The Elliptic Double Box and its Symbol

Based on the recent work 2106.14902 with Alexander Kristensson and Matthias Wilhelm

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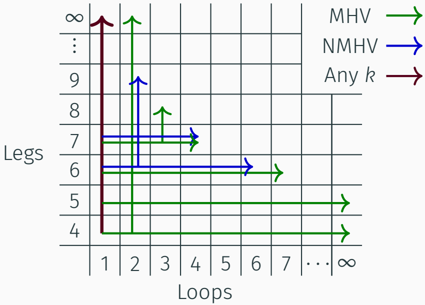
Introduction and Motivation

Amplitudes in planar $\mathcal{N} = 4$ sYM



[Bern, Caron-Huot, Del Duca, Dixon, Drummond, Duhr, Foster, Golden, Goncharov, Gürdoğan, He, Henn, von Hippel, Kosower, Li, McLeod, Papathanasiou, Pennington, Roiban, Smirnov, Spradlin, Vergu, Volovich, CZ, ...]

- High loop results at 6 and 7 point achieved by the bootstrap programme.
  [Caron-Huot, Dixon, Dulat, McLeod, von Hippel, Papathanasiou][Drummond, Foster, Gurdogan, Papathanasiou] . . .
- High multiplicity results at 2-loop MHV and NMHV achieved by the $\bar{Q}$-equation.
  [Caron-Huot][Caron-Huot, He][He, Li, CZ]

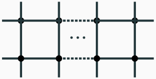
Remarkably, all these results are Multiple Polylogarithms (MPLs)

Symbol made MPLs simple:

- Simplify MPLs: 17 pages becomes a few lines for 2-loop hexagon remainder function! [Del Duca, Duhr, Smirnov][Goncharov, Spradlin, Vergu, Volovich]
- Manifest the singularity structures *e.g.* first entry conditions and Steinmann relations $\Rightarrow$ High loop hexagon and heptagon bootstrap programme!
- Relation with Cluster algebras and Grassmannian. [Golden, Goncharov, Spradlin, Vergu, Volovich] [Drummond, Foster, Gurdogan, Kalousios] [Henke, Papathanasiou] [Arkani-Hamed, Lam, Spradlin] [also see Volovich's talk]. . .
- Bootstrap general Feynman integrals and form factors. [Caron-Huot, Dixon, von Hippel, McLeod, Papathanasiou][Henn, Herrmann, Parra-Martinez][Dixon, McLeod, Wilhelm][He, Li, Yang]. . .

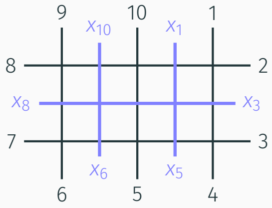
General Amplitudes are much richer than MPLs (even in planar $\mathcal{N} = 4$ sYM):

- Calabi-Yau manifolds starting at two-loops [Bloch, Kerr, Vanhove][Bourjaily, He, McLeod, von Hippel, Wilhelm] [also see Klemm's talk] . . .



In this talk, I will focus on the simplest Feynman integral in planar $\mathcal{N} = 4$ sYM beyond MPLs: the 2-loop 10-pt double box (Calabi-Yau 1-manifold: elliptic curve).

Elliptic double box at 10 point



Kinematics:

$$u_1 = \frac{x_{1,3}^2 x_{5,8}^2}{x_{1,5}^2 x_{3,8}^2}\,, \quad u_2 = \frac{x_{3,6}^2 x_{8,10}^2}{x_{3,8}^2 x_{6,10}^2}\,, \quad v_1 = \frac{x_{1,8}^2 x_{5,8}^2}{x_{3,5}^2 x_{3,8}^2}\,, \quad v_2 = \frac{x_{6,8}^2 x_{3,10}^2}{x_{3,8}^2 x_{6,10}^2}\,,$$

$$u_3 = \frac{x_{1,3}^2 x_{5,10}^2}{x_{1,5}^2 x_{3,10}^2}\,, \quad u_4 = \frac{x_{1,6}^2 x_{3,5}^2}{x_{1,5}^2 x_{3,6}^2}\,, \quad u_5 = \frac{x_{1,5}^2 x_{6,10}^2}{x_{1,6}^2 x_{5,10}^2}$$

The planar variable: $x_{i,j}^2 = (p_i + \cdots + p_{j-1})^2$

$$I_{\text{db}}^{\text{ell}} = \int \frac{d^4x_a d^4x_b \quad x_{1,5}^2 x_{6,10}^2 x_{3,8}^2}{x_{1,a}^2 x_{3,a}^2 x_{5,a}^2 x_{a,b}^2 x_{6,b}^2 x_{8,b}^2 x_{10,b}^2}$$

- The only contribution to one component of the 2-loop 10-pt N^{3}MHV amplitude. [Caron-Huot, Larsen]
- The first two-loop Feynman integral which is expected to be some elliptic function in planar $\mathcal{N} = 4$ sYM. [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich][Caron-Huot, Larsen][Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm][Vergu, Volk]. . .

The emergence of the elliptic curve

Elliptic curve: $y^2 = P(x)$ with $\deg P(x) = 3$ or 4.

1. Using the differential equation related to the one-loop hexagon in 6D [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich]:

   $\partial_{u_5}\Big($  $\Big) = \dfrac{1}{\sqrt{\Delta_6}}\Big($  $\Big)$

   where the (normalized) Gram determinant Δ_6 is a cubic polynomial in u_5.

2. Directly integrating the Feynman parameter rep [Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm]

    $= \displaystyle\int_0^\infty \frac{d\alpha}{\sqrt{Q(\alpha)}} H_3(\alpha)$

   $Q(\alpha)$ is a quartic polynomial in α.

The obstacle to perform the last integration: the extra square roots in the hexagon integral and $H_3(\alpha)$.

Motivation and Outline

The problems that motivate our work:

1. The elliptic double box is expected to be the iterated integral on an elliptic curve (or torus) — elliptic polylogarithms, *but How*?
2. What is the "Symbology" for the elliptic double box?
    - Symbol letters?
    - Analogue of $\log(ab) = \log(a) + \log(b)$?
    - First entry conditions and Steinmann relations?

In the rest of the talk:

1. Review the framework of elliptic polylogarithms.
2. Solve Problem 1 by a variable substitution in Feynman parameter rep.
3. Partial answer to the Problem 2 by studying the symbol.

Elliptic Polylogarithms

Polylogarithms of weight n are n-fold iterated integrals with logarithmic singularities on the Riemann sphere: [Poincaré 1884, Lappo-Danilevsky 1927][Chen][Goncharov]

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dx_1}{x_1 - a_1} G(a_2, \ldots, a_n; x_1) \quad a_i \in \mathbb{C} \qquad \text{with } G(;x) := 1.$$

The differential of $G(a_1, a_2, \ldots, a_n; a_{n+1})$

$$dG(a_1, a_2, \ldots, a_n; a_{n+1}) = \sum_{i=1}^{n} G(a_1, \ldots, \hat{a}_i, \ldots, a_n; a_{n+1}) d\log \frac{a_{i-1} - a_i}{a_{i+1} - a_i}, \qquad \text{with } a_0 = 0.$$

The singularity structures of MPLs can be characterized by the symbol:
[Goncharov][Goncharov, Spradlin, Vergu, Volovich]

- Derivatives only act on the last entries:

$$dF_n = \sum F_{i,n-1} \times d\log R_i \quad \Rightarrow \quad \mathcal{S}(F_n) = \sum \mathcal{S}(F_{i,n-1}) \otimes \log R_i$$

- Symbol letters R_i are algebraic functions.
- The first entries indicate the loci of branch cuts.
- Easy to manipulate: $\cdots \otimes \log(ab) \otimes \cdots = \cdots \otimes \log(a) \otimes \cdots + \cdots \otimes \log(b) \otimes \cdots$

Elliptic Generalizations I

eMPLs are iterated integrals with logarithmic singularities on elliptic curves $y^2 = P_4(x)$:
[Broedel, Duhr, Dulat, Tancredi]

$$\mathrm{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right) = \int_0^x dx' \, \psi_{n_1}(c_1, x') \mathrm{E}_4\left(\begin{smallmatrix} n_2 & \dots & n_k \\ c_2 & \dots & c_k \end{smallmatrix}; x'\right) \qquad \text{with } \mathrm{E}_4(;x) = 1,$$

where

$$\psi_1(c,x) = \frac{1}{x-c},$$
$$\psi_0(0,x) = \frac{1}{y},$$
$$\psi_{-1}(c,x) = \frac{y_c}{y(x-c)}, \qquad \psi_{-1}(\infty,x) = \frac{x}{y},$$

and other $\psi_n(c,x)$ to make a complete basis for the general kernel $R_1(x) + \frac{R_2(x)}{y}$

- Advantage: Directly related to Feynman integrals.
- Disadvantage: Difficulty to compute the differential and not pure.

Side remark: Similarly, there are E_3's defined on $y^2 = P_3(x)$. [Broedel, Duhr, Dulat, Tancredi] eMPLs are iterated integrals with logarithmic singularities on torus $\mathbb{C}/\Lambda$:

[Brown, Levin] [Broedel, Mafra, Matthes, Schlotterer] [Broedel, Duhr, Dulat, Tancredi] [also see Weinzierl's talk]

$$\widetilde{\Gamma}\left(\begin{smallmatrix} n_1 & \dots & n_k \\ w_1 & \dots & w_k \end{smallmatrix}; w\right) = \int_0^w dw' \, g^{(n_1)}(w' - w_1) \widetilde{\Gamma}\left(\begin{smallmatrix} n_2 & \dots & n_k \\ w_2 & \dots & w_k \end{smallmatrix}; w'\right) \qquad \text{with} \quad \widetilde{\Gamma}(;w) = 1.$$

The integration kernels $g^{(n)}(z)$ are generated by the *Eisenstein-Kronecker series*

$$\frac{\partial_z \theta_1(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)} = \sum_{n\geq 0} \alpha^{n-1} g^{(n)}(z) \, .$$

θ_1: odd Jacobi θ-function. $g^{(0)} = 1$, $g^{(1)}(z) = \partial_z \log \theta_1, \dots$

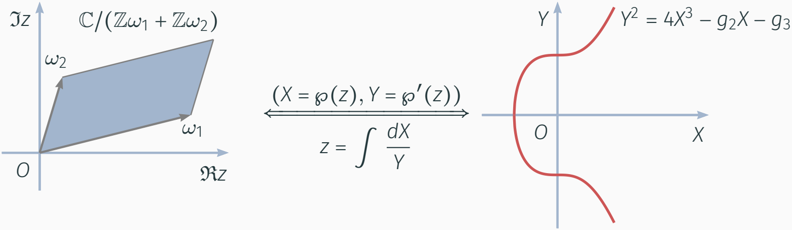
- Manifestly pure.
- Length k $\neq$ Weight $n = \sum n_i$
- Easy to compute the differential [Broedel, Duhr, Dulat, Penante, Tancredi]

$$d\widetilde{\Gamma}_k^{(n)} = \sum_i \widetilde{\Gamma}_{k-1}^{(n-j_i)} \omega^{(j_i)}(z_i) \quad \Rightarrow \quad \mathcal{S}\left(\widetilde{\Gamma}_k^{(n)}\right) = (2\pi i)^{j_i - 1} \sum_i \mathcal{S}\left(\widetilde{\Gamma}_{k-1}^{(n-j_i)}\right) \otimes \Omega^{(j_i)}$$

  where $\omega^{(j\geq -1)}(z) = (2\pi i)^{j-1} d\Omega^{(j)}(z) = g^{(j)}(z,\tau)dz + n(2\pi i)^{-1} g^{(j+1)}(z,\tau)d\tau$

- Elliptic Symbol letter: $\Omega^{(-1)} = -2\pi i \tau$, $\Omega^{(0)}(z) = 2\pi i z$, $\Omega^{(1)}(z) = \log(\theta_1(z,\tau)/\eta(\tau)), \dots$
  $\eta(\tau)$: Dedekind eta function.

Tori ↔ Elliptic curves



The Weierstrass map and Abel's map for an elliptic curve $y^2 = x^4 + \sum_{i=0}^{3} a_i x^i$:

Torus → Elliptic Curve $\quad x = \kappa(z) = \dfrac{6a_1 - a_2a_3 + 12a_3\wp(z) - 24\wp'(z)}{3a_3^2 - 8a_2 - 48\wp(z)}, \quad y = \kappa'(z)$

Elliptic Curve → Torus $\quad z = \displaystyle\int \frac{dx}{y}$

Remark:

- Each point c in kinematic space → $(c, \pm y_c) \leftrightarrow z_c^{\pm}$.

Transformation of kernels

Relations between integration kernels for E_4 functions and those for $\widetilde{\Gamma}$ functions:

$$
\begin{aligned}
\psi_0 dx &= \omega_1 dw\,, \\
\psi_1(c,x)dx &= \Big(g^{(1)}(w-w_c^+) + g^{(1)}(w-w_c^-) - g^{(1)}(w-w_\infty^+) - g^{(1)}(w-w_\infty^-)\Big)dw\,, \\
\psi_{-1}(c,x)dx &= \Big(g^{(1)}(w-w_c^+) - g^{(1)}(w-w_c^-) + g^{(1)}(w_c^+) - g^{(1)}(w_c^-)\Big)dw\,, \\
\psi_{-1}(\infty,x)dx &= \Big(g^{(1)}(w-w_\infty^-) - g^{(1)}(w) + g^{(1)}(w_\infty^-) - \omega_1 a_3/4\Big)dw\,.
\end{aligned}
$$

$w_c^\pm := z_c^\pm/\omega_1$ are the images of c on the normalized torus $[1:\tau=\omega_2/\omega_1]$

General strategy for computing Feynman Integrals as eMPLs:

1. Compute the Feynman integrals as E_4 functions.
2. Use the above rules to rewrite E_4 functions as $\widetilde{\Gamma}$ functions

Side remark: Another way to package eMPLs: $\mathcal{E}_4\left(\begin{smallmatrix} n_1 & \dots & n_k \\ c_1 & \dots & c_k \end{smallmatrix}; x\right)$ [Broedel, Duhr, Dulat, Penante, Tancredi] which are also manifest pure.

Direct integration to eMPLs

The resolution of linear reducibility problem in the double box

1. Starting Point: The dual conformally invariant Feynman parameter rep. for the double box [Bourjaily, McLeod, Spradlin, von Hippel, Wilhelm]:

$$= \int_0^\infty \frac{d^4\beta}{f_1 f_2 f_3} \qquad \left\{ \begin{array}{l} f_1 = \beta_4(1+\beta_1)+\beta_1\,, \quad f_2 = 1+u_1\beta_4+v_1\beta_1+u_2\beta_2+v_2\beta_2, \\ f_3 = (1+u_3\beta_4)\beta_2+(1+u_4\beta_1)\beta_3+\beta_2\beta_3+u_3u_4u_5f_1. \end{array} \right\}$$

2. Integrating out β_3 and β_4:

$$\int_0^\infty \frac{d\beta_1 d\beta_2}{\mathcal{P}(\beta_1,\beta_2)} G_2(\beta_1,\beta_2)\,,$$

   - the polynomial $\mathcal{P}$ is cubic in β_1 and quadratic in β_2: gives the elliptic curve.
   - 3 letters of $\mathcal{S}\big(G_2(\beta_1,\beta_2)\big)$ are quadratic in β_1 and β_2: gives extra square roots.

3. These extra square roots can be avoided by the following variable substitution:

$$x = \beta_1 v_1 + \beta_2 u_2\,, \qquad \widetilde{\beta}_2 = u_2\beta_2/v_1.$$

   All letters of $\mathcal{S}\big(G_2(x,\widetilde{\beta}_2)\big)$ are linear in $\widetilde{\beta}_2$

4. Now the integration over $\widetilde{\beta}_2$ gives

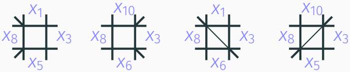
$$= \int_0^\infty \frac{dx}{y}\, G_3(x,y)\,,$$

where $y^2 = x^4 + \sum_{i=0}^{3} a_i x^i$ defines the elliptic curve and G_3 is a MPL of weight 3, whose letters are rational in x and y.

5. No obstacle to compute in terms of E_4 functions,

$$= \mathrm{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & -u_2 & -u_2 \end{smallmatrix}; \infty\right) + \cdots$$

- only consist of $\psi_1(c) = \frac{1}{x-c}$, $\psi_0 = \frac{1}{y}$, $\psi_{-1}(c) = \frac{y_c}{y(x-c)}$, $\psi_{-1}(\infty) = \frac{x}{y}$.
- 26 choices for c, containing 4 square roots from four-mass box sub-diagrams:



6. . . . and in terms of $\widetilde{\Gamma}$ functions (Recall $\int dx/y = \omega_1 \int dw$)

$= \omega_1 \times \Big($A pure combination of $\widetilde{\Gamma}$'s of length 4 and weight 3$\Big)$.

. . . and Symbology

Simplifying the symbol

Initially, $\mathcal{S}\big($$/\omega_1\big) \sim 10^6$ terms, alphabet $\sim 10^3$ letters consisting of $\Omega^{(0,1,2,3)}$'s.
Recall that $\Omega^{(0)}(w) = 2\pi i w$, $\Omega^{(1)}(w) = \log(\theta_1(z)/\eta(\tau)), \dots$

The analogue of $\log(ab) = \log(a) + \log(b)$ for the elliptic letters:

1. Addition theorem $\wp(u+v) = \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 - \wp(u) - \wp(v)$ gives

   30 linear relations among $\Omega^{(0)}(w_c)$.

2. $\int_a^b \psi_1(c,x)dx = \log\frac{c-a}{c-b}$ gives

$$\log\frac{c-a}{c-b} = \sum_{\sigma\in\pm} \Omega^{(1)}(w_\infty^\sigma - w_b^+) - \Omega^{(1)}(w_\infty^\sigma - w_a^+) + \Omega^{(1)}(w_c^\sigma - w_a^+) - \Omega^{(1)}(w_c^\sigma - w_b^+)\,.$$

3. 28 relations found numerically by PSLQ algorithm. For example,

$$\sum_{i=1}^{6}(-1)^{i+1}\Big(\Omega^{(1)}(w_{d_i}^- - w_\infty^+) - \Omega^{(1)}(w_{d_i}^- - w_0^+)\Big) \equiv \log\frac{d_2}{d_3 d_5} + \Omega^{(0)}(w_\infty^+ - w_0^+)$$

   with $d_i \in \{\infty, -v_1/u_4,\ \bar{z}_{1,3,5,8}-1,\ z_{1,3,5,8}-1,\ -z_{3,6,8,10},\ -\bar{z}_{3,6,8,10}\}$,

Notations: $z_{a,b,c,d}\bar{z}_{a,b,c,d} = \frac{x_{a,b}^2 x_{c,d}^2}{x_{a,c}^2 x_{b,d}^2}$, $(1-z_{a,b,c,d})(1-\bar{z}_{a,b,c,d}) = \frac{x_{a,d}^2 x_{b,c}^2}{x_{a,c}^2 x_{b,d}^2}$

The Symbol of the double box

A dramatic simplification happens, all $\Omega^{(3)}$'s drop out:

$\mathcal{S}\Big($  $/\omega_1\Big) = \mathcal{S}\Big($  $\Big) \otimes \left(w^+_{c_{\text{hex}}} - \frac{w^+_\infty}{2}\right) + \mathcal{S}(F_-) \otimes \left(w^-_\infty - \frac{w^+_\infty}{2}\right) + \mathcal{S}(F_+) \otimes w^+_\infty$

$$+ \left[\mathcal{S}(F_{d_4}) \otimes \left(w^+_{d_4} - \frac{w^+_\infty}{2}\right) + \text{reflections}\right],$$

where $c_{\text{hex}} = \frac{\langle 9,10,1(7,8)\cap(2,3,5)\rangle}{\langle 1,5,9,10\rangle\langle 2,3,7,8\rangle}$ and $d_4 = z_{1,3,5,8} - 1$.

- Last entries are elliptic integrals: $w^+_c = \frac{1}{\omega_1}\int_{-\infty}^{c} dx/y$.
- Manifestly satisfy the first entry conditions: first entries are $\log u_i$ or $\log v_i$
- Particular patterns for the first-two entries: $\text{Li}_2(1-u)$, $\log u \log v$ and four-mass boxes
  $\Rightarrow$ Manifest Steinmann relations
- Apart from the last entries, the elliptic letters only occur in the third entries of $\mathcal{S}(F_+)$.
- Manifest the reflection symmetries:  and 
- Manifest the differential equation [Paulos, Spradlin, Volovich][Nandan, Paulos, Spradlin, Volovich]:

$\omega_1 \partial_{u_5}\left(w^+_{c_{\text{hex}}} - \frac{w^+_\infty}{2}\right) = \frac{1}{\sqrt{\Delta_6}} \quad \Rightarrow \quad \partial_{u_5}\Big($  $\Big) = \frac{1}{\sqrt{\Delta_6}}\Big($  $\Big)$

Symbol Alphabet

Symbol Alphabet for the elliptic double box:

$$36 \text{ rational letters} + 24 \text{ algebraic letters} + (13+7) \text{ elliptic letters.}$$

Which are new comparing with the alphabets for the known 2-loop amplitudes [Caron-Huot] [He, Li, CZ]:

- Rational letter: None.
- Algebraic letter:

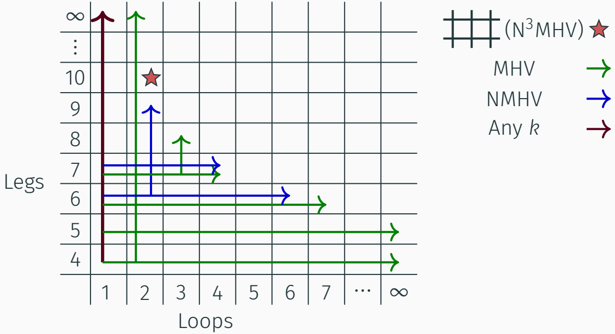
$$\frac{\bar{z}_{1,3,5,8} - z_{1,3,6,8}}{\bar{z}_{1,3,5,8} - \bar{z}_{1,3,6,8}}, \quad \frac{z_{1,3,5,8} - z_{1,3,6,8}}{z_{1,3,5,8} - \bar{z}_{1,3,6,8}} \quad \text{and reflection images.}$$

- 7 simple elliptic integrals as last entries.
  13 complicated linear independent combinations of $\Omega^{(0,1,2)}$.

Further understanding is still needed for these 13 elliptic letters.

Conclusion and Outlook

Conclusion



We are starting to understand the elliptic aspect of 2-loop integrated amplitudes in planar $\mathcal{N} = 4$ sYM by studying the special component — the elliptic double box:

- The double box is indeed an eMPL.
- First entry conditions and Steinmann relations are the same as the cases of MPLs.
- The elliptic letters only appear at the 3^{rd} and 4^{th} entries and last entries are elliptic integrals.

Outlook

1. Further understanding of the symbol for the double box.
   - A lot of unproven identities in the simplification of $\mathcal{S}($$/\omega_1)$.
   - Understanding the 13 elliptic letters and the elliptic analogue of $\frac{a+\sqrt{\Delta}}{a-\sqrt{\Delta}}$?
   - Simplified symbol → Simplified function?
2. Bootstrapping the elliptic double box via its differential equation:
   - $\mathcal{S}($$) = \mathcal{S}($$) \otimes \int_{u_5}^{\infty} \frac{du_5'}{\sqrt{\Delta_6(u_5')}} +$ integrability.

   Symbols of more complicated integrals via differential equations?

   $\partial_u \partial_v($$) = \frac{1}{\sqrt{\Delta_8}}($$)$ [Nandan, Paulos, Spradlin, Volovich]
3. Elliptic letters via cluster algebras?
4. Compute/Bootstrap general 2-loop amplitudes in planar $\mathcal{N} = 4$ sYM.

Thank You

The solution for the linear reducibility of the double box

3 quadratic letters in $\mathcal{S}\big(G_2(\beta_1, \beta_2)\big)$:

$$q_1 = \beta_1(\beta_2 u_2 + \beta_1 v_1) + \text{linear terms}\,,$$
$$q_2 = -u_3(\beta_2 + \beta_1 u_4 u_5)(\beta_2 u_2 + \beta_1 v_1) + \text{linear terms}\,,$$
$$q_3 = (\beta_2 + \beta_1 u_4 u_5)(\beta_2 u_2 + \beta_1 v_1) + \text{linear terms}\,,$$

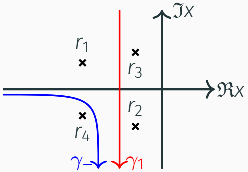
A natural variable substitution:

$$x = \beta_1 v_1 + \beta_2 u_2, \qquad \widetilde{\beta}_2 = u_2 \beta_2 / v_1.$$

Then q_1, q_2, q_3 are linear in $\widetilde{\beta}_2$.

Backup: Abel's map

Suppose the roots of $y^2(x)$ come in complex conjugate pairs:



- The image z_c^+ of $c \in \mathbb{R}$ on the torus is given by Abel's map: $z_c^+ = \int_{-\infty}^{c} \frac{dx}{y}$
- Two periods of the torus are $\omega_1 = \int_{\gamma_1} \frac{dx}{y}$ and $\omega_2 = \int_{\gamma_2=[-\infty,\infty]} \frac{dx}{y} = z_\infty^+$.
- The other image of ∞ on the torus is $z_\infty^- = \int_{\gamma_-} \frac{dx}{y}$.

The shuffle regularization for MPLs:

$$G(0;x) = \int_0^x dt/t := \log(x)\,.$$

The eMPL $\widetilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};w)$ needs to be regularized since $g^{(1)}(z) \sim \pi\cot(\pi z)$.

Denote the primitive of $g^{(1)}(z)$ by $\Omega^{(1)}(z)$, then

$$\Omega^{(1)}(\epsilon) = \log(1-\mathrm{e}^{2\pi i\epsilon}) + 2\log\eta(\tau) + O(\epsilon)$$

The usual shuffle regularization takes [Broedel, Mafra, Matthes, Schlotterer][Broedel, Duhr, Dulat, Penante, Tancredi]

$$\Omega^{(1)}(0) := 2\log\eta(\tau)$$

To be consistent with the shuffle regularization for MPLs, we take

$$\widetilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};w) = \Omega(w) - 2\log\eta(\tau) - \log\frac{2\pi i}{\omega_1 y_0}$$