E^2$$

$$c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right]$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+ \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg], \end{aligned}$$

$$\begin{aligned} c_4 = {}&\frac{M^7\nu^2}{4\xi E^2}\left[\mathcal{M}_4^{\mathrm{p}} + \nu\left(\frac{\mathcal{M}_4^{\mathrm{t}}}{\epsilon} + \mathcal{M}_4^{\mathrm{f}} - 10\mathcal{M}_4^{\mathrm{t}}\right)\right] + \mathcal{D}^3\left[\frac{E^3\xi^3}{3}c_1^4\right] + \mathcal{D}^2\left[\left(\frac{E^3\xi^3}{\boldsymbol{p}^2} + \frac{E\xi(3\xi-1)}{2}\right)c_1^4 - 2E^2\xi^2c_1^2c_2\right] \\ &+ \left(\mathcal{D}+\frac{1}{\boldsymbol{p}^2}\right)\left[E\xi(2c_1c_3+c_2^2) + \left(\frac{4\xi-1}{4E} + \frac{2E^3\xi^3}{\boldsymbol{p}^4} + \frac{E\xi(3\xi-1)}{\boldsymbol{p}^2}\right)c_1^4 + \left((1-3\xi) - \frac{4E^2\xi^2}{\boldsymbol{p}^2}\right)c_1^2c_2\right], \end{aligned}$$

$$\mathcal{D} = \frac{d}{d\boldsymbol{p}^2}$$

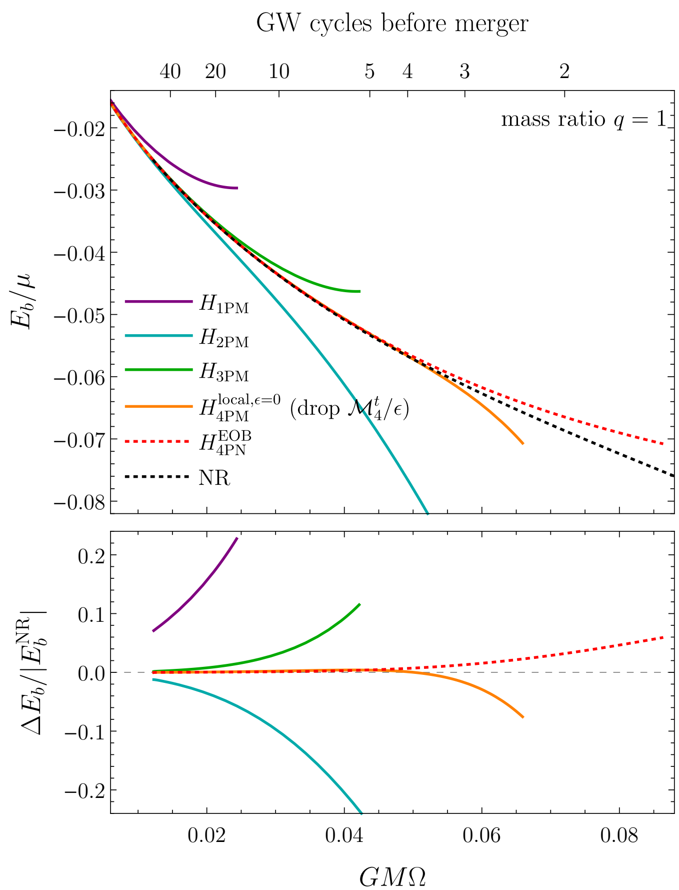
# NEW! from Alessandra Buonanno's recent talk at GGI

**Crucial** to **push PM** calculations at **higher order,** and **resum them** in EOB formalism.

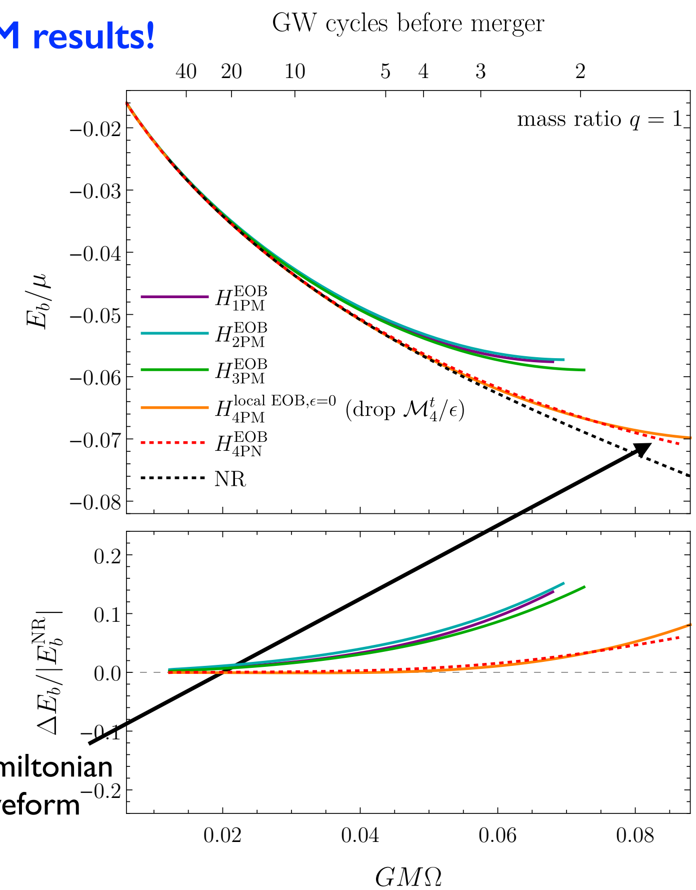
*(Damour 19, Antonelli, AB, Steinhoff, van de Meent & Vines 19, Khalil, AB, Steinhoff & Vines in prep 21)*

**Encouraging** (local-in-time) **4PM results!**

*(Khalil, AB, Steinhoff & Vines in prep 21)*




binding energy




binding energy

current (uncalibrated) Hamiltonian used to build EOBNR waveform models for LIGO/Virgo

## checks:

Bini, Damour, Geralico 2020

Blümlein, Maier, Marquard, Schäfer 2020

Blümlein, Maier, Marquard, Schäfer 2021

Dlapa, Kälin, Liu, Porto 2021

Gravitational wave science has opened up a new direction in theoretical high energy physics.

Classical binary dynamics has the hallmarks of a great problem in theoretical high energy physics.

This program is in a nascent stage.