Amplitudes 20th Aug. '21

Moduli space of tropical curves & Feynman Integrals.

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Theme : Interpret amplitudes (or Feynman integrals) as volumes of cells on some natural geometric object.

More info : 2101.04419v3

- talk at Harvard mathematical picture seminar (youtube)

<u>Today</u> : $\mathcal{M}_g^{trop}$ = moduli space of tropical curves.

Cells $\longleftrightarrow$ weighted graphs

$C(G,w)$ (G,w)

<u>Overview</u> : We will define the following:

- Canonical differential forms on $\mathcal{M}_g^{trop}$, singular along $w \neq 0$

$$\omega \in \Omega^{\bullet}_{can}$$

- Canonical integrals

$$I_G(\omega) = \int_{C(G,w)/\mathbb{R}_{>0}} \omega$$

Always finite!

- Connect:
  - Feynman residues in massless ϕ^4
  - cohomology of Graph complexes, $\mathcal{M}_g$, $\mathcal{A}_g$, $GL_g(\mathbb{Z})$, ...

I. Tropical curves

(…, Brannetti–Melo–Viviani, …)

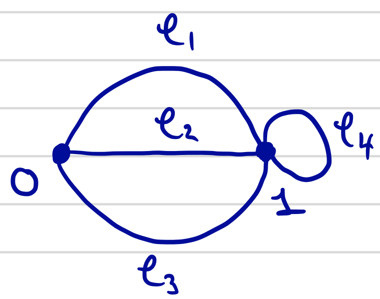
A weighted graph (G, w) is assignment of a weight $w(v) \geq 0$ to each vertex $v \in V(G)$.

- Stable if $\deg(v) \geq 3$ if $w(v) = 0$.
  $\deg(v) \geq 1$ if $w(v) = 1$
- Metric graph is $\ell : E(G) \to \mathbb{R}_{>0}$ length of each edge.

A tropical curve of genus

$$g = h_G + \sum_{V(G)} w(v)$$

is a stable weighted metric graph G.



Space of possible lengths

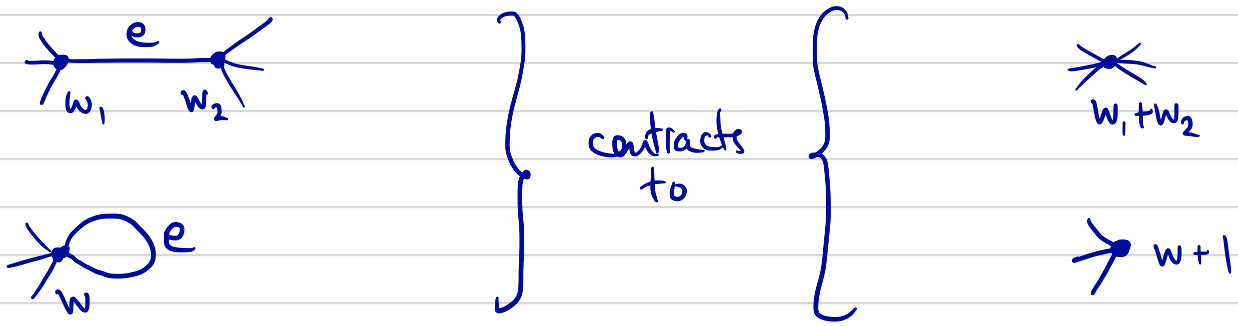
$$C(G, w) = \{ \ell_e \in \mathbb{R}_{>0} \ , \ e \in E_G \} \simeq \mathbb{R}_{>0}^{E_G}$$

Contractions

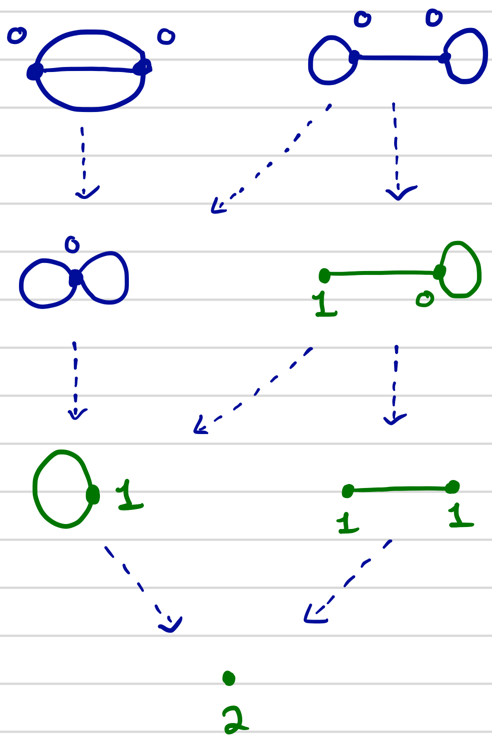
Letting edge length $\ell_e \longrightarrow 0$

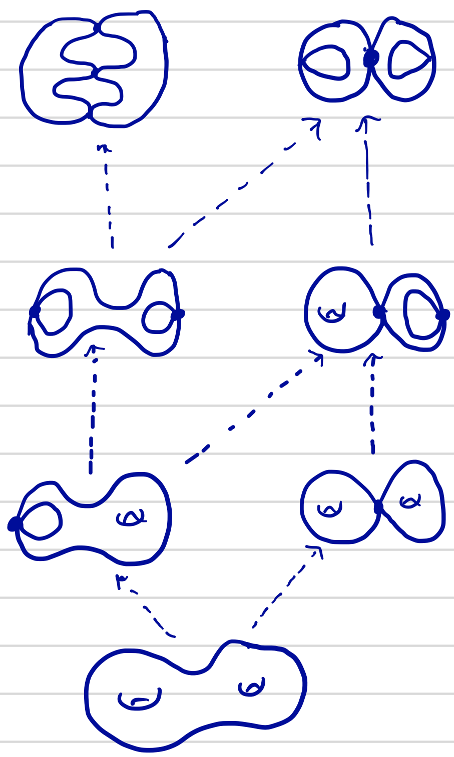
$\updownarrow$

Contract edge e.



Eg: $g=2$





$\overline{\mathcal{M}}_2$

II. Moduli space $\mathcal{M}_g^{trop}$

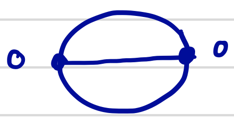
Cells $\qquad C(G,w) = \mathbb{R}_{>0}^{E_G}$

$\mathrm{Aut}(G,w)$ permutes edges & acts on $\mathbb{R}_{>0}^{E_G}$

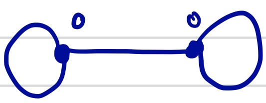
$$\overline{C(G,w)} = \mathbb{R}^{E_G} / \mathrm{Aut}(G,w)$$

$$\mathcal{M}_g^{trop} = \coprod_{(G,w)} \overline{C(G,w)} / \sim$$

Glue different cells along common specialisations

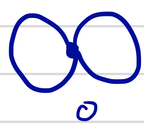


$$\overline{C(G,w)} = \mathbb{R}_{>0}^3 / S_3$$



$$\overline{C(G,w)} = \mathbb{R}_{>0}^3 / S_2$$

Glue these two cells along



$$\overline{C(G,w)} = \mathbb{R}_{>0}^2 / S_2$$

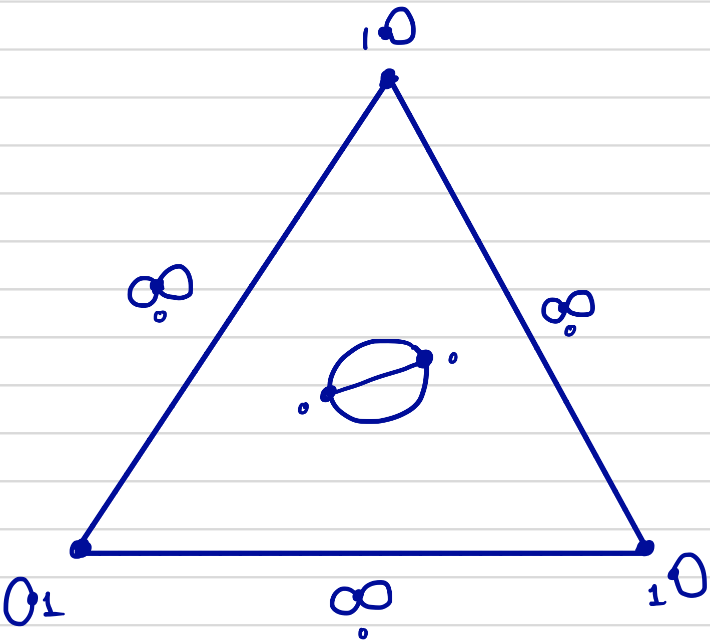
Easier to visualise the links $LC(G,w)$

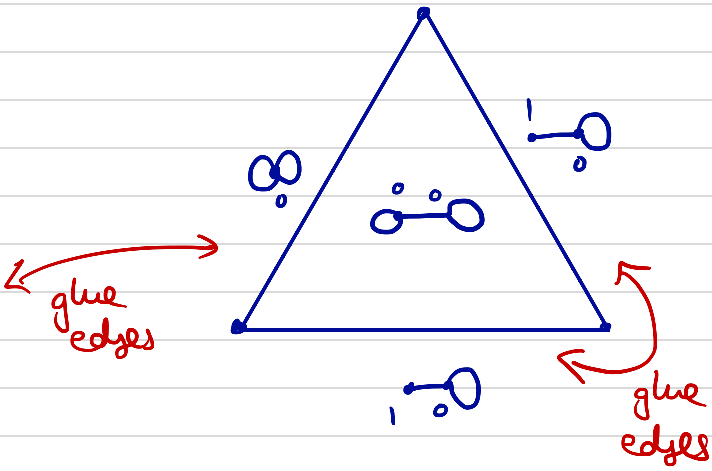
$\hookrightarrow$ impose $\sum_{e \in E_G} \ell_e = 1$

Drops dimension by 1.

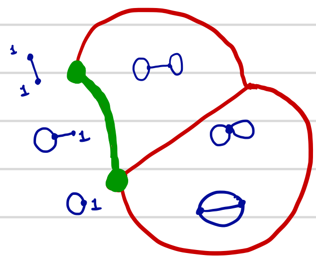
$$LC(G,w) \cong \{ \ell_e \in \mathbb{R}_{>0} \,,\ \sum \ell_e = 1 \}$$

$\cong$ simplex.





Glue all 3 sides together



III. Differential forms

Want collections $\{\omega_G\}$ of differential forms

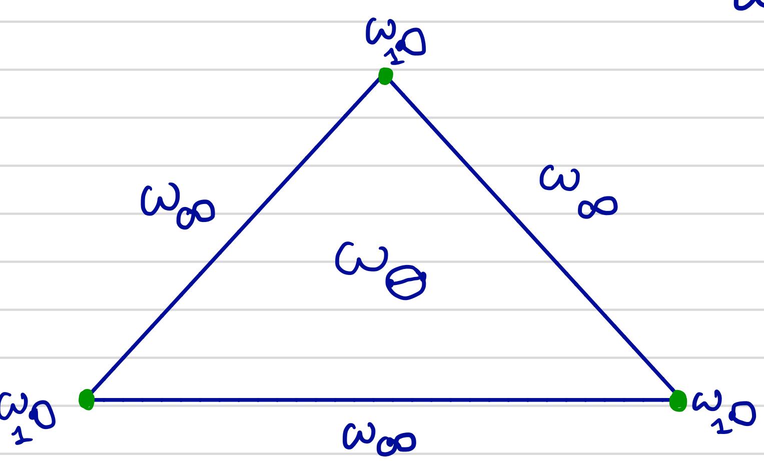
$\forall G$, ω_G a smooth k-form on $LC(G,w)$

such that

$$\begin{cases} \bullet \quad \pi^* \omega_G = \omega_G & \forall \pi \in \mathrm{Aut}(G,w) \\ \bullet \quad \omega_G \text{ compatible} & \text{(agree on boundaries of cells)} \end{cases}$$

Eg :

$$\begin{tikzcd} \omega_{\ominus}(e_1,e_2,e_3) \arrow[dr, "e_2 \to 0"'] & & \omega_{\text{o-o}}(e_1,e_2,e_3) \arrow[dl, "e_2 \to 0"] \\ & \omega_{\infty}(e_1,e_3) & \end{tikzcd}$$



Our forms will have poles along $LC(G,w)$ $w \neq 0$

We can throw away all (G,ω) where ω non-zero.

ie " contracting tadpole O $\Rightarrow$ zero. "

IV. Biinvariant forms

X invertible $k \times k$ matrix, generic entries.

X entries $(x_{ij})_{ij}$

dX . . . . . . $(dx_{ij})_{ij}$

$\mu_X = X^{-1} dX$ analogue of $d\log z$.

It is left-invariant: $\mu_{AX} = X^{-1} A^{-1} d(AX)$

if $A \in GL_k(\mathbb{Z})$ $= X^{-1} \cancel{A^{-1}} \cancel{A} dX = \mu_X$

Not right-invariant.

Define: $\beta^n_X = \mathrm{tr}(\mu^n_X)$

- $\beta^{2n}_X = 0$
- $\beta^{4k+3}_X = 0$ if X symmetric
- $d\beta^n_X = 0$ closed
- $\beta_{AX} = \beta_{XA} = \beta_X$ bi-invariant

Interesting forms: $\beta^5_X, \beta^9_X, \beta^{13}_X, \dots$ X symmetric

Examples

- $X = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}$ generic

$$\beta^3_X = 3 \frac{\sum_{i=1}^{4} (-1)^i a_i \, da_1 \ldots \widehat{da_i} \ldots da_4}{(a_1 a_2 - a_3 a_4)^2}$$

- $X = \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{pmatrix}$ symmetric

$$\beta^3_X = 0$$

$$\beta^5_X = -10 \sum_{i=1}^{6} (-1)^i \frac{a_i \, da_1 \ldots \widehat{da_i} \ldots da_6}{\det(X)^2}$$

V. Canonical forms

G a graph

$\rightsquigarrow$

Λ_G graph Laplacian

$\rightsquigarrow$

$$\omega_G^{4k+1} = \mathrm{tr}\left(\left(\Lambda_G^{-1} d\Lambda_G\right)^{4k+1}\right) \qquad \forall\, k \geq 1$$

Theorem: Well-defined, equivariant with respect to all automorphisms, compatible (glue together).

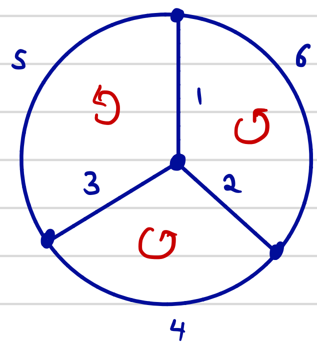
$\hookrightarrow$ For any $k \geq 1$, $\omega_\bullet^{4k+1}$ glue together to give a differential of degree $4k+1$ on the link of $\left(\mathcal{M}_g^{trop}\right)_{w=0} \subseteq \mathcal{M}_g^{trop}$

locus of graphs with zero weights

Take exterior products.

Define: Ω^{can} = Exterior algebra on $\omega^5, \omega^9, \omega^{14}, \ldots$

Ex :

$G =$ 

wheel with 3 spokes

Laplacian matrix

$$\Lambda_G = \begin{pmatrix} \alpha_1+\alpha_3+\alpha_5 & -\alpha_1 & -\alpha_3 \\ -\alpha_1 & \alpha_1+\alpha_2+\alpha_6 & -\alpha_2 \\ -\alpha_3 & -\alpha_2 & \alpha_2+\alpha_3+\alpha_4 \end{pmatrix}$$

$$\omega_G^5 = \sum_{i=1}^{6} (-1)^i \, \alpha_i \, \frac{d\alpha_1 \wedge \cdots \wedge \widehat{d\alpha_i} \wedge \cdots \wedge d\alpha_6}{\psi_G^2}$$

$\psi_G = \det \Lambda_G =$ graph polynomial

Recall $\psi_G = \det \Lambda_G$ graph polynomial
(Kirchhoff, Symanzik)

Define : Canonical integrals.

Let $\omega \in \Omega^{can}$ (eg $\omega^5 \wedge \omega^9$).

G st $E_G = \deg(\omega) + 1$

$$\boxed{I_G(\omega) = \int_{LC(G,0)} \omega = \int_{\alpha_e \geq 0} \omega_G}$$

Projective integral.

Theorem: • $I_G(\omega)$ is finite.

• The integrand always has the form

$$\omega_G = N_G \frac{\Omega_G}{\psi_G^{k+1}} \qquad \Omega_G = \sum_i (-1)^i \alpha_i \, d\alpha_1 \dots \widehat{d\alpha_i} \dots d\alpha_{E_G}$$

numerator

graph polynomial.

(c.f. residues in massless theories in parametric form)

• There is a Stokes formula relating

$$\begin{cases} I_{G/e}(\omega_1 \wedge \omega_2) \\ I_\gamma(\omega_1)\, I_{G/\gamma}(\omega_2) \end{cases}$$

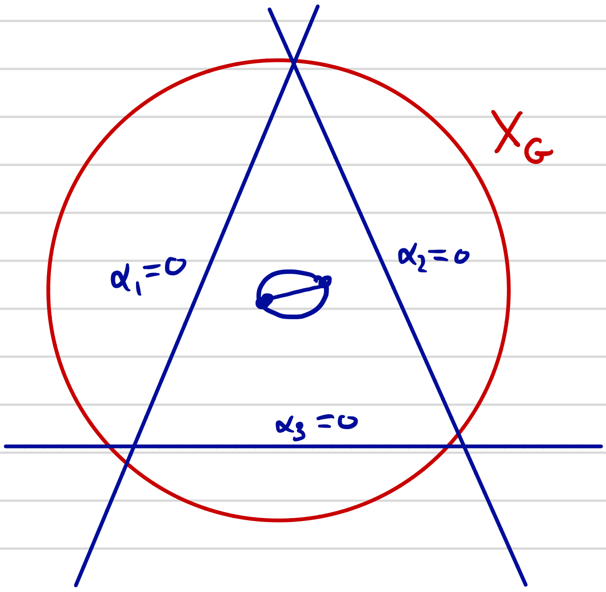
i.e canonical integrals for different graphs.

(can prove relations between canonical integrals topologically)

Idea of proof

$X_G \subseteq \mathbb{P}^{E_G-1}$ graph hypersurface

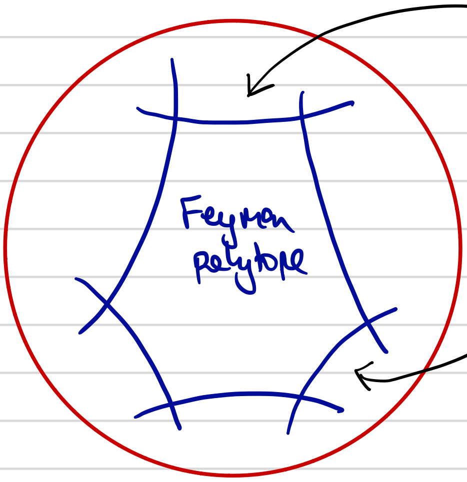
locus where $\psi_G = 0$.



Blow up π

$$\begin{array}{c} \mathbb{P}^{E_G-1} \\ \uparrow \pi \\ \mathbb{P}^{E_G-1} \end{array}$$

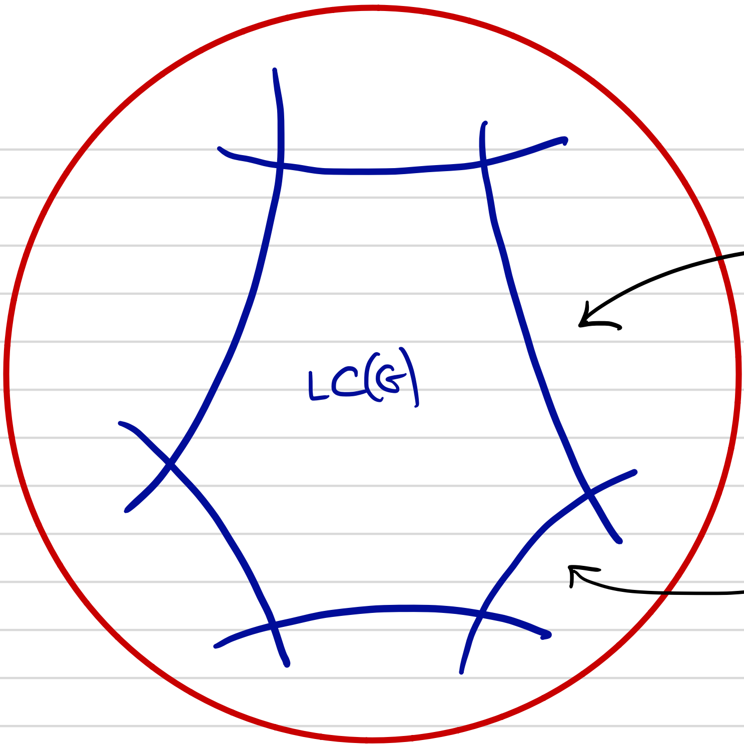
Exceptional divisors $\mathcal{E}$



$\pi^*(\omega)$ has no poles along $\mathcal{E}$

(Totally different from case of Feynman integrals, which have poles along $\mathcal{E}$

Renormalization cancels these poles.)



Apply Stokes: $\int_{LC(G)} d\omega = 0$

$$\boxed{\sum_e \int_{G/e} \omega + \sum_{\substack{\gamma \subseteq G \\ 1PI}} \sum_i \underbrace{\int_\gamma \omega_i' \int_{G/\gamma} \omega_i''}_{\text{Connes-Kreimer coproduct.}} = 0}$$

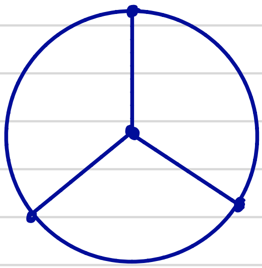
$$\Delta \omega = \sum_i \omega_i' \otimes \omega_i''$$ coproduct

Refined by

$$\Delta \omega^{4k+1} = 1 \otimes \omega^{4k+1} + \omega^{4k+1} \otimes 1$$

Known examples

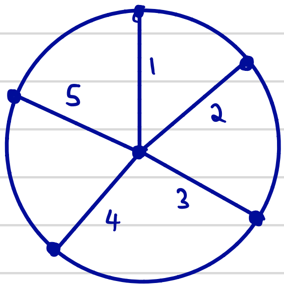
W_3



$$\omega_G^5 = 10\,\frac{\Omega_G}{\psi_G^2}$$

$$I_{W_3}(\omega^5) = 60\,\zeta(3)$$

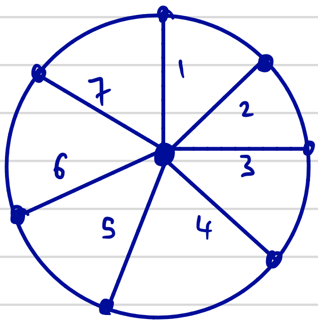
W_5



$$\omega_G^9 = 18\left(\frac{\Omega_G}{\psi_G^2} + 12\,\frac{\alpha_1\cdots\alpha_5\,\Omega_G}{\psi_G^3}\right)$$

$$I_{W_5}(\omega^9) = 1260\,\zeta(5)$$

W_7



$$\omega_G^{13} = 26\left(\frac{\Omega_G}{\psi_G^2} + 60\,\frac{\alpha_1\cdots\alpha_7\,\Omega_G}{\psi_G^3} + 360\,\frac{(\alpha_1\cdots\alpha_7)^2\,\Omega_G}{\psi_G^4}\right)$$

$$I_{W_7}(\omega^{13}) = 24024\,\zeta(7)$$

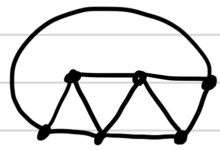
Conj.:
$$I_{W_{2n+1}}\left(\omega^{4n+1}\right) = (2n+1)\binom{4n+2}{2n+1}\zeta(2n+1)$$

Schnetz.

Rem : $I_{W_{2n+1}}(\omega^{4n+1}) \neq 0 \Rightarrow$ new results in geometry

More examples

- Zig-zag Z_5

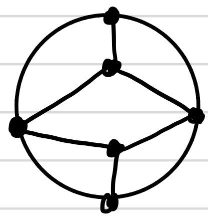


$$2\, I_{Z_5}(\omega^9) = I_{W_5}(\omega^9)$$

Stokes relation

(graph complexes)

- $W_3 : W_3$



$$I_{W_3:W_3}(\omega^9) = 0$$

- K_6 complete graph with 6 vertices

$$\omega^5_{K_6} \wedge \omega^9_{K_6} = \frac{9!}{8} \frac{\prod_e \alpha_e}{\psi^3_{K_6}} \Omega_{K_6}$$

$$I_{K_6}(\omega^5 \wedge \omega^9) = \frac{9!}{16} \left(360\, \zeta(3,5) + 690\, \zeta(3)\zeta(5) - \frac{29\pi^8}{315} \right)$$

(Borinsky-Schnetz)

Compare with:

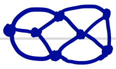
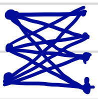
Feynman residues

$$I_G^{\text{Feyn}} = \int_{\alpha_e \geq 0} \frac{\Omega_G}{\psi_G^2}$$

Residue in ε, in $4-2\varepsilon$ dimensions.

Finite if and only if:

- $h_G = 2E_G$
- $h_\gamma < 2E_\gamma \qquad \forall\ \gamma \subsetneq G$

Feynman integrals I^{Feyn}		Canonical integrals	
W_3 	$6\,\zeta(3)$	W_3 	$60\,\zeta(3)$
W_4 	$20\,\zeta(5)$	W_5 	$1260\,\zeta(5)$
W_5 	$70\,\zeta(7)$	W_7	$24024\,\zeta(7)$
W_n	$*\,\zeta(2n-3)$	W_{2n+1}	$*\,\zeta(2n+1)$ (Conj.)
$W_3 : W_4$ 	$120\,\zeta(3)\zeta(5)$	$\mathcal{I}_{3,5}$	$\zeta(3)\zeta(5)$
$K_{3,4}$ 	$*\,\zeta(3)\zeta(5) + *\left(\zeta(3,5) - \frac{29}{12}\zeta(8)\right)$	$I_{K_6}(\omega^5 \wedge \omega^9)$ complete graph	$*\,\zeta(3)\zeta(5) + *\left(\zeta(3,5) - \frac{29}{12}\zeta(8)\right)$

Broadhurst-Kreimer

Q: Do canonical integrals come from $I^{Feyn}_{\bullet}$ via IBP?

i.e., $\mathbb{Q}\langle I^{can}\rangle \subseteq \mathbb{Q}\langle I^{Feyn}_{\bullet}\rangle$?

Mechanism to relate canonical & Feynman integrals.

eg
$$I^{\mathrm{can}}_{W_5}(\omega^9) = * \; I^{\mathrm{Feyn}}_{W_4}$$

Sketch proof:

$$\omega^9 \in H^9(\mathbb{P}^9 \setminus X_{W_5})$$

must be zero (weight reasons). Hence $\omega^9 = d\alpha^8$.

$$\int_{LC(W_5)} \omega^9 \overset{\text{Stokes}}{=} \int_{\partial LC(W_5)} \alpha^8$$

But $\partial LC(W_5) = \bigcup_G LC(G)$ where

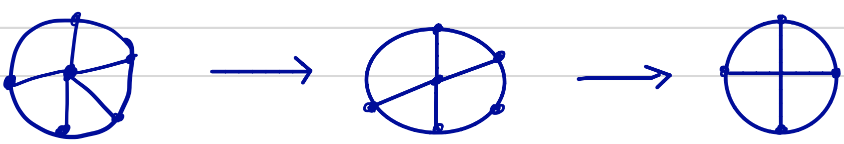
$$\begin{cases} G = W_5 \setminus e \quad \text{or} \quad W_5/e \\ G = \gamma \times W_5/\gamma \end{cases}$$

No cohomology in correct weights, so $\alpha^8\big|_{LC(G)} = d\beta^7$ exact on each facet.

Double boundary: $\partial LC(W_5 \setminus e)$ contains $LC(W_4)$

$\left[\beta^7\big|_{LC(W_4)}\right]$ is proportional to $\left[\Omega^{\mathrm{Feyn}}_{W_4}\right] \in H^7(\mathbb{P}^7 \setminus X_{W_4})$

$\square$



A taster of Graph homology (Kontsevich)

G graph, ω orientation

$\omega = e_1 \wedge e_2 \wedge \dots \wedge e_n$

= ordering on set of edges / alternating permutations

- $(G, \omega) = -(G, -\omega)$
- $(G, \eta) = (G, \sigma\eta)$ $\quad \sigma : G \xrightarrow{\sim} G$ auto.

$$d(G,\omega) = \sum_{i=1}^{n} (-1)^i \left(G /\!\!/ e_i ,\ e_1 \wedge \dots \wedge \hat{e}_i \wedge \dots \wedge e_n \right)$$

contract edge i

$d : \mathbb{Q}\langle \text{oriented graphs} \rangle \longrightarrow \mathbb{Q}\langle \text{oriented graphs} \rangle$

$d^2 = 0.$

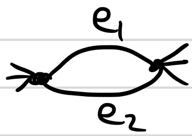
Graph homology: $\quad H(GC) = \dfrac{\ker d}{\operatorname{Im} d}$

Huge, mysterious.

Related to $\begin{cases} \mathfrak{grt} & \text{(Willwacher)} \\ \bigoplus_g H^{\bullet}(\mathcal{M}_g) & \text{(Chan – Galatius – Payne)} \end{cases}$

Examples:

(i) If G contains bubble

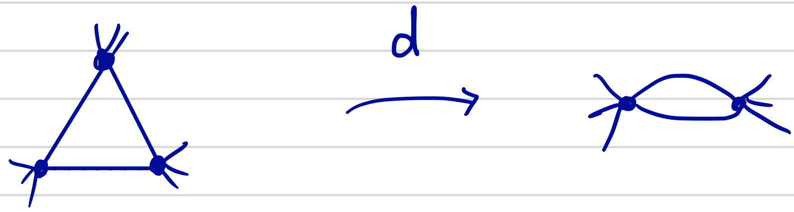


$\exists\ \sigma : G \xrightarrow{\sim} G$ switch e_1, e_2.

$$(G,\ e_1 \wedge e_2 \wedge \dots) \overset{\sigma}{=} (G,\ e_2 \wedge e_1 \wedge \dots) = -(G,\ e_1 \wedge e_2 \wedge \dots)$$

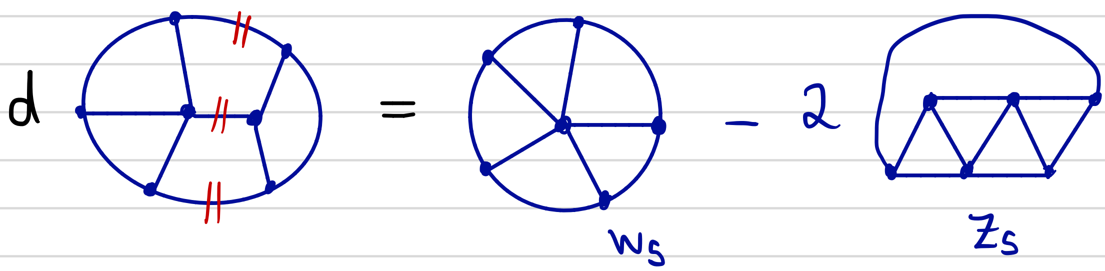
$\Longrightarrow$ (G, ω) is zero in graph complex.

(ii) If every edge of G contained in a triangle, then $d(G,\omega) = 0$



vanishes.

(iii)



Explains, (via Stokes) $\quad 0 = I_{W_5}(\omega^9) - 2\, I_{Z_5}(\omega^9)$

$\longrightarrow$ Relations between canonical integrals closely related to graph homology

Conclusion

- On moduli space $\mathcal{M}_g^{trop}$, there exist natural differential forms with poles

$$\omega \in \Omega^{can}$$

st volumes of every cell

$$I_G(\omega) = \int_{LC(G,0)} \omega \quad \text{are } \underline{\underline{\text{finite}}}.$$

- Have same shape & values, as Feynman integrals.
- Proving that a canonical integral $I_G(\omega)$ is non-zero can lead to new theorems in geometry.