Karlsruher Institut für Technologie

Precision calculations for collider physics: the subtraction contribution

Chiara Signorile-Signorile

Amplitudes 2021

Based on: *Magnea, Maina, Pelliccioli, C.S., Torrielli, Uccirati, JHEP12(2018)107, JHEP02(2021)037*

Buccioni, Caola, Chawdhry, Devoto, Melnikov, Röntsch, C.S., In preparation

Goal of this talk:

Why the *amplitudes community* should care about infrared subtraction?

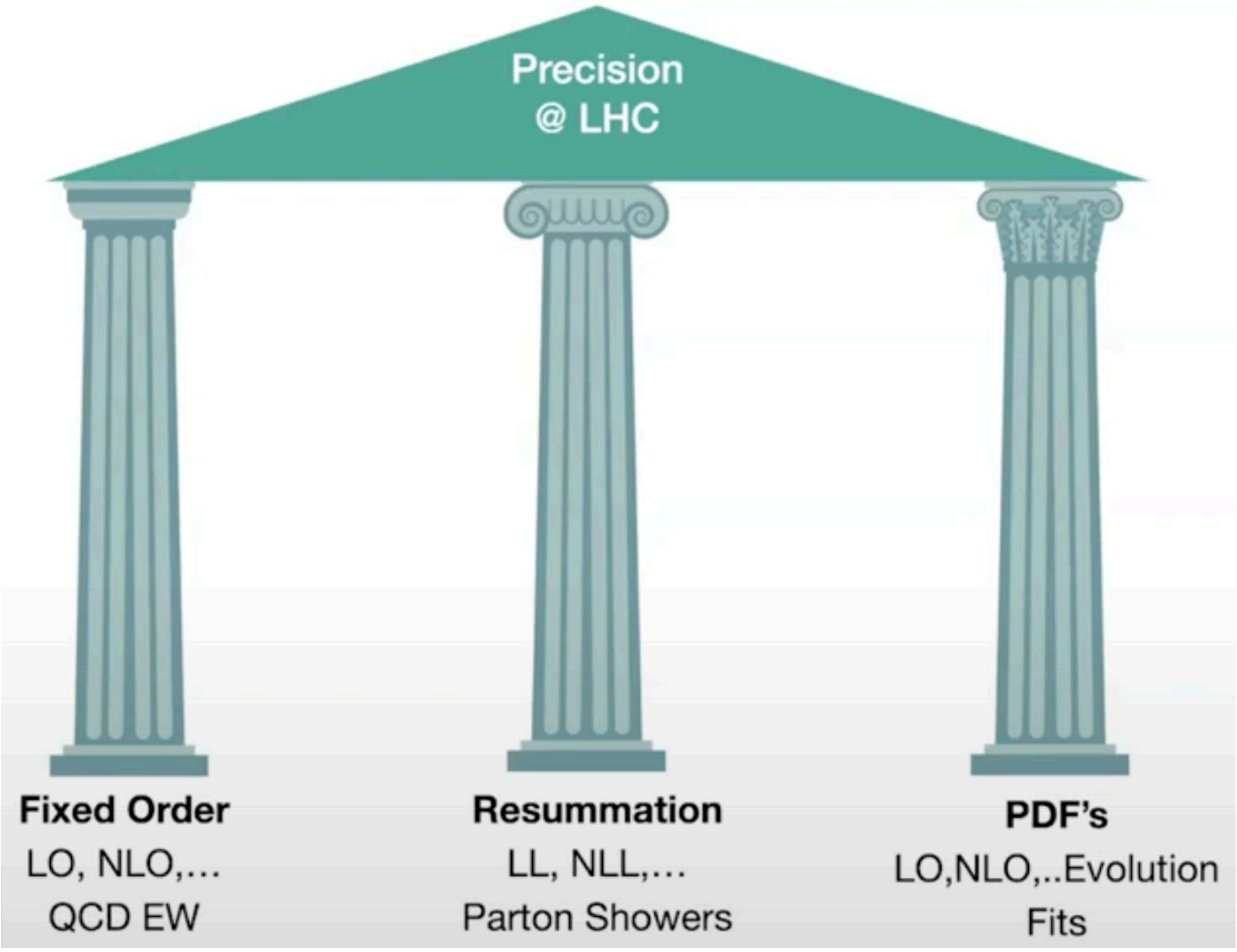
Motivations

Hard collisions at the LHC are described in terms of quark and gluon cross sections

→ Collinear factorisation theorem *[Collins, Soper, Sterman 0409313]*

$$d\sigma = \int dx_1\, dx_2\, f_i(x_1) f_j(x_2)\, d\sigma_{ij}\, \mathcal{F}\left(1 + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{Q}\right)^n\right), \quad n \geq 1$$

Fixed-order predictions represent one of the pillars for precision physics at LHC.



[Maltoni, GGI Tea Breaks '21]

$$d\sigma_{ij} = d\sigma_{ij,\rm LO}\left(1 + \alpha_s\, \Delta^{QCD}_{ij,\rm NLO} + \alpha_{ew}\, \Delta^{EW}_{ij,\rm NLO} + \alpha_s^2\, \Delta^{QCD}_{ij,\rm NNLO} + \alpha_s\, \alpha_{ew}\, \Delta^{QCD\otimes EW}_{ij,\rm NNLO} + \ldots\right)$$

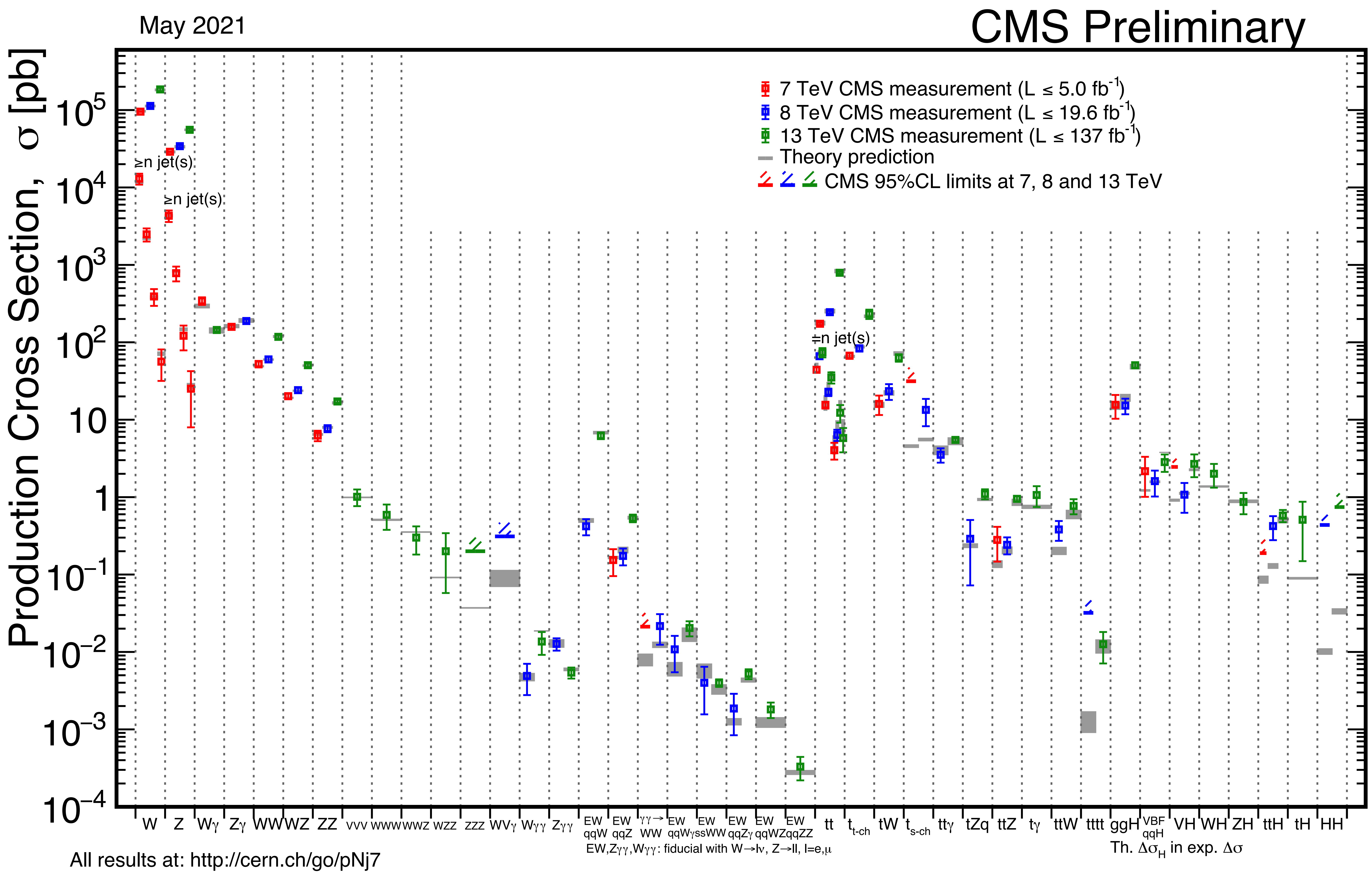
NNLO QCD: rapidly becoming the required accuracy standard.

Mixed QCD-EW corrections becoming important to match experimental precision.
[Dittmaier, Huss, Schwinn '14, Delto, Jaquier, Melnikov, Röntsch '20, Dittmaier, Schmidt, Schwarz '20, Behring, Buccioni, Caola, Delto, Jaquier, Melnikov, Röntsch '20, Buonocore, Grazzini, Kallweit, Savoini, Tramontano '21]

LHC: many processes known at few percent level → few-percent precision measurements expected for several complex final states

Motivations

LHC continues to confirm the Standard Model

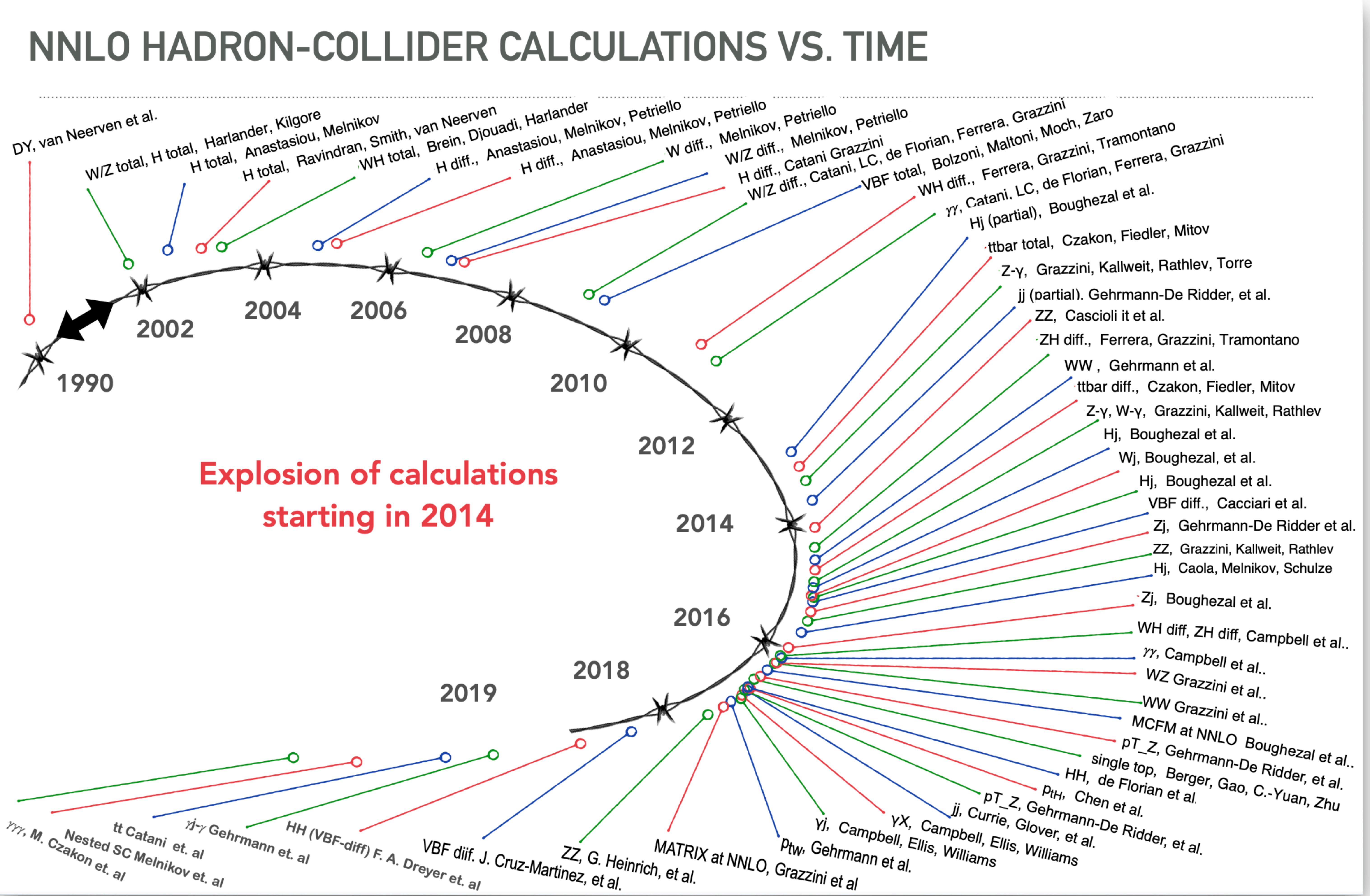


- Direct search for BSM:
  many proposal,
  no obvious candidate

- Indirect search for BSM:
  small corrections to SM

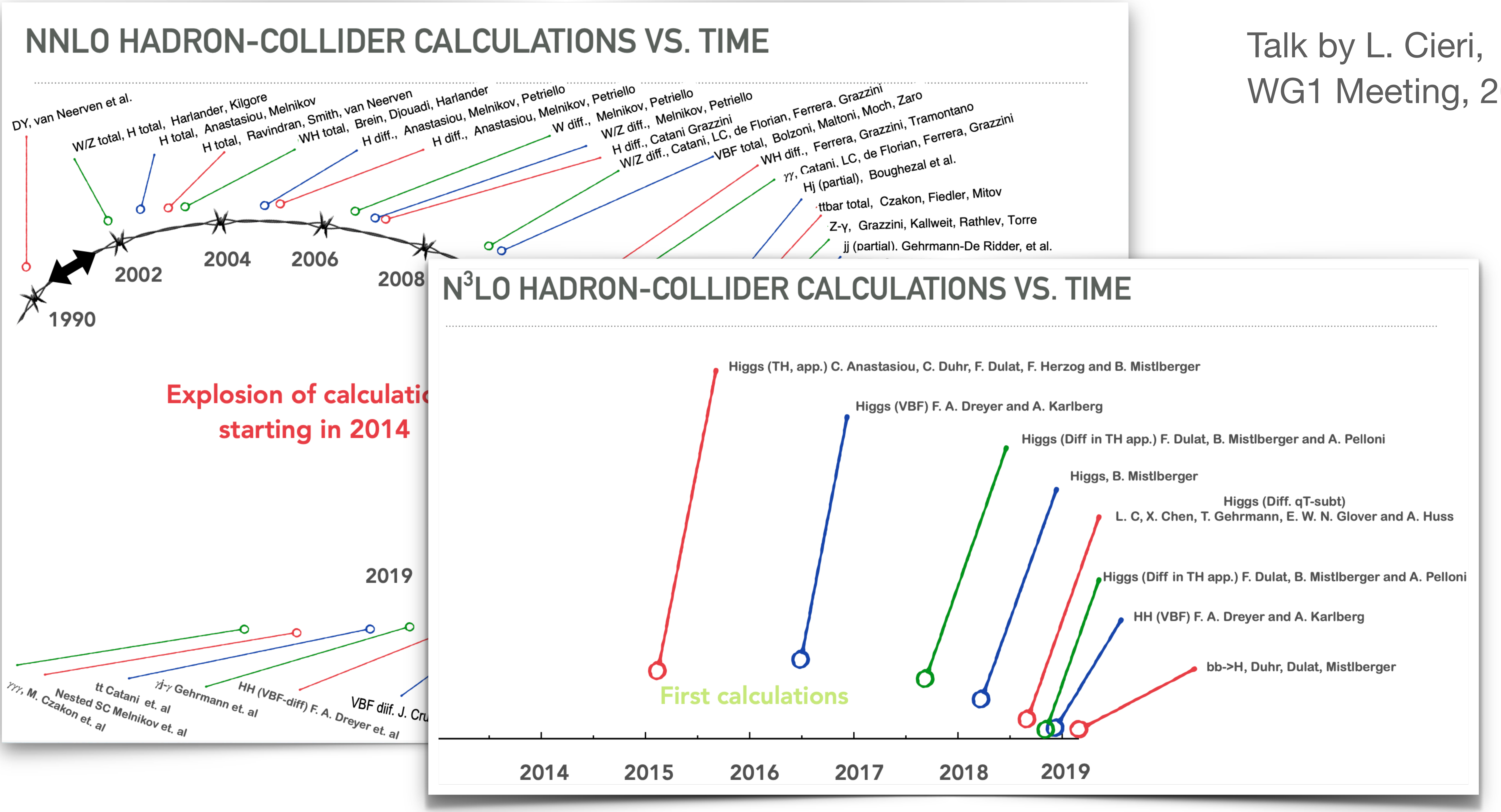
High precision theoretical predictions

Motivations



Talk by L. Cieri,
WG1 Meeting, 2019

Motivations



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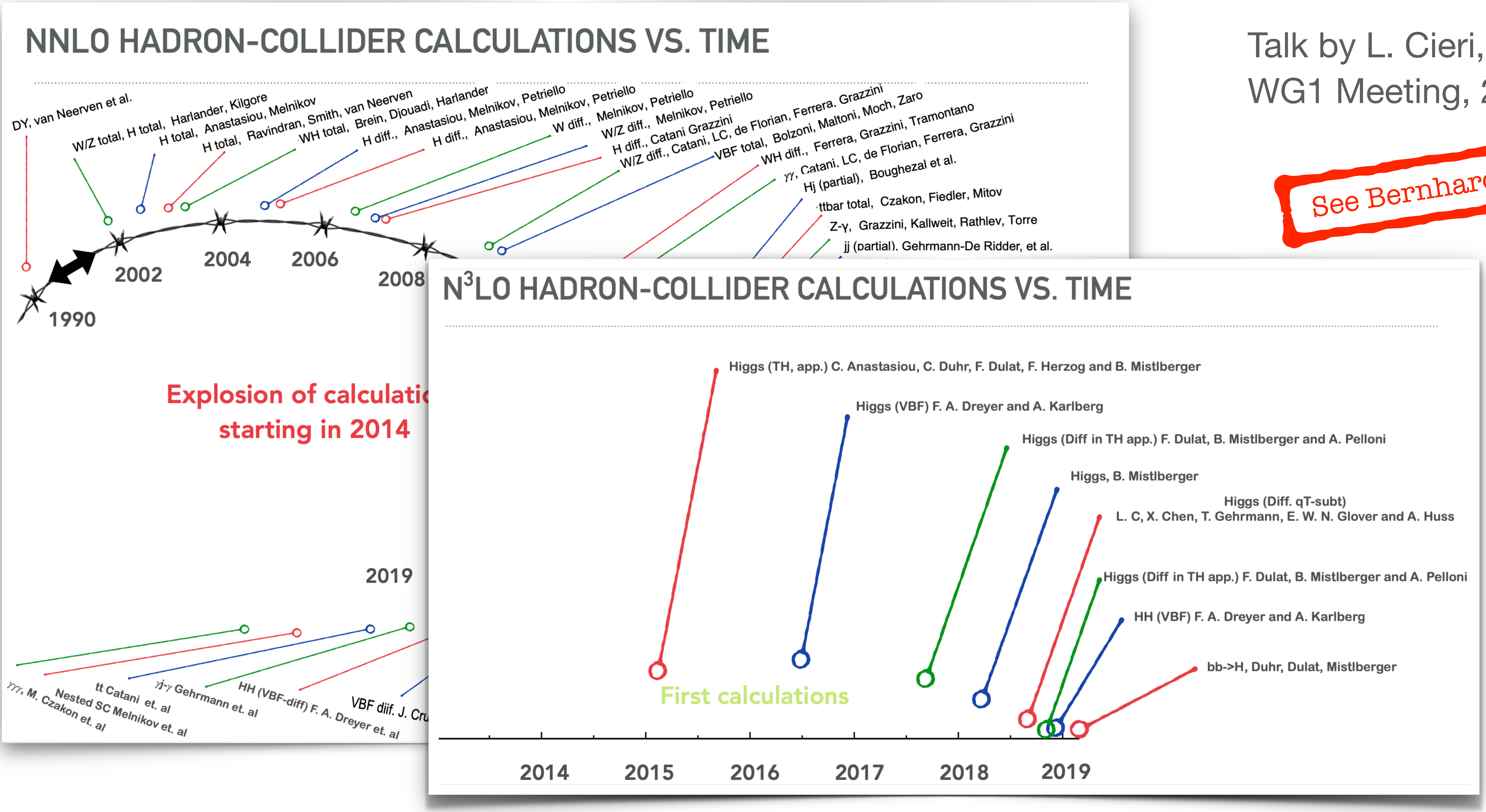

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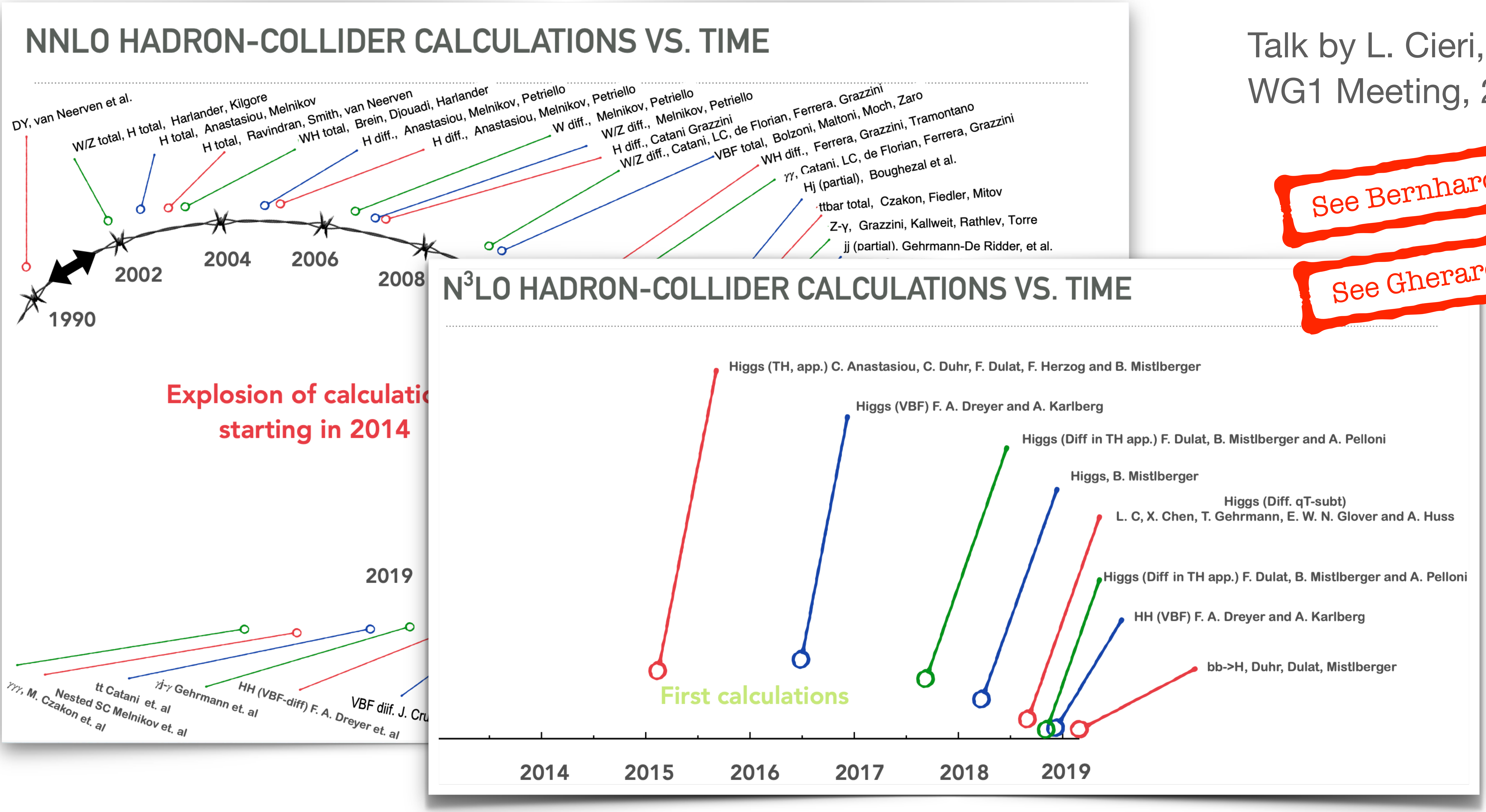
See Bernhard's talk

Motivations



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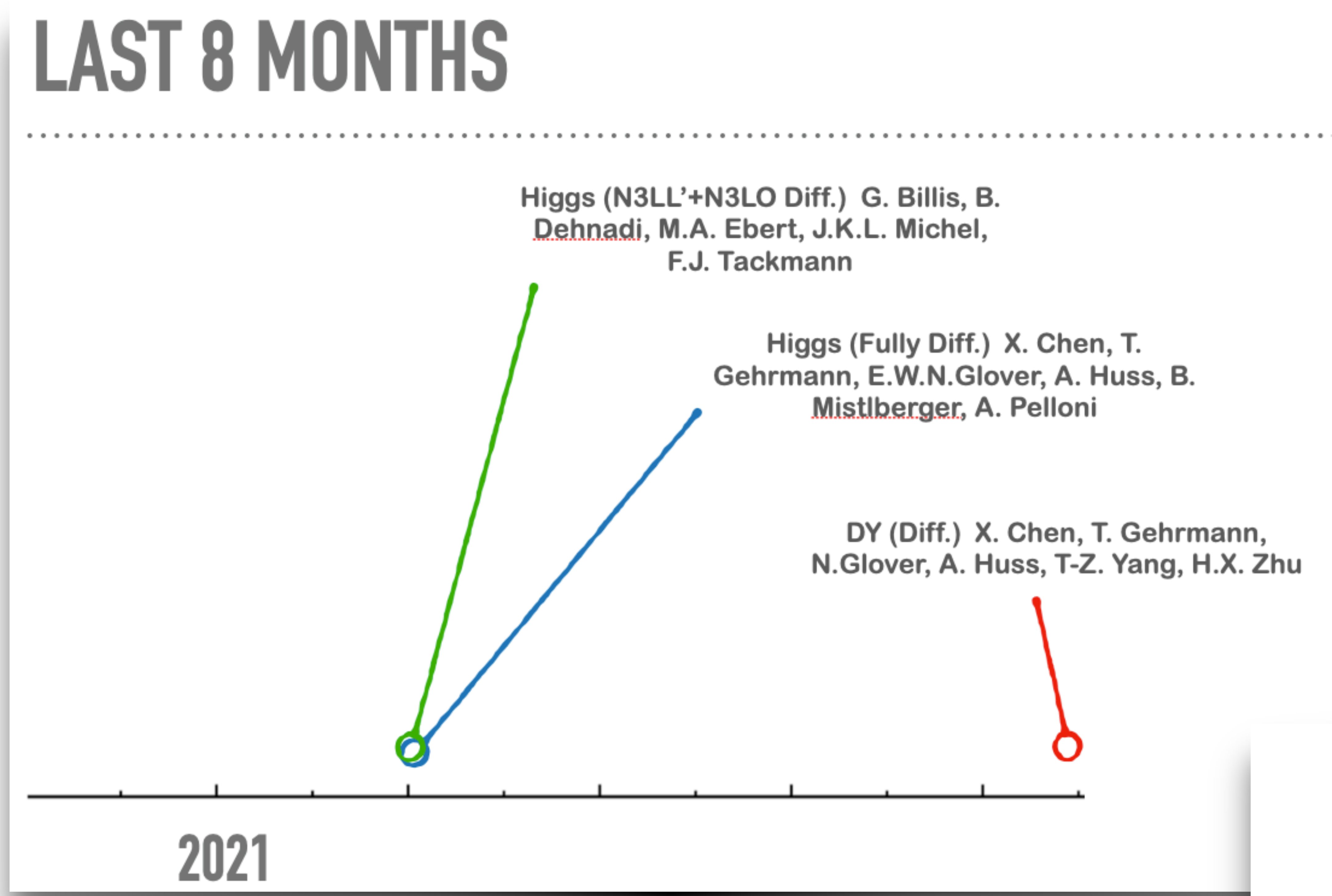

Towards DIS at N4LO

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August 3, 2021

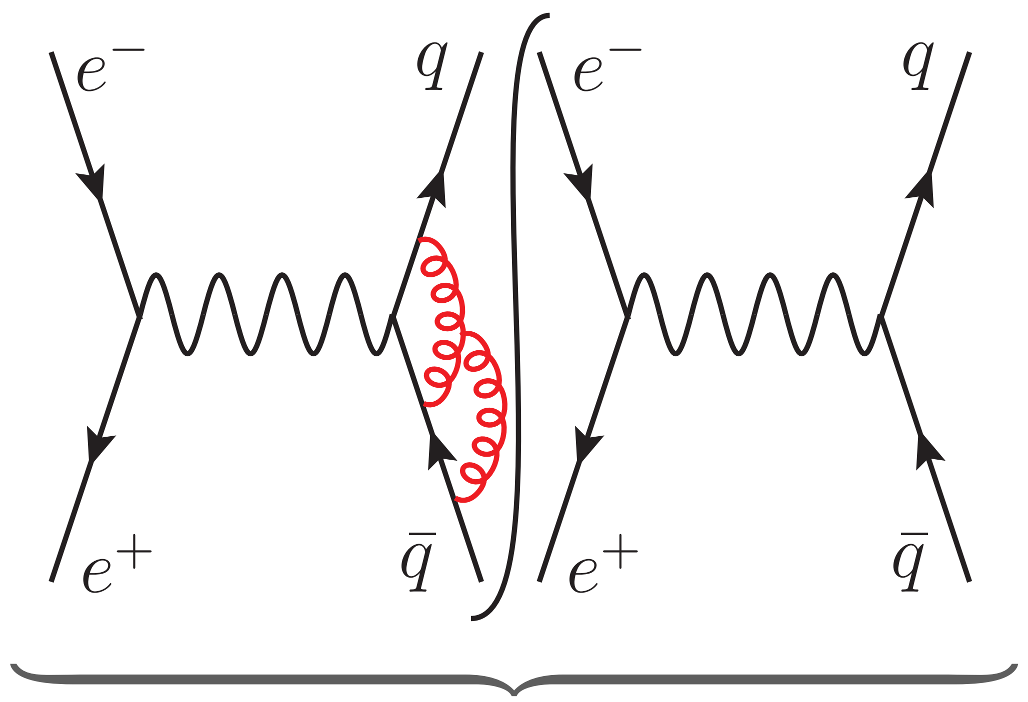
Proceedings for the XXVIII International Workshop on Deep-Inelastic Scattering and Related Subjects, Stony Brook University, New York, USA, 12-16 April 2021
doi:10.21468/SciPostPhysProc.?

NNLO generalities

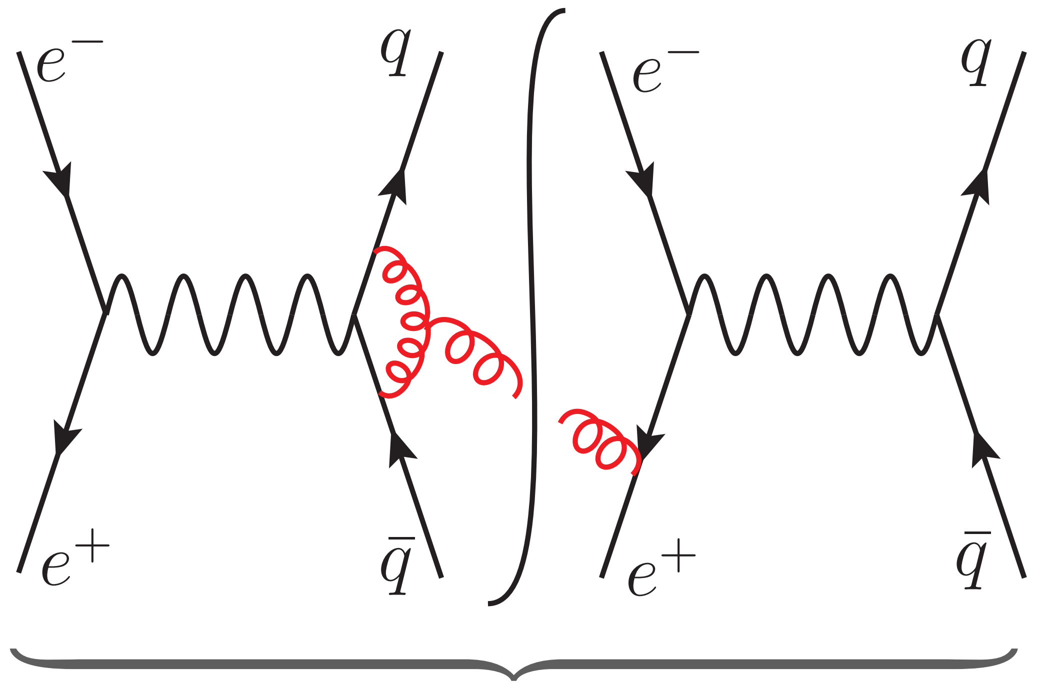
Ingredients for NNLO correction to $pp \to X$

- **two-loop** matrix element for $ff \to X$
- **one-loop** matrix element for $ff \to X + f'$
- **tree-level** matrix element for $ff \to X + f'f'$

$$\frac{d\sigma^{\mathrm{N^2LO}}}{dX} = \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) + \int d\Phi_n\, VV_n\, \delta_n(X)$$

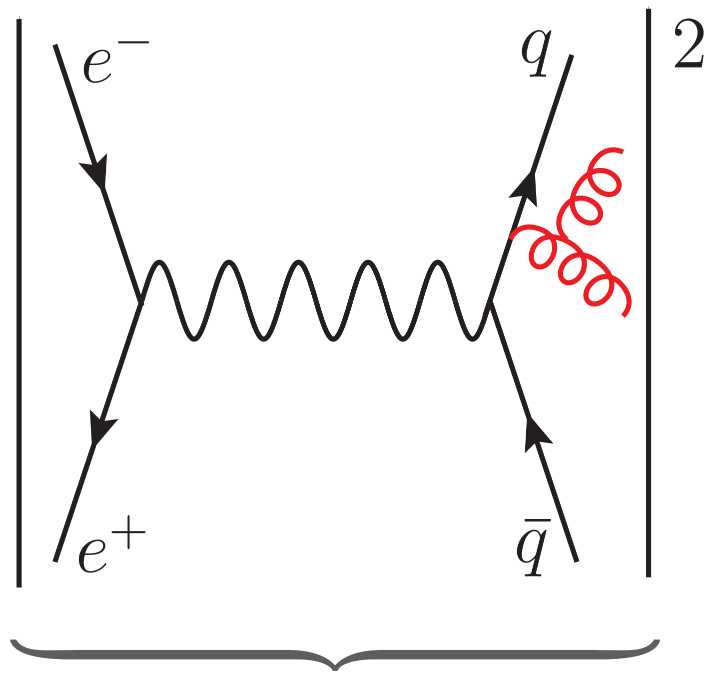


Explicit poles



Explicit poles from virtual corrections

Phase space singularities



Well defined in the non-degenerate kinematics

- Significant progress in calculations of **two-loop amplitudes** (both analytic and numerical methods)
- Almost all relevant amplitudes for $2 \to 2$ massless processes
- First results for $2 \to 3$ amplitudes

See Simon's talk

- **One-loop amplitudes in degenerate kinematics**
- OpenLoops, Recola

- **Real emission corrections finite in the bulk of the allowed PS**
- IR singularities arise upon integration over energies and angles of emitted partons

The problem

1. **Extract** infrared $1/\epsilon$ poles in d-dimension **without integrating over the resolved phase space**

   → **fully differential** predictions for IR-safe observables

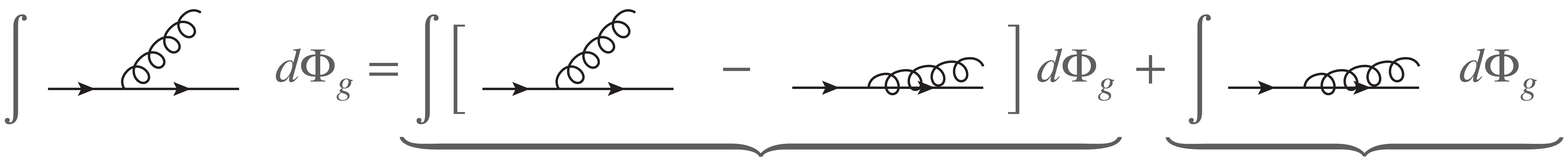
2. **Cancel** the $1/\epsilon$ poles stemming from the phase space integration against the poles of the **virtual contributions**

Fully general solution?

- **Phase space singularities** of the real radiation
- Explicit **poles from virtual contributions**

} Known **independently** of the hard subprocess

→ A general procedure seems to be practicable, although non-trivial to implement → **Slicing** / **Subtraction** ← This talk



Finite in d=4, integrable numerically

exposes the same $1/\epsilon$ poles as the virtual correction

Well established schemes at NLO

- **Catani-Seymour (CS)** *[9602277]*
- **Frixione-Kunst-Signer (FKS)** *[9512328]*
- Nagy-Soper *[0308127]*

Currently implemented in full generality in fast and efficient NLO generators
[Gleisberg, Krauss ’07, Frederix, Gehrmann, Greiner ’08, Hasegawa, Moch, Uwer ’09, Frederix, Frixione, Maltoni, Stelzer ‘09, Alioli, Nason, Oleari, Re ‘10, Reuter et al. ‘16

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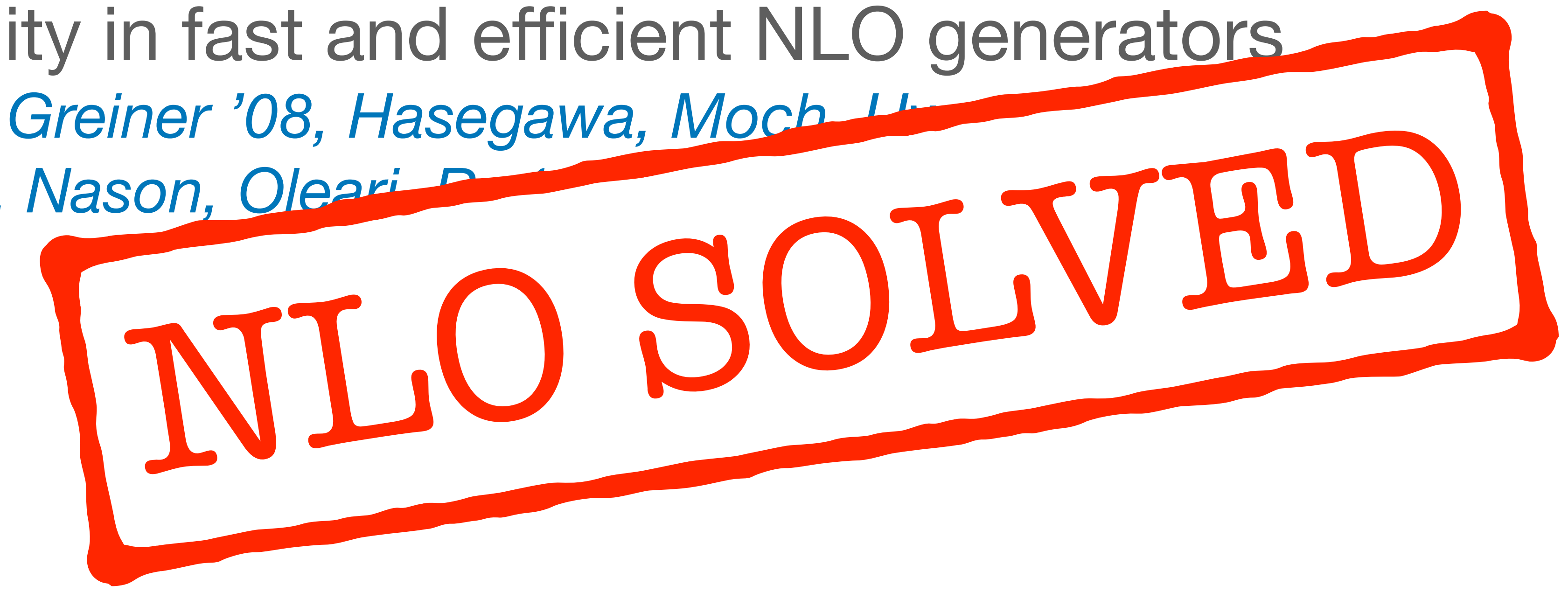

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NLO SOLVED

What about NNLO?

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NLO SOLVED

What about NNLO?

Extraction of real-emission singularities was the main bottleneck for NNLO predictions.

<u>Example</u>: di-jet two-loop amplitudes ~ 20 years ago *[Glover, Oleari, Tejeda-Yeomans '01]*,
di-jet production at NNLO ~ 4 ago *[Currie, De Ridder, Gehrmann, Glover, Huss, Pires '17]*

Many schemes are available:

Antenna *[Gehermann-De Ridder et al. 0505111]*

ColorfullNNLO *[Del Duca et al. 1603.08927]*

Nested-soft-collinear subtraction *[Caola et al. 1702.01352]*

Residue subtraction *[Czakon 1005.0274]*

New strategies have been explored:

Analytic Sector Subtraction *[Magnea et al. 1806.09570]*

Geometric IR subtraction *[Herzog 1804.07949]*

Unsubtraction *[Sborlini et al. 1608.01584]*

FDR *[Pittau, 1208.5457]*

Universal Factorisation *[Sterman et al.2008.12293]*

None of the existing subtraction schemes satisfies **all the '5 criteria'**

1) **Physical transparency**
2) **Generality**
3) **Locality**
4) **Analyticity**
5) **Efficiency**

Why is NNLO so difficult?

At NLO two main strategies have been implemented

Catani Seymour:

- Counterterm contribution: reproduces the **IR singularities** related to a dipole in **all of the phase space** [complicated structure]
- Full counterterm: sum of **contributions**, each **parametrised differently**
- **Analytic integration** of each term [non trivial, complicated structure of the counterterm]

FKS:

- **Partition** of the radiative phase space with sector functions
- **Different parametrisation** for each sector
- **Analytic integration**, after getting rid of sector functions [non trivial, non optimised parametrisation]

Detail informations of NNLO kernels also available ~ 20 years ago

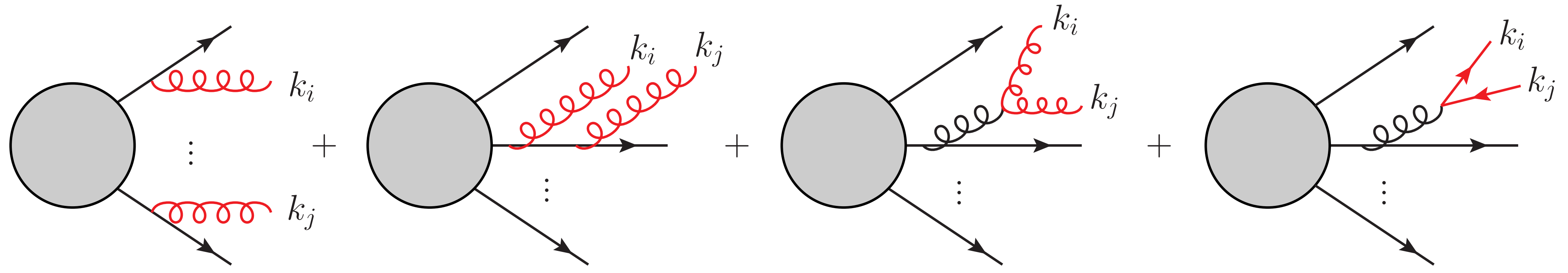
(N3LO kernels partially available *[Catani, Colferai, Torrini 1908.01616, Del Duca, Duhr, Haindl, Lazopoulos Michel 1912.06425, Dixon, Herrmann, Kai Yan, Hua Xing Zhu 1912.09370Yu Jiao Zhu 2009.08919]*)

Why is NNLO so difficult?

Under IR singular limits, the RR factorise into: (universal kernel) x (lower multiplicity matrix elements)

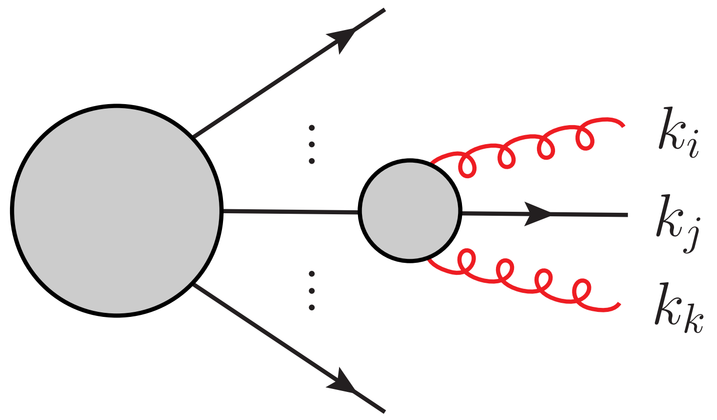
Double soft limit *[Catani, Grazzini 9903516,9810389]*

$$\lim_{k_i, k_j \to 0} RR_{n+2}\left(\{k\}_n, k_i, k_j\right) \sim \mathrm{Eik}\left(\{k\}_n, k_i, k_j\right) \otimes B_n\left(\{k\}_n\right)$$



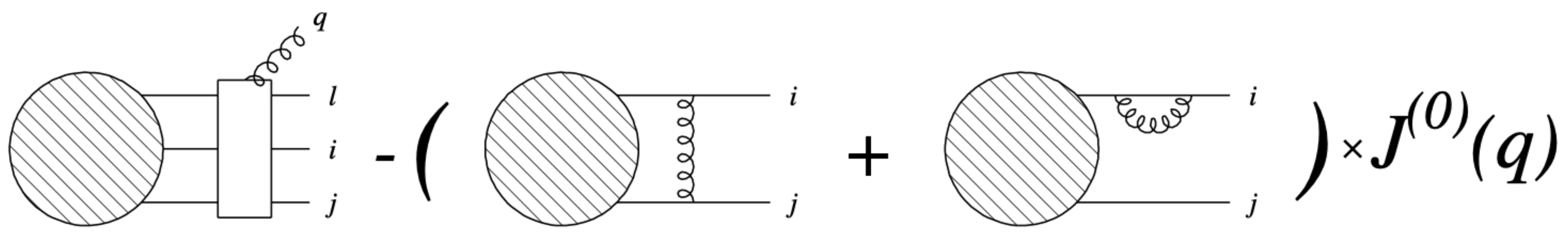
Triple collinear limit *[Catani, Grazzini 9903516,9810389]*

$$\lim_{k_i \| k_j \| k_k} RR_{n+2}\left(\{k\}_{n-1}, k_i, k_j, k_k\right) \sim \frac{1}{s_{ijk}^2} P\left(k_i, k_j, k_k\right) \otimes B_n\left(\{k\}_{n-1}, k_{ijk}\right)$$



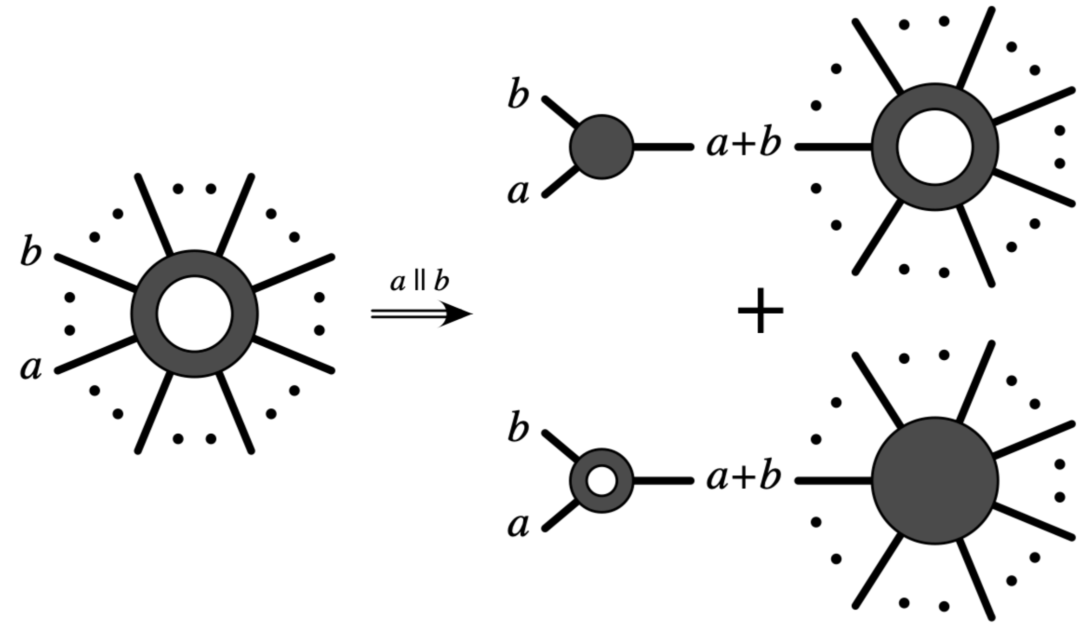
One loop single soft limit *[Catani, Grazzini 0007142]*

$$\lim_{k_i \to 0} RV_{n+1}\left(\{k\}_n, k_i\right) \sim \mathrm{Eik}\left(\{k\}_n, k_i\right) \otimes V_n\left(\{k\}_n\right) + \widetilde{\mathrm{Eik}}\left(\{k\}_n, k_i\right) \otimes B_n\left(\{k\}_n\right)$$



One loop single collinear limit *[Kosower 9901201, Bern, Del Duca, Kilgore, Schmidt 9903516]*

$$\lim_{k_i \| k_{\to} 0} RV_{n+1}\left(\{k\}_n, k_i\right) \sim \frac{1}{s_{ij}}\left[P\left(k_i, k_j\right) \otimes V_n\left(\{k\}_n\right) + \widetilde{P}\left(k_i, k_j\right) \otimes B_n\left(\{k\}_n\right)\right]$$



$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \right]$$

$$I^{(i)}_{cd} = \frac{s_{cd}}{s_{ic}\, s_{id}} \qquad I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar{q})(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd} \qquad s_{ab} = 2p_a \cdot p_b$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})} \qquad I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})} \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big) \qquad P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \Bigg\{ \frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right]$$

$$+ \frac{s_{ijk}^2}{2s_{ij}s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2z_i z_{jk}}{z_j z_k} + 2z_j z_k + z_i(1 + 2z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \Bigg\} + perm\,.$$

$$z_a = \frac{s_{ar}}{s_{ir} + s_{jr} + s_{kr}}\,, \; z_{ab} = z_a + z_b$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \Bigg\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_i z_j}{z_{ij} z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \Bigg\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivia**l and various **structures**, to be integrated over a **6-dim PS.**

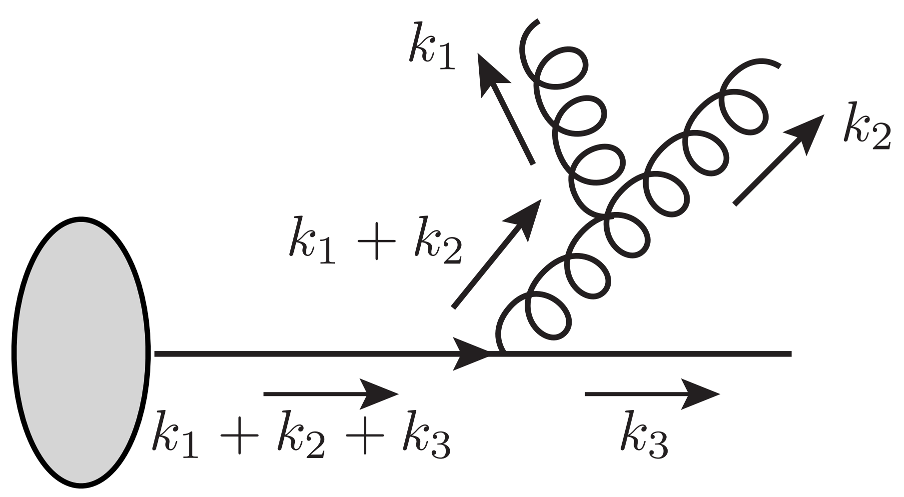
Why is NNLO so difficult?

1. Clear understanding of which singular configurations do actually contribute
2. Get to the point where the problem is well defined
3. Solve the phase space integrals of the relevant limits

Message

Amplitudes community may be interested in the key ideas behind subtraction, and in particular to the phase space integration.

1. Clear understanding of which singular configurations do actually contribute



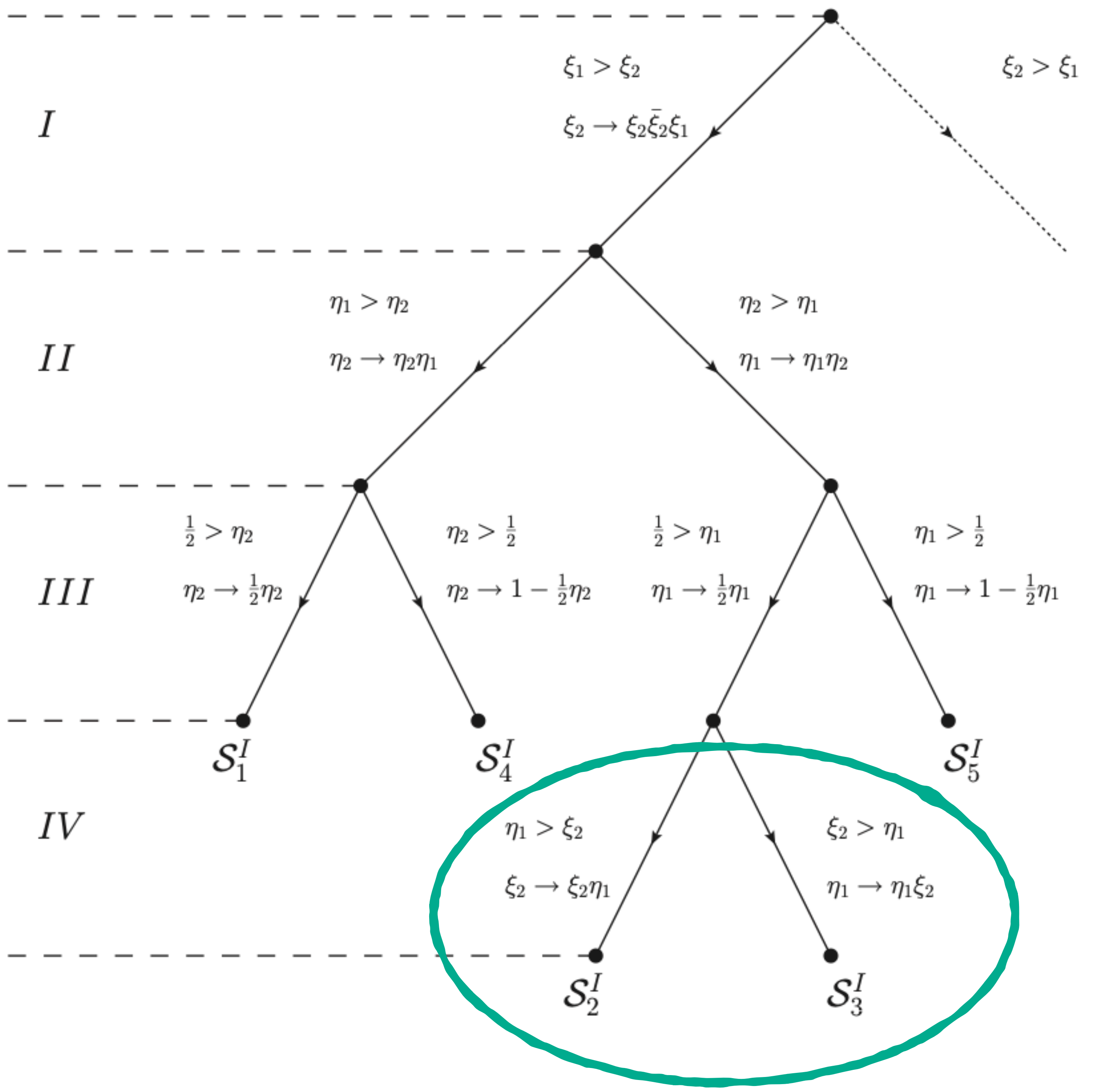
$$\sim \frac{1}{(k_1+k_2)^2}\,\frac{1}{(k_1+k_2+k_3)^2} = \frac{1}{2k_1\cdot k_2}\,\frac{1}{2k_1\cdot k_2 + 2k_1\cdot k_3 + 2k_2\cdot k_3} \iff k_1 \to 0 \text{ and } k_2 \parallel k_3$$

Entangled soft-collinear limits of diagrams can not be treated in a process-independent way.
Do non-commutative limits actually contribute?

STRIPPER was implemented taking into account all the possible choices of soft and collinear limits order -> redundant configurations were included

Gauge invariant amplitudes are free of entangled singularities
thanks to color coherence: soft parton does not resolve angles of the collinear partons

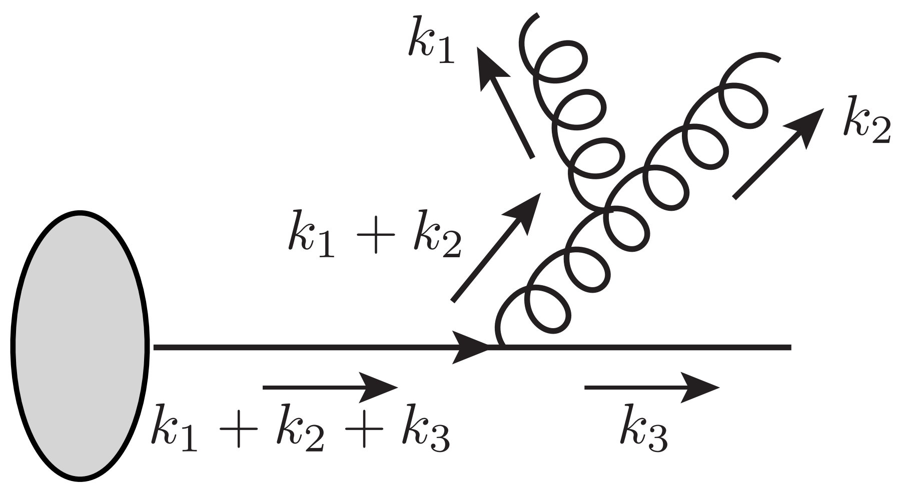
Soft-collinear limits can be described by taking the known soft and collinear limits sequentially



2. Get to the point where the problem is well defined

a) Identify the overlapping singularities

b) Regulate them



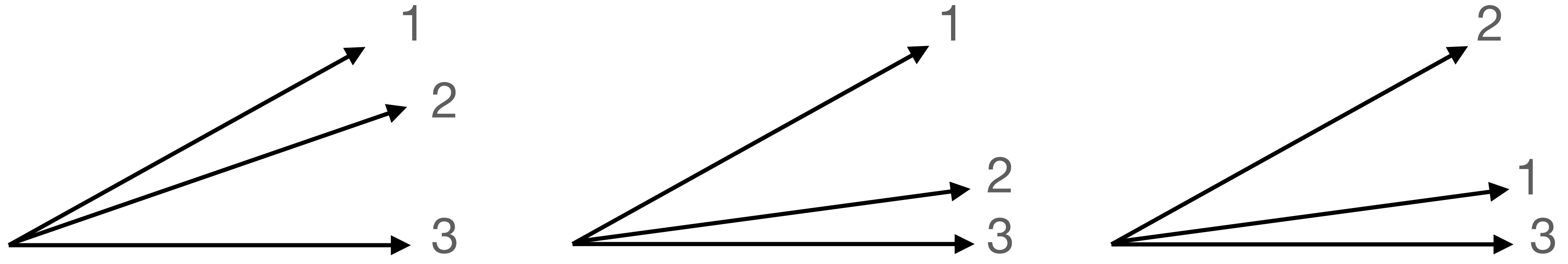
$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

Soft origin: $\overbrace{E_1 \to 0 \quad E_2 \to 0 \quad E_1, E_2 \to 0}$

Collinear origin: $\overbrace{\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3}$

$$E_1 \ll E_2 \,, \quad E_2 \ll E_1$$

Includes **strongly ordered** configurations



$\vec{n}_1 \cdot \vec{n}_2 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_2 \cdot \vec{n}_3 < \vec{n}_1 \cdot \vec{n}_3$ $\quad$ $\vec{n}_1 \cdot \vec{n}_3 < \vec{n}_2 \cdot \vec{n}_3$

Soft and collinear modes do not intertwine: soft subtraction can be done globally. Collinear singularities have still to be regulated. Strongly ordered configurations have to be properly taken into account.

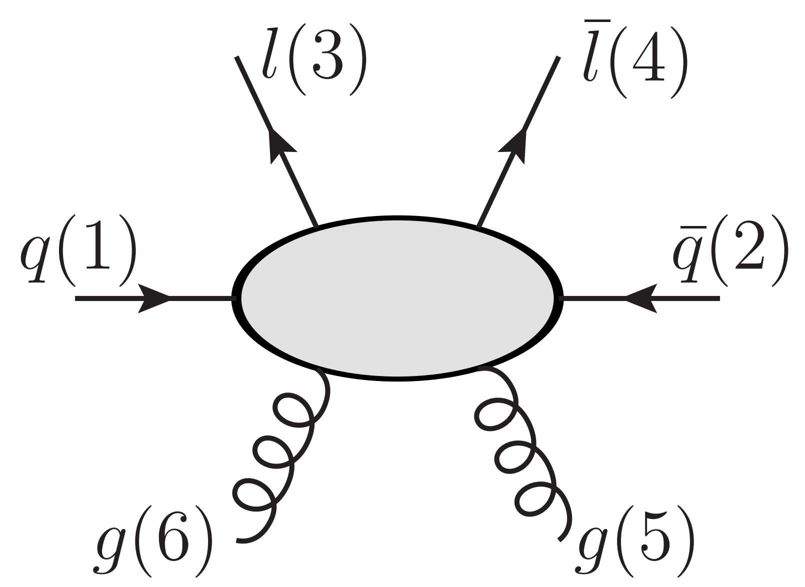
Phase space partitions

Efficient way to simplify the problem: introduce **partition functions** (following FKS philosophy):

- **Unitary partition**
- Select a **minimum number of singularities** in each sector
- Do **not affect** the **analytic integration** of the counterterms

Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ \, g \, g$ [Caola, Melnikov, Röntsch 1702.01352]



$$1 = \omega^{51,61} + \omega^{52,62} + \omega^{51,62} + \omega^{52,61}$$

$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}}$$

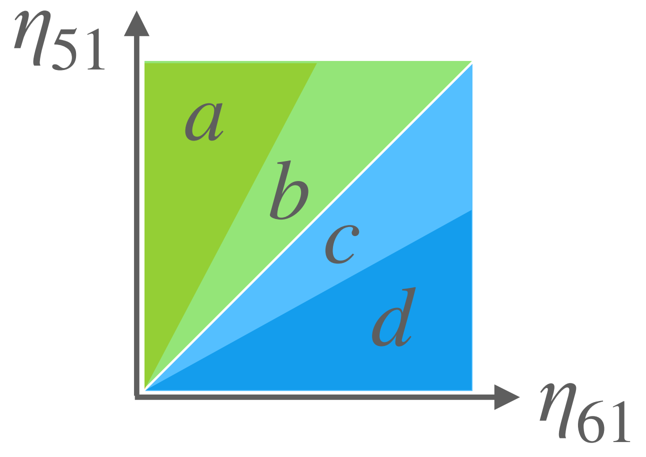
$$\omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$

$$\rho_{ab} = 1 - \cos\vartheta_{ab}\,,\ \eta_{ab} = \rho_{ab}/2$$

$$d_{i=5,6} = \rho_{1i} + \rho_{2i} = 2$$

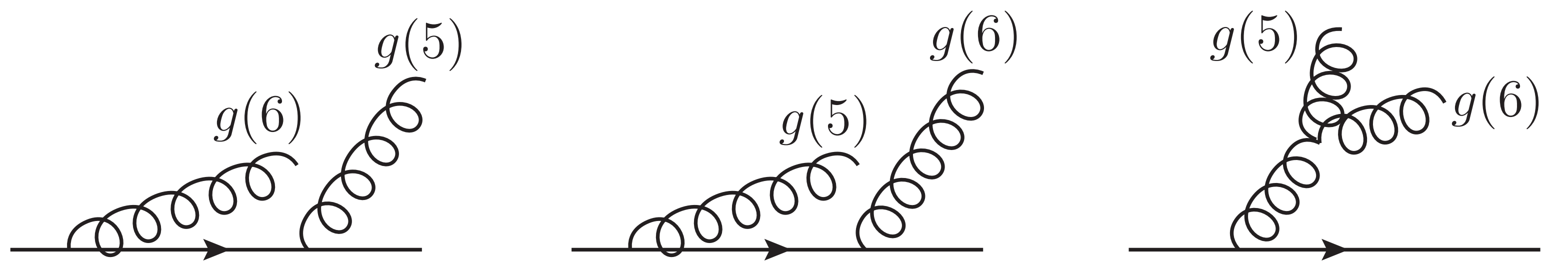
$$d_{5621} = \rho_{56} + \rho_{52} + \rho_{61}$$

$$d_{5612} = \rho_{56} + \rho_{51} + \rho_{62}$$



$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right)$$

$$= \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}$$



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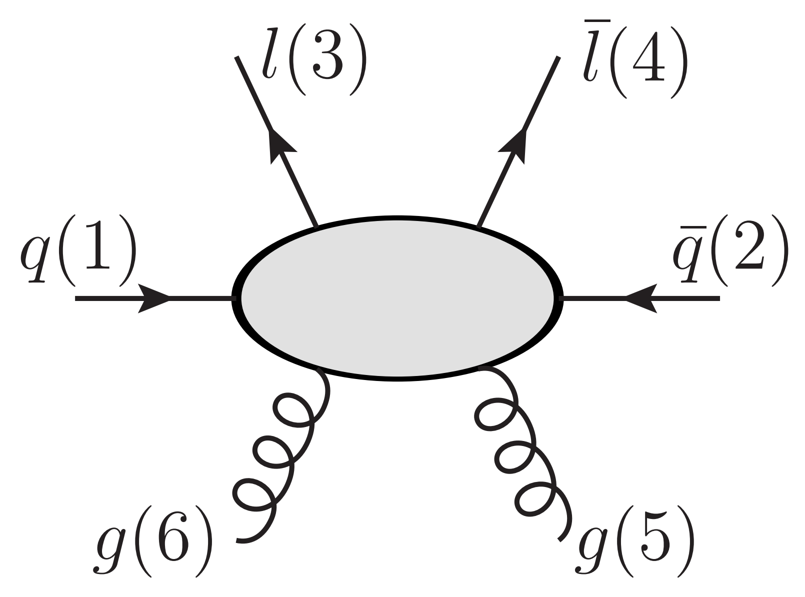
Definition of partition functions benefits from remarkable degree of **freedom**: different approaches can be implemented

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ \, g \, g$ *[Caola, Melnikov, Röntsch 1702.01352]*



Advantages:

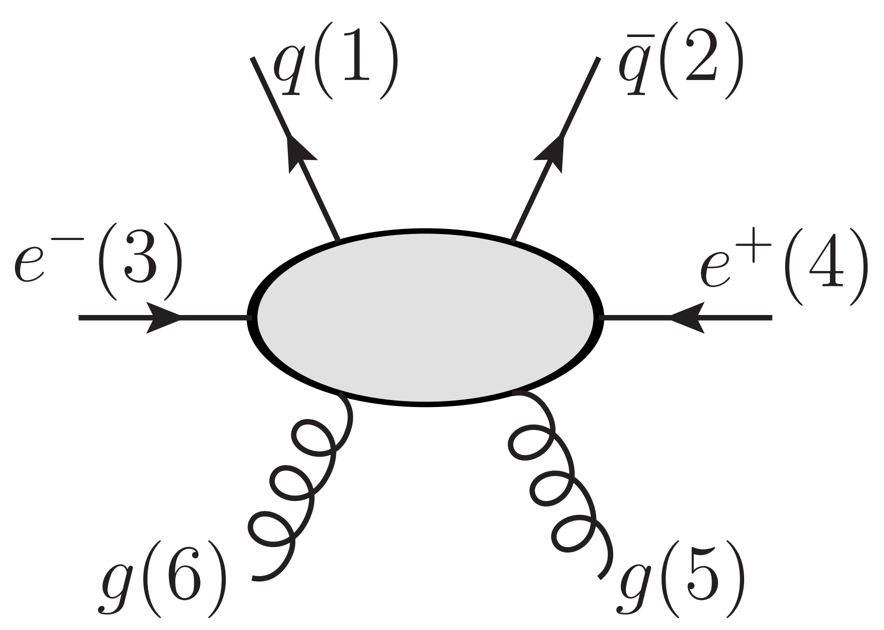
1. Simple definition
2. Structure of collinear singularities fully defined
3. Same strategy holds for NNLO mixed QCDxEW processes
4. **Minimum number of sector**

Disadvantages:

1. Partition based on angular ordering -> Lorentz invariance not preserved
   -> angles defined in a given reference frame
2. Theta function

Phase space partitions

Examples: Local Analytic Sector Subtraction $e^+e^- \to \gamma^* \to q\bar{q} + g\,g$ *[Magnea, C.S. et al. 1806.09570]*



$$1 = \mathscr{W}_{1225} + \mathscr{W}_{1226} + \mathscr{W}_{1252} + \mathscr{W}_{1256} + \ldots + \mathscr{W}_{6152}$$

$$\mathscr{W}_{abcd} = \frac{\sigma_{abcd}}{\sum_{m,n,p,q} \sigma_{mnpq}} \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}, \quad \alpha > 1$$

$$e_i \propto s_{qi}, \qquad w_{ij} \propto \frac{s_{ij}}{s_{qi}\, s_{qj}} \qquad q^\mu = (\sqrt{s}, \vec{0}), \qquad s_{ab} = 2k_a \cdot k_b$$

Advantages:

1. Compact definition
2. Triple-collinear sectors do not require further partition
3. Structure of collinear singularities fully defined
4. Valid for arbitrary number of FS partons
5. **Defined in terms of Lorentz invariants**

Disadvantages:

1. Numerous sectors -> consequence of being fully general -> non minimal structure
2. Non-trivial recombination before integration

3. Solve the PS integrals

The problem is now well defined:

A. **Singular kernels** and their nested limits have to be **subtracted from the double real correction** to get integrable object

$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2}\, \underbrace{\left[RR_{n+2} - K_{n+2}\right]}_{\text{Fully regulated real emission contribution}} + \int d\Phi_{n+2}\, K_{n+2} \qquad K_{n+2} \supset C_{ij},\, C_{kl},\, S_i,\, S_{ij},\, C_{ijk}$$

Fully regulated real emission contribution ⟶ Numerical evaluation

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ [Caola, Melnikov, Röntsch]

$$[\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}\,(2\pi)\,\delta_+(k_i^2)$$

$$\mathrm{d}\hat{\sigma}^{NNLO}_{\text{resolv.}} = \int \theta(E_5 - E_6)\,\theta(E_{\max} - E_5)\,\Big\{ \sum_{i,j\in\{1,2\},i\neq j} \big(1 - C_{5i}\big)\,\big(1 - C_{6j}\big)\,\big(1 - S_{56}\big)\,\big(1 - S_6\big)\,[\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\omega^{5i,6j}\,B\big(\{k\}_{1\ldots6}\big)$$

$$+ \sum_{i\in\{1,2\}} \Big[\theta^{(a)}\big(1 - C_{i56}\big)\,\big(1 - C_{6i}\big) + \theta^{(b)}\big(1 - C_{i56}\big)\,\big(1 - C_{56}\big)$$

$$+\theta^{(c)}\big(1 - C_{i56}\big)\,\big(1 - C_{5i}\big) + \theta^{(d)}\big(1 - C_{i56}\big)\,\big(1 - C_{56}\big)\Big][\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\omega^{5i,6i}\,B\big(\{k\}_{1\ldots6}\big)\Big\}$$

Explicit expression depends on the scheme

3. Solve the PS integrals

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$$\int d\Phi_{n+2}\, RR_{n+2} = \int d\Phi_{n+2} \left[RR_{n+2} - K_{n+2} \right] + \int d\Phi_{n+2}\, K_{n+2} \qquad K_{n+2} \supset C_{ij}, C_{kl}, S_i, S_{ij}, C_{ijk}$$

B. **Counterterms** have to be **integrated over the unresolved phase space**

$$I = \int \mathrm{PS}_{\mathrm{unres.}} \otimes \mathrm{Limit} \otimes \mathrm{Constraints}$$

The ‘Limit’ component is universal and known. The phase space is well defined. Constraints may vary depending on the scheme.

Several kinematic structures have to be integrated **analytically** over a 6-dim PS.

Different approximations and techniques can be applied: the results assume different form depending on the adopted strategy

Two main structure are the most complicated ones and affect most of the physical processes:

- **Double soft**
- **Triple collinear**

Kernels integration

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Delto, Frellesvig, Melnikov 1807.05835, Delto, Melnikov 1901.05213]*

Two soft parton (5,6) and two hard massless radiator (1,2): arbitrary relative angle between the three-momenta of the radiators

$$I_{12}^{(gg)(56)} = \frac{(1-\epsilon)(s_{51}s_{62} + s_{52}s_{61}) - 2s_{56}s_{12}}{s_{56}^2(s_{51} + s_{61})(s_{52} + s_{62})} + s_{12}\,\frac{s_{51}s_{62} + s_{52}s_{61} - s_{56}s_{12}}{s_{56}s_{51}s_{62}s_{52}s_{61}}\left[1 - \frac{1}{2}\,\frac{s_{51}s_{62} + s_{52}s_{61}}{(s_{51} + s_{61})(s_{52} + s_{62})}\right]$$

$$I_{S_{56}}^{(gg)} = \int [\mathrm{d}k_5]\,[\mathrm{d}k_6]\,\theta(E_{\max} - E_5)\,\theta(E_5 - E_6)\,I_{12}^{(gg)(56)}\big(k_1, k_2, k_5, k_6\big) \qquad [\,\mathrm{d}f_i\,] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}\,(2\pi)\,\delta_+(k_i^2)$$

$$E_5 = E_{\max}\,\xi \qquad E_6 = E_{\max}\,\xi\, z \qquad 0 < \xi < 1\,,\, 0 < z < 1$$

Reverse unitarity: **map phase space integrals onto loop integrals** *[Anastasiou, Melnikov 0207004]*
after defining integral families, integration-by-part identities. Differential equations w.r.t. the ratio of energies of emitted gluons at fixed angle.
Boundary conditions for z=0, and arbitrary angle

Kernels integration

Examples: Nested soft-collinear subtraction $q\bar{q} \to Z \to e^- e^+ g\, g$ [Caola, Delto, Frellesvig, Melnikov 1807.05835, Delto, Melnikov 1901.05213]

$$
\begin{aligned}
I^{(gg)}_{S_{56}} = (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \Bigg\{ & \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left[\frac{11}{12} - \ln(s^2)\right] \\
& + \frac{1}{\epsilon^2}\left[2\mathrm{Li}_2(c^2) + \ln^2(s^2) - \frac{11}{6}\ln(s^2) + \frac{11}{3}\ln 2 - \frac{\pi^2}{4} - \frac{16}{9}\right] \\
& + \frac{1}{\epsilon}\Bigg[6\mathrm{Li}_3(s^2) + 2\mathrm{Li}_3(c^2) + \left(2\ln(s^2) + \frac{11}{3}\right)\mathrm{Li}_2(c^2) - \frac{2}{3}\ln^3(s^2) \\
& \quad + \left(3\ln(c^2) + \frac{11}{6}\right)\ln^2(s^2) - \left(\frac{22}{3}\ln 2 + \frac{\pi^2}{2} - \frac{32}{9}\right)\ln(s^2) \\
& \quad - \frac{45}{4}\zeta_3 - \frac{11}{3}\ln^2 2 - \frac{11}{36}\pi^2 - \frac{137}{18}\ln 2 + \frac{217}{54}\Bigg] \\
& + 4\mathrm{G}_{-1,0,0,1}(s^2) - 7\mathrm{G}_{0,1,0,1}(s^2) + \frac{22}{3}\mathrm{Ci}_3(2\delta) + \frac{1}{3\tan(\delta)}\mathrm{Si}_2(2\delta) \\
& + 2\mathrm{Li}_4(c^2) - 14\mathrm{Li}_4(s^2) + 4\mathrm{Li}_4\left(\frac{1}{1+s^2}\right) - 2\mathrm{Li}_4\left(\frac{1-s^2}{1+s^2}\right) \\
& + 2\mathrm{Li}_4\left(\frac{s^2-1}{1+s^2}\right) + \mathrm{Li}_4(1-s^4) + \Bigg[10\ln(s^2) - 4\ln\left(1+s^2\right) \\
& + \frac{11}{3}\Bigg]\mathrm{Li}_3(c^2) + \left[14\ln(c^2) + 2\ln(s^2) + 4\ln\left(1+s^2\right) + \frac{22}{3}\right]\mathrm{Li}_3(s^2) \\
& + 4\ln(c^2)\mathrm{Li}_3(-s^2) + \frac{9}{2}\mathrm{Li}_2^2(c^2) - 4\mathrm{Li}_2(c^2)\mathrm{Li}_2(-s^2) + \Bigg[7\ln(c^2)\ln(s^2) \\
& - \ln^2(s^2) - \frac{5}{2}\pi^2 + \frac{22}{3}\ln 2 - \frac{131}{18}\Bigg]\mathrm{Li}_2(c^2) + \left[\frac{2}{3}\pi^2 - 4\ln(c^2)\ln(s^2)\right]\times \\
& \mathrm{Li}_2(-s^2) + \frac{\ln^4(s^2)}{3} + \frac{\ln^4\left(1+s^2\right)}{6} - \ln^3(s^2)\left[\frac{4}{3}\ln(c^2) + \frac{11}{9}\right] \\
& + \ln^2(s^2)\left[7\ln^2(c^2) + \frac{11}{3}\ln(c^2) + \frac{\pi^2}{3} + \frac{22}{3}\ln 2 - \frac{32}{9}\right] - \frac{\pi^2}{6}\ln^2\left(1+s^2\right) \\
& + \zeta_3\left[\frac{17}{2}\ln(s^2) - 11\ln(c^2) + \frac{7}{2}\ln\left(1+s^2\right) - \frac{21}{2}\ln 2 - \frac{99}{4}\right] + \ln(s^2)\times \\
& \left[-\frac{7\pi^2}{2}\ln(c^2) + \frac{22}{3}\ln^2 2 - \frac{11}{18}\pi^2 + \frac{137}{9}\ln 2 - \frac{208}{27}\right] - 12\mathrm{Li}_4\left(\frac{1}{2}\right) \\
& + \frac{143}{720}\pi^4 - \frac{\ln^4 2}{2} + \frac{\pi^2}{2}\ln^2 2 - \frac{11}{6}\pi^2\ln 2 + \frac{125}{216}\pi^2 + \frac{22}{9}\ln^3 2 \\
& + \frac{137}{18}\ln^2 2 + \frac{434}{27}\ln 2 - \frac{649}{81} + \mathcal{O}(\epsilon)\Bigg\},
\end{aligned}
$$

$$
\delta = \frac{\delta_{12}}{2}, \; s = \sin\frac{\delta_{12}}{2}, \; c = \cos\frac{\delta_{12}}{2}
$$

$$
\mathrm{Ci}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) + \mathrm{Li}_n(e^{-iz})}{2}, \; \mathrm{Si}_n(z) = \frac{\mathrm{Li}_n(e^{iz}) - \mathrm{Li}_n(e^{-iz})}{2i}
$$

Kernels integration

Examples: Local analytic sector subtraction $e^+e^- \to \gamma^* \to q\bar{q} + g\,g$ [Magnea, C.S. et al. 2010.14493]

Two soft parton (5,6) and two hard massless radiator (1,2)

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Adapt the phase space parametrisation to the invariants appearing in the kernel!

$$d\Phi_{n+2}\big(\{k\}\big) = d\Phi_n\big(\{\bar{k}\}^{(ijcd)}\big)\; d\Phi_{\text{rad},2}^{(ijcd)} \qquad I_{S_{56}}^{(gg)} = \int \mathrm{d}\phi_{\text{rad},2}^{(5612)}\; I_{12}^{(gg)(56)}(k_5,k_6,k_1,k_2)$$

Parametrise the phase space using **Catani-Seymour variables**

$$y' = \frac{s_{56}}{s_{56}+s_{51}+s_{61}}\,, \qquad z' = \frac{s_{51}}{s_{51}+s_{61}}\,, \qquad y = \frac{s_{56}+s_{51}+s_{61}}{s_{56}+s_{51}+s_{61}+s_{52}+s_{62}}\,, \qquad z = \frac{s_{52}+s_{62}-\frac{s_{56}}{s_{52}+s_{56}}\,s_{12}}{s_{52}+s_{62}-\frac{s_{56}+s_{51}}{s_{51}+s_{56}}\,s_{12}}$$

Use phase space symmetries, partial fractioning and Hypergeometric function properties

$$I_{S_{56}}^{(gg)} = \frac{1}{2}\frac{1}{\epsilon^4} + \frac{35}{12}\frac{1}{\epsilon^3} + \left(\frac{487}{36} - \frac{2}{3}\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{1562}{27} - \frac{269}{72}\pi^2 - \frac{77}{6}\zeta_3\right)\frac{1}{\epsilon} + \frac{19351}{81} - \frac{3829}{216}\pi^2 - \frac{1025}{18}\zeta_3 - \frac{23}{240}\pi^4 + \mathcal{O}(\epsilon)$$

Conclusions

1. Subtraction schemes are necessary ingredients to achieve precise theoretical predictions

2. Efforts to implement an “ideal” subtraction scheme are still ongoing

3. Nested-soft collinear and Local analytic sector subtractions provide efficient methods that aim to accomplish all the five criteria

4. Phase space integrals of soft and collinear limits of QCD amplitudes are non-trivial and crucial ingredients that all the subtraction schemes need

Thank you!

Backup

IR regularisation: subtraction vs slicing

$F(x)$ arbitrary complicated function

$$I = \lim_{\epsilon\to 0}\left[\int_0^1 \frac{dx}{x}\, x^\epsilon F(x) - \frac{1}{\epsilon}F(0)\right]$$

Goal: compute *I* without relying on the analytic evaluation of the integral

Slicing

$$I \sim \lim_{\epsilon\to 0}\left[F(0)\int_0^\delta \frac{dx}{x}\, x^\epsilon + \int_\delta^1 \frac{dx}{x}\, x^\epsilon F(x) - \frac{1}{\epsilon}F(0)\right] = F(0)\,\log\delta + \int_\delta^1 \frac{dx}{x}\, x^\epsilon F(x)$$

Slicing parameter $\delta \ll 1$ $\to$ power dependence on the slicing parameter in the result

Subtraction

$$I = \lim_{\epsilon\to 0}\left[\underbrace{\int_0^1 \frac{dx}{x}\, x^\epsilon\Big(F(x) - F(0)\Big)}_{\text{Regulated, finite for } \epsilon\to 0} + \underbrace{\int_0^1 \frac{dx}{x}\, x^\epsilon F(0)}_{\text{Extract } 1/\epsilon \text{ pole}} - \frac{1}{\epsilon}F(0)\right]$$

Counterterm: the definition may be involved!

Local Analytic Sector Subtraction

Go **back to NLO** to implement a new scheme featuring **key properties** that can be **exported at NNLO**.
(This talk: massless partons, FSR only, arbitrary number of FS particles)

$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \lim_{d\to 4}\left\{ \int d\Phi_n \; V_n \, \delta_n(X) + \int d\Phi_{n+1} \, R_{n+1}\delta_{n+1}(X) \right\}$$

X IR safe observable

$$\frac{d\sigma^{\mathrm{NLO}}_{ct}}{dX} = \int d\Phi_{n+1} \, \overline{K}_{n+1}$$

Counterterm

$$I_n = \int d\Phi_{\mathrm{rad}} \, \overline{K}_{n+1}$$

Integrated Counterterm

$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \int d\Phi_n \left(V_n + I_n \right) \delta_n(X) + \int d\Phi_{n+1} \left(R_{n+1} \, \delta_{n+1}(X) - \overline{K}_{n+1} \, \delta_n(X) \right)$$

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Properties of the scheme:

Analytically calculable
(possibly with standard techniques)

Minimal structure and **simple integration**

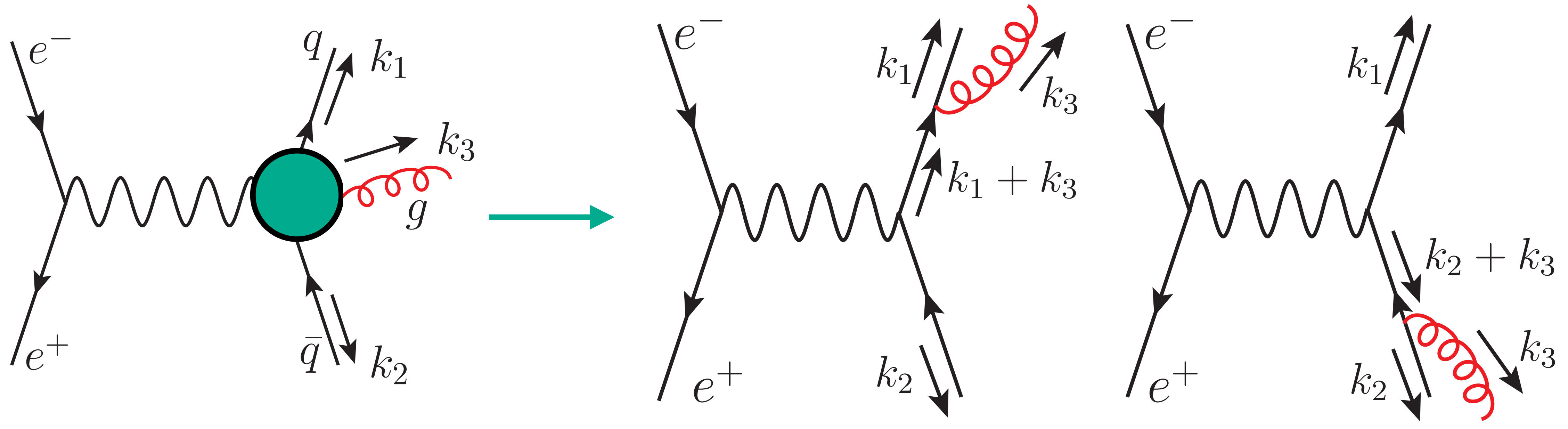
Require:

Choose an **optimise parametrisation** of the phase space

Organise all the overlapping **singularities** and choose an **appropriate kinematics**

Ingredients of the subtraction

- *Phase space partitioning (FKS) : multiple singular configuration that overlap*



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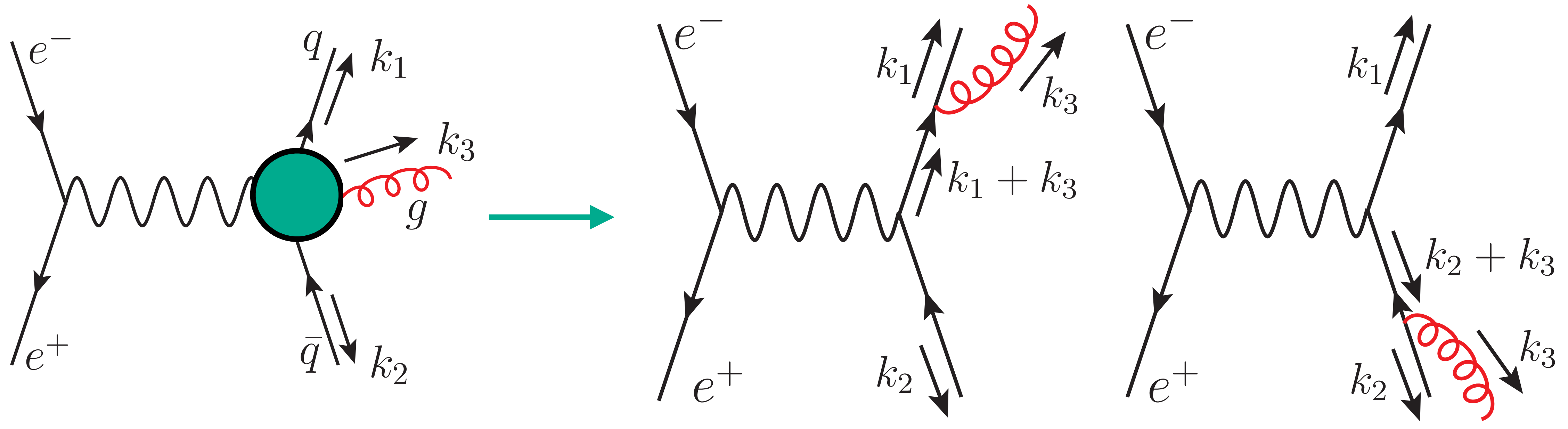

$$R \sim \frac{1}{(k_1+k_3)^2} + \frac{1}{(k_2+k_3)^2} \sim \frac{1}{E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3)} + \frac{1}{E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

$$R \to \infty \quad \begin{cases} E_3 \to 0 & \to S_3 \\ \vec{n}_1 \parallel \vec{n}_3 & \to C_{13} = C_{31} \\ \vec{n}_2 \parallel \vec{n}_3 & \to C_{23} = C_{32} \end{cases}$$

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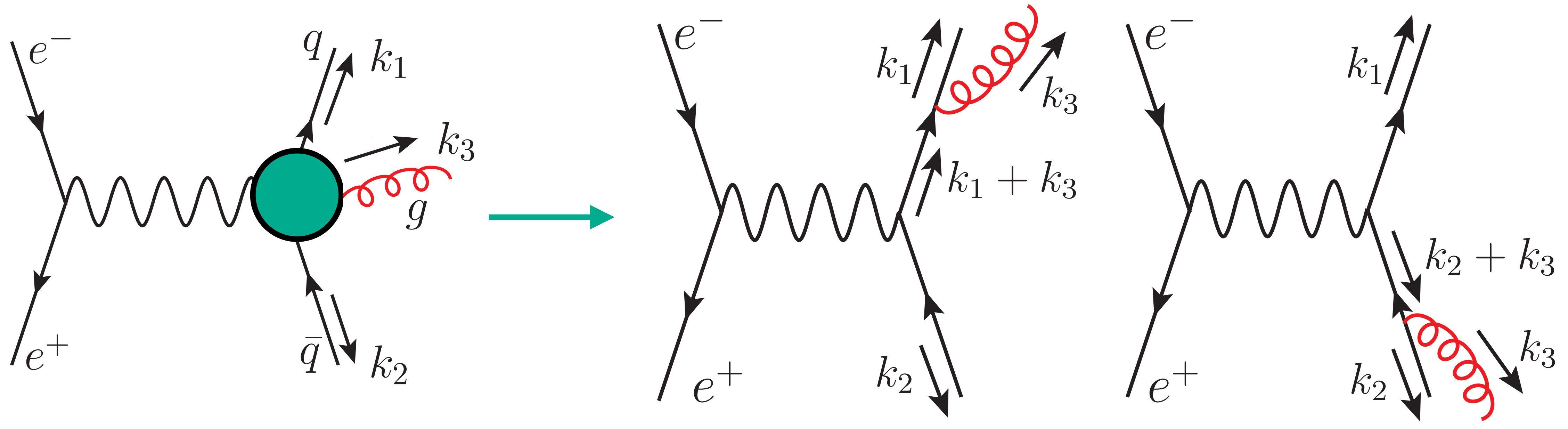

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Sector functions $\mathscr{W}_{ij}$:

At most one soft and/or two collinear partons in each sector

$$R = \sum_{i,j} R\,\mathscr{W}_{ij} = R\,\mathscr{W}_{31} + R\,\mathscr{W}_{32} + \ldots$$

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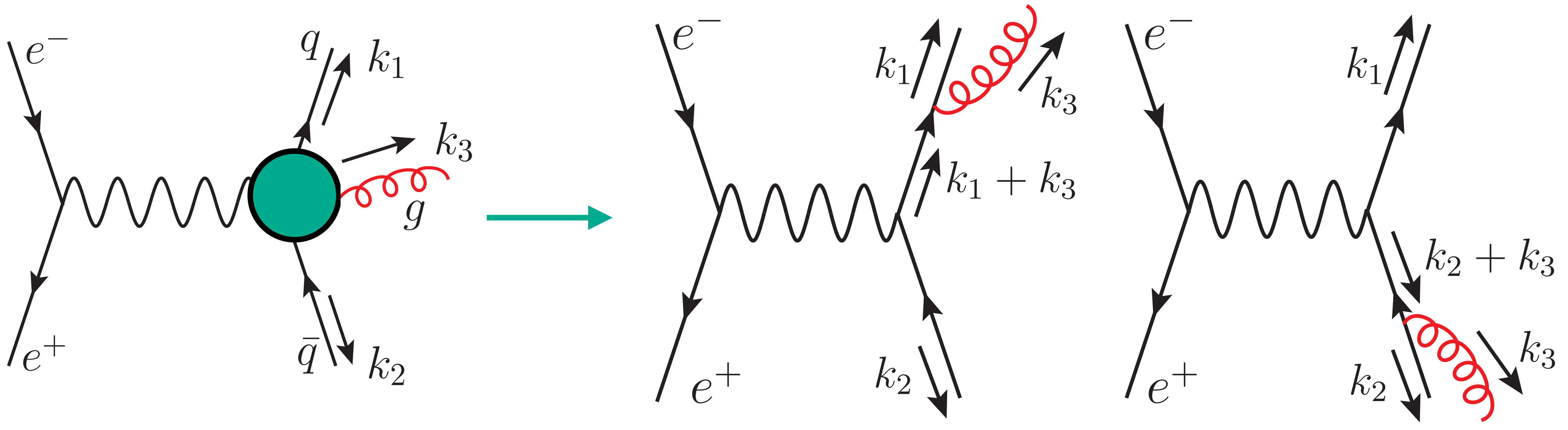

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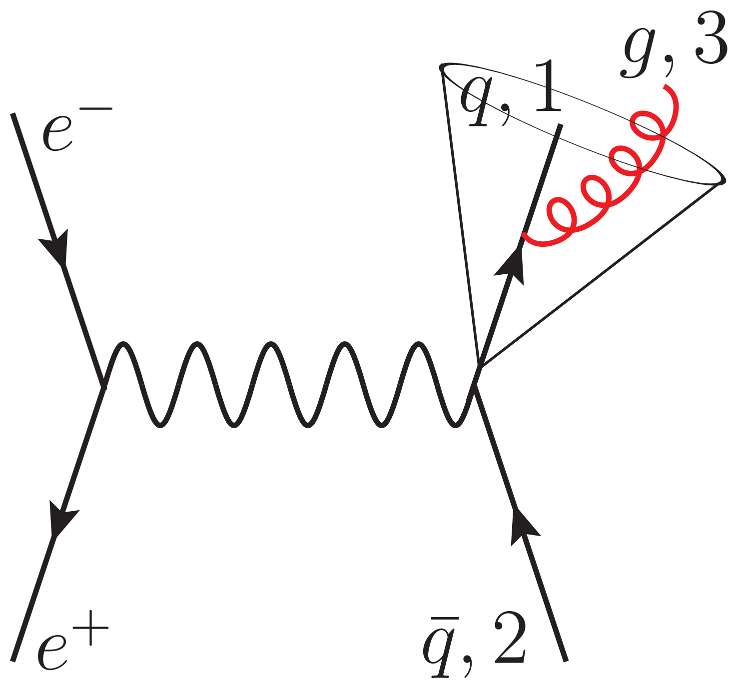

Damp: $\vec{n}_2 \parallel \vec{n}_3$

Enhance: $\vec{n}_1 \parallel \vec{n}_3$

$\mathscr{W}_{31} \sim \frac{1}{s_{31}}$

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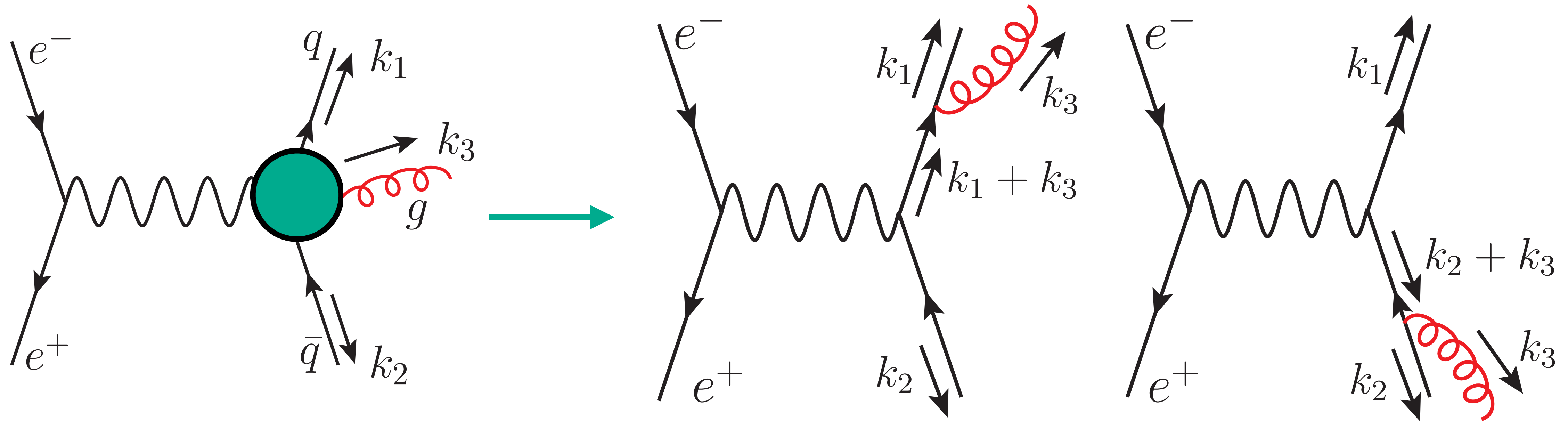

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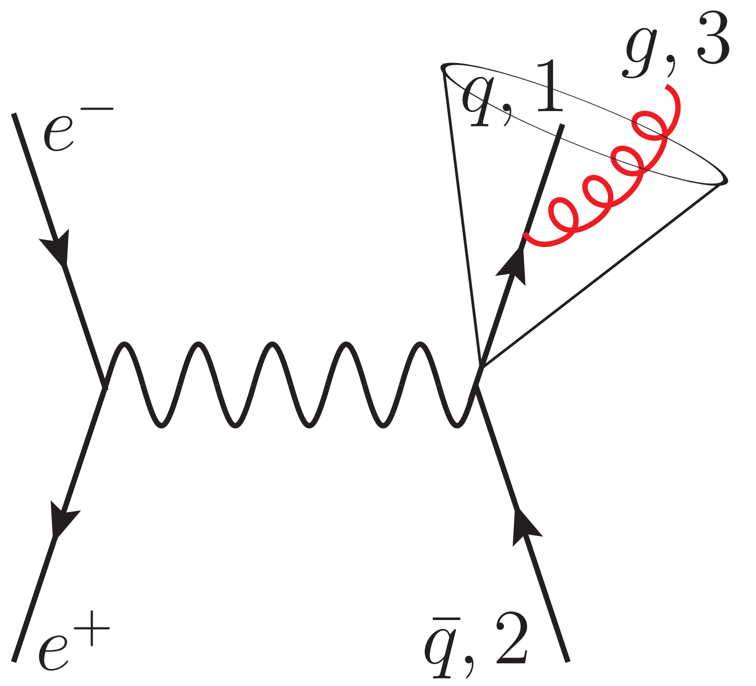

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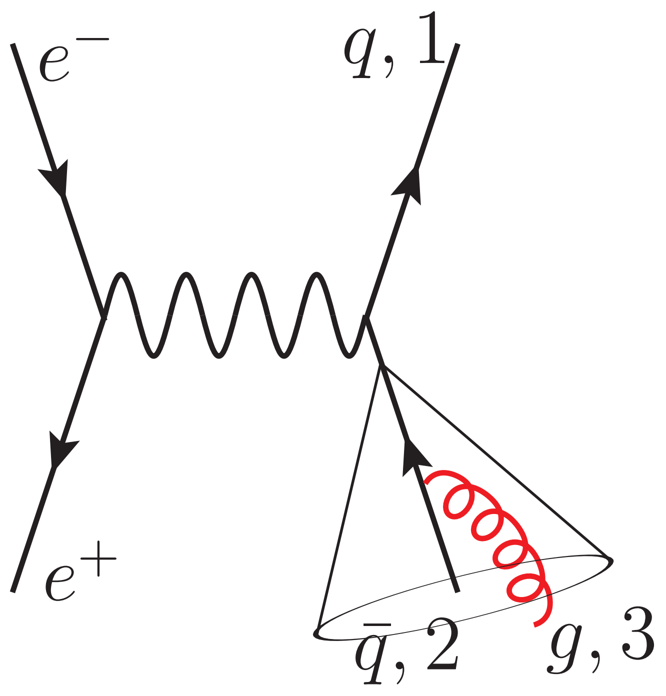

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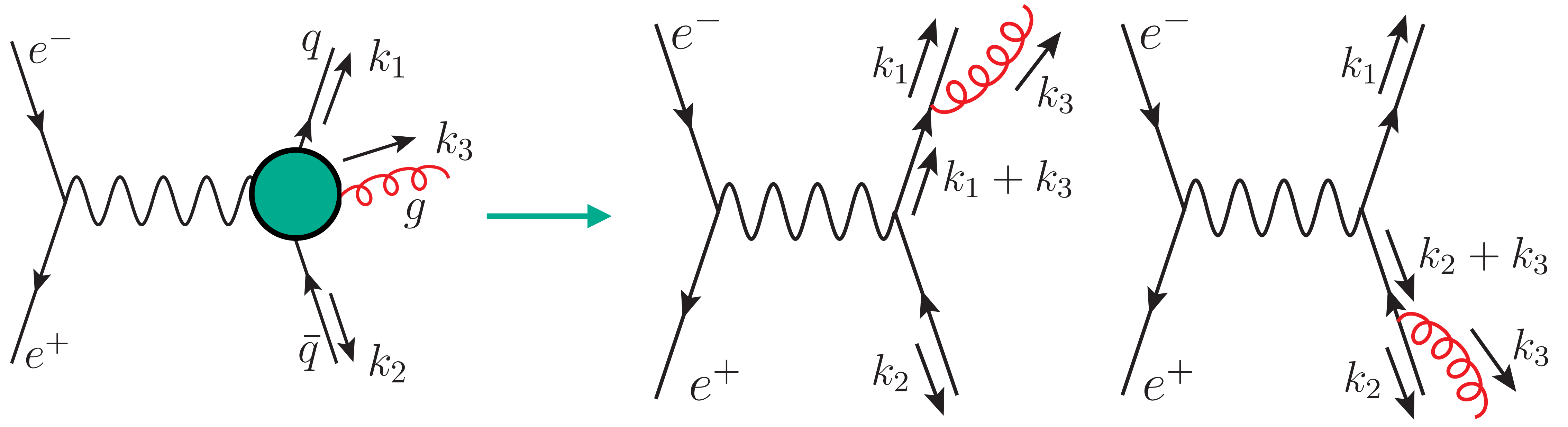

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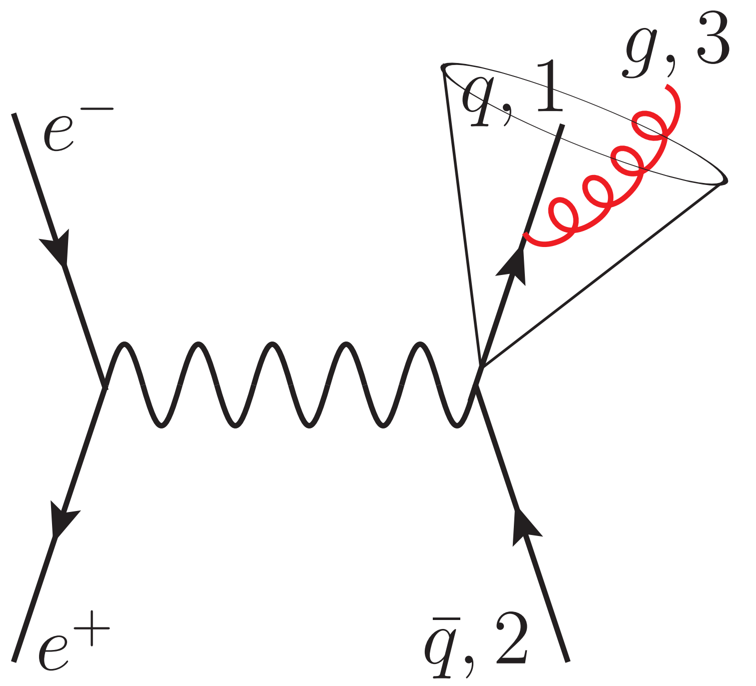

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Enhance: $\vec{n}_1 \parallel \vec{n}_3$
$\mathcal{W}_{31} \sim \dfrac{1}{s_{31}}$

Damp: $\vec{n}_1 \parallel \vec{n}_3$
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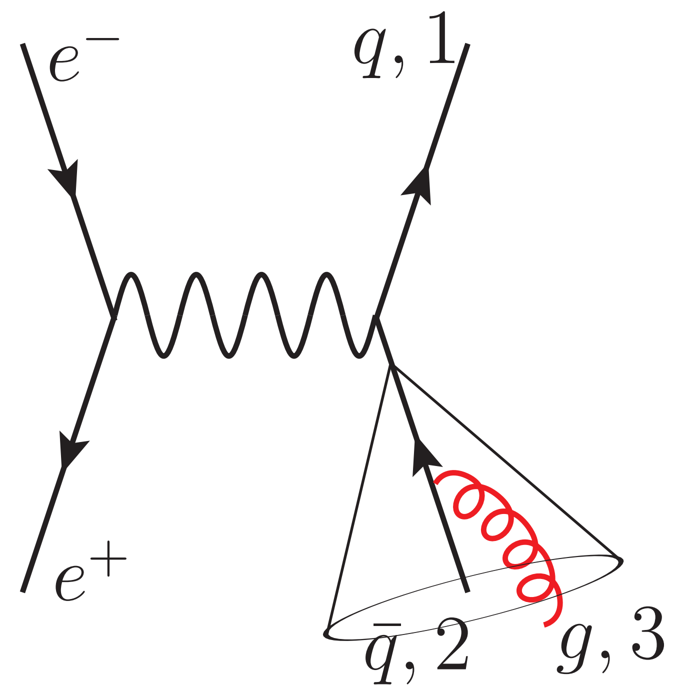
$\mathcal{W}_{32} \sim \dfrac{1}{s_{32}}$



preserving Lorentz invariance

The idea of mappings

$$\int d\Phi_{n+1}\left(R_{n+1} - K_{n+1}\right) \xrightarrow{\{k\}_{n+1} \to \{\bar{k}_n\}^{(abc)}} \int d\Phi_{n+1}\left(R_{n+1} - \overline{K}_{n+1}\right)$$

$$S_i\, R_{n+1}\big(\{k\}\big) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}} B_n\big(\{k\}_{\not i}\big) \qquad \bar{S}_i\, R_{n+1}\big(\{k\}\big) \propto \sum_{a,c\neq i} \frac{s_{cd}}{s_{ci}\, s_{di}} B_n\big(\{\bar{k}\}^{(icd)}\big)$$

$$C_{ij}\, R_{n+1}\big(\{k\}\big) \propto \frac{1}{s_{ij}} P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\, B^{\mu\nu}_n\big(\{k\}_{\not i \not j}, k_{ij}\big) \qquad \overline{C}_{ij}\, R_{n+1}\big(\{k\}\big) \propto \frac{1}{s_{ij}} P^{\mu\nu}_{ij}(s_{ir}, s_{jr})\, B^{\mu\nu}_n\big(\{\bar{k}\}^{(ijr)}\big)$$

$$S_i\, C_{ij}\, R_{n+1}\big(\{k\}\big) \propto \frac{s_{jr}}{s_{ij}\, s_{ir}} B_n\big(\{k\}_{\not i}\big) \qquad \bar{S}_i\, \overline{C}_{ij}\, R_{n+1}\big(\{k\}\big) \propto \frac{s_{jr}}{s_{ij}\, s_{ir}} B_n\big(\{\bar{k}\}^{(ijr)}\big)$$

Why a mapping?

1. $\{k\}_{\not i}$ is a set of n momenta that do not satisfy n-body momentum conservation away from the exact S_i limit
2. $\{k\}_{\not i \not j}, k_{ij}$ is a set of n momenta where $k_{ij} = k_i + k_j$ is off-shell away from the exact C_{ij} limit
3. Factorise the $n+1$-body PS intro a n-body and radiation phase space is necessary to integrate K only in the latter

Collinear limit: single mapping > *dipole = (ijr)*
Soft limit: different mapping for each contribution > *dipole = (icd)*

The idea of mappings

Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

On-shell particle **conserving momentum** in the entire PS

The idea of mappings

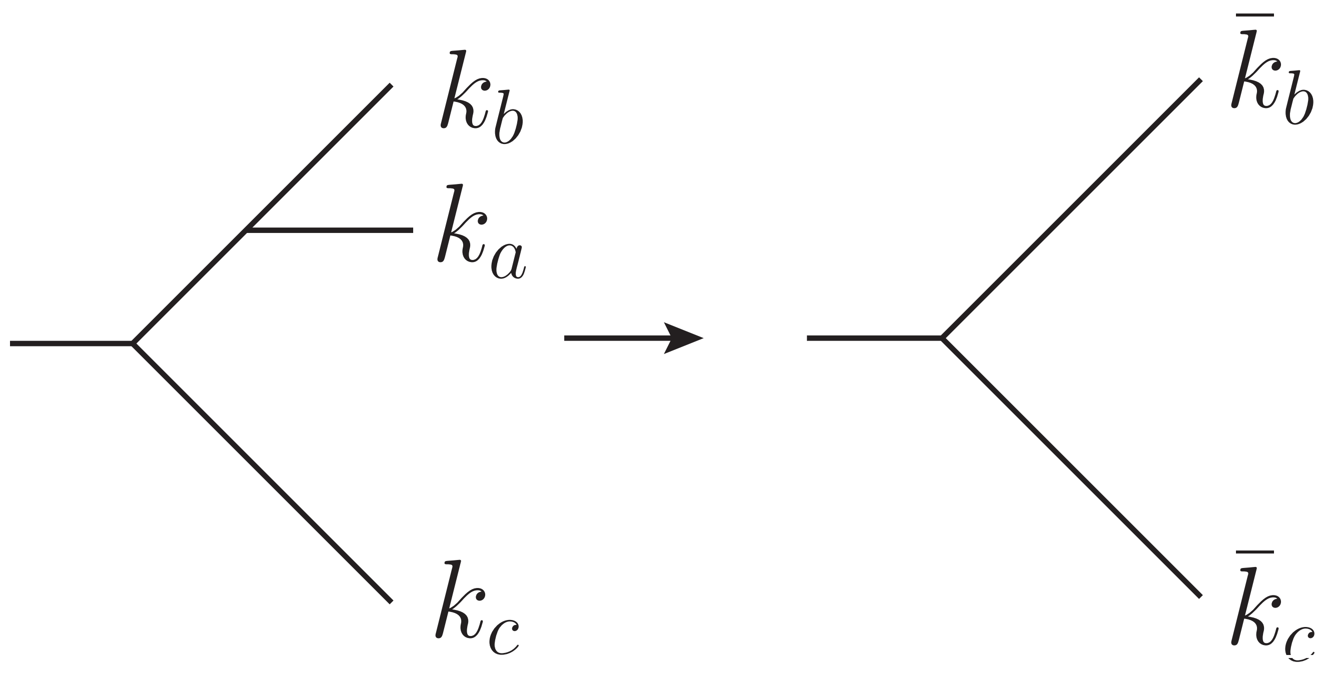
Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\mathrm{rad}}$

On-shell particle **conserving momentum** in the entire PS

↓

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not a \not b \not c}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$



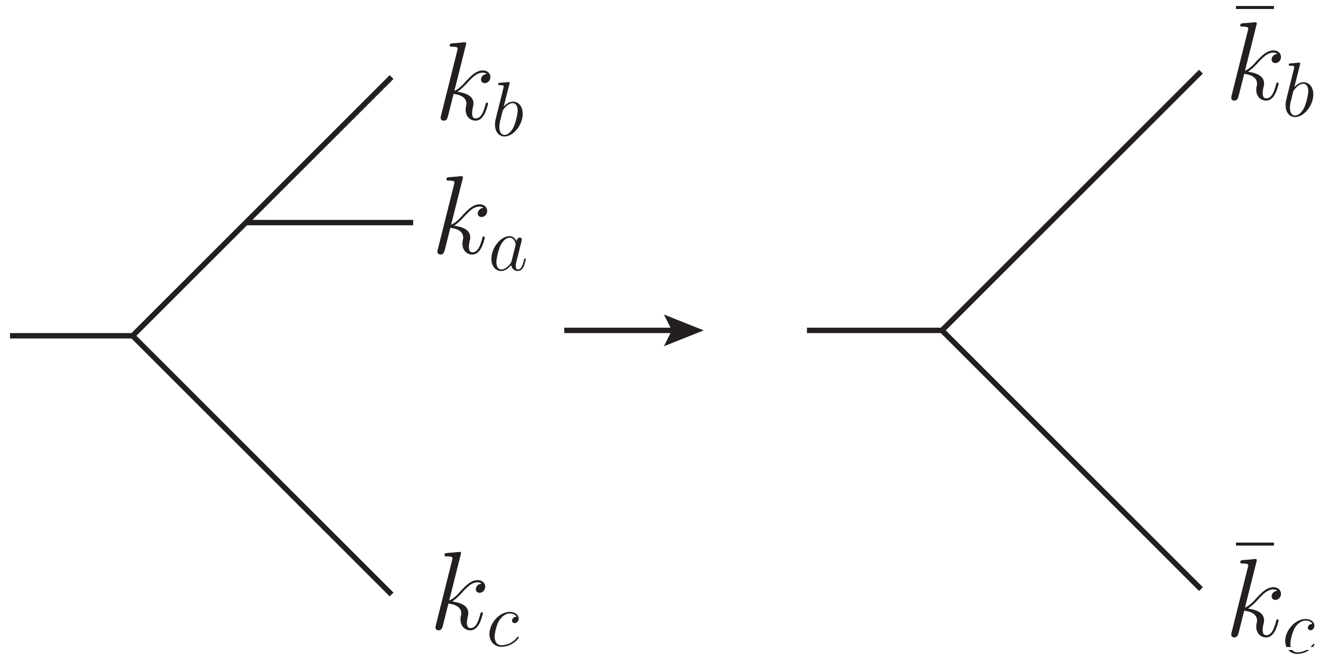
Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

The idea of mappings

Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\text{rad}}$

On-shell particle **conserving momentum** in the entire PS



$\downarrow$

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$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$

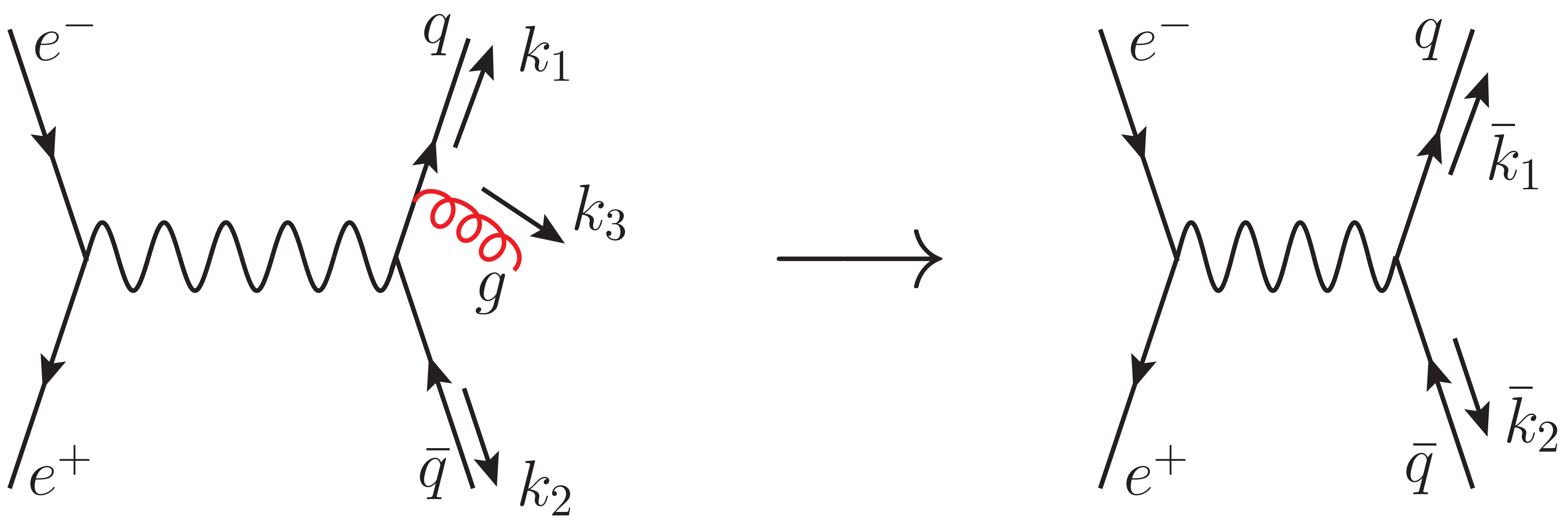
Different ways to combine momenta, depending on the **choice** of the dipole (abc)

$\rightarrow$ Freedom to choose the momenta to **simplify the integration**

$$k_1, k_2, k_3, \; k_i^2 = 0$$

$$\bar{k}_2^{(312)} = \frac{s_{312}}{s_{32} + s_{12}} k_2$$

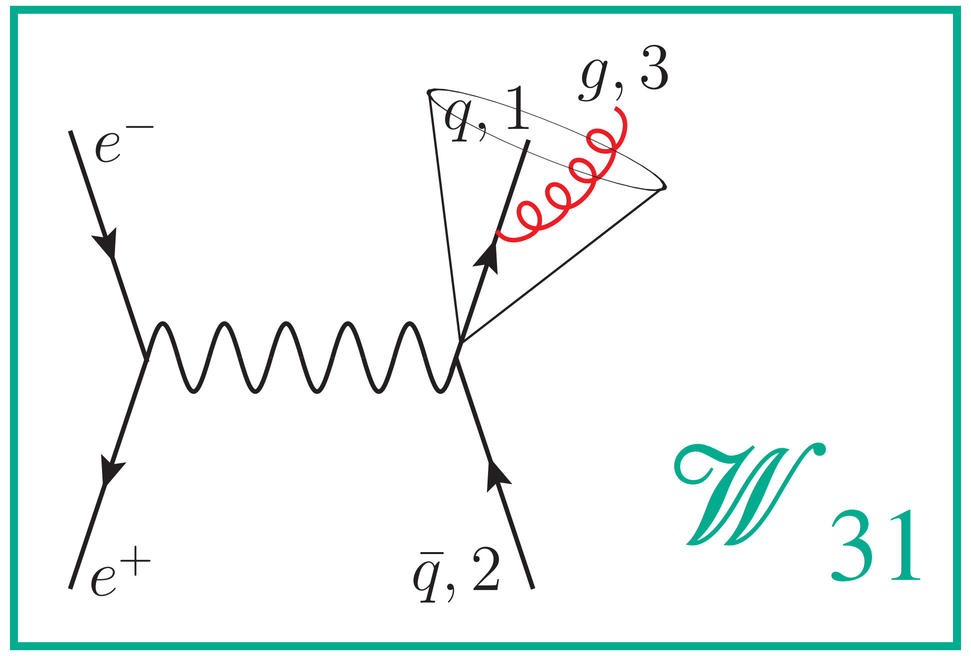
$$\bar{k}_1^{(312)} = k_3 + k_1 - \frac{s_{31}}{s_{32} + s_{12}} k_2$$



Ingredients of the subtraction

- *Candidate counterterm:*

Defined **sector by sector** as the collection of all the contributing limits (correct multiplicity!)



iterative definition $\left(1-\overline{\mathbf{S}}_3\right)\left(1-\overline{\mathbf{C}}_{13}\right) R\,\mathcal{W}_{31}$ = finite

$$\overline{K}_{31} = \left[\overline{\mathbf{S}}_3 + \overline{\mathbf{C}}_{13}\left(1-\overline{\mathbf{S}}_3\right)\right] R\,\mathcal{W}_{31} \;\to\; R\,\mathcal{W}_{31} - \overline{K}_{31} = \text{finite}$$

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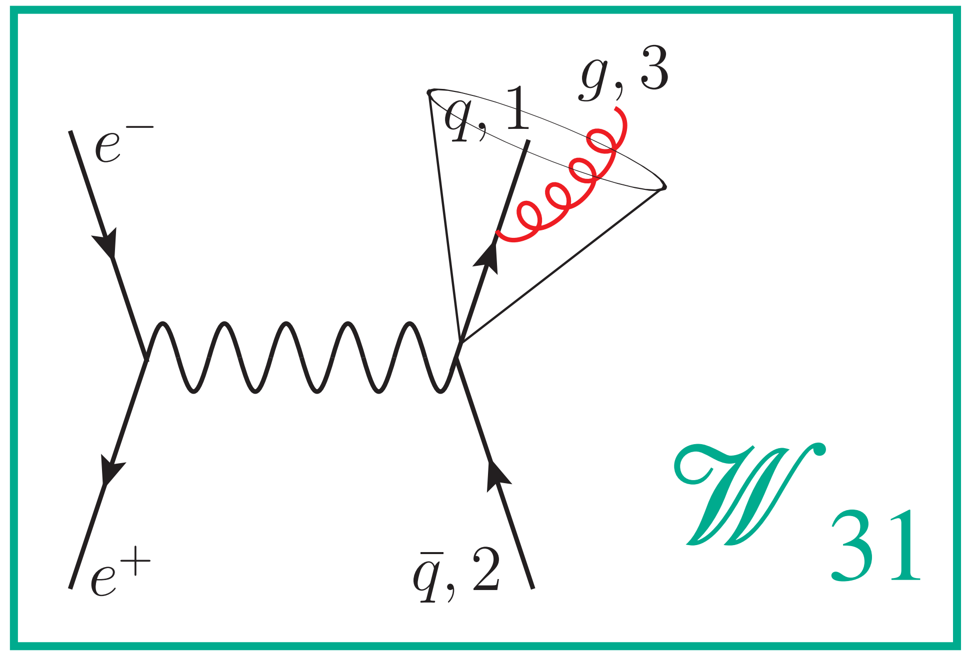

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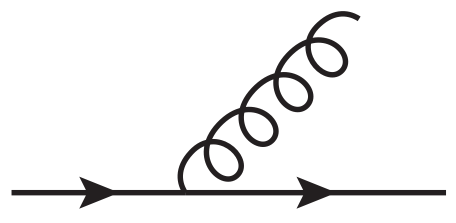
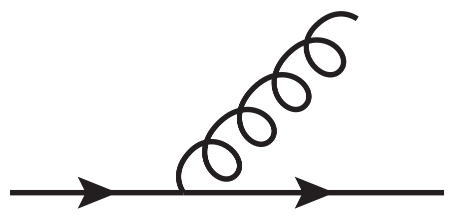
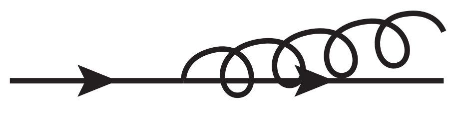
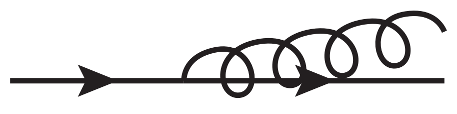
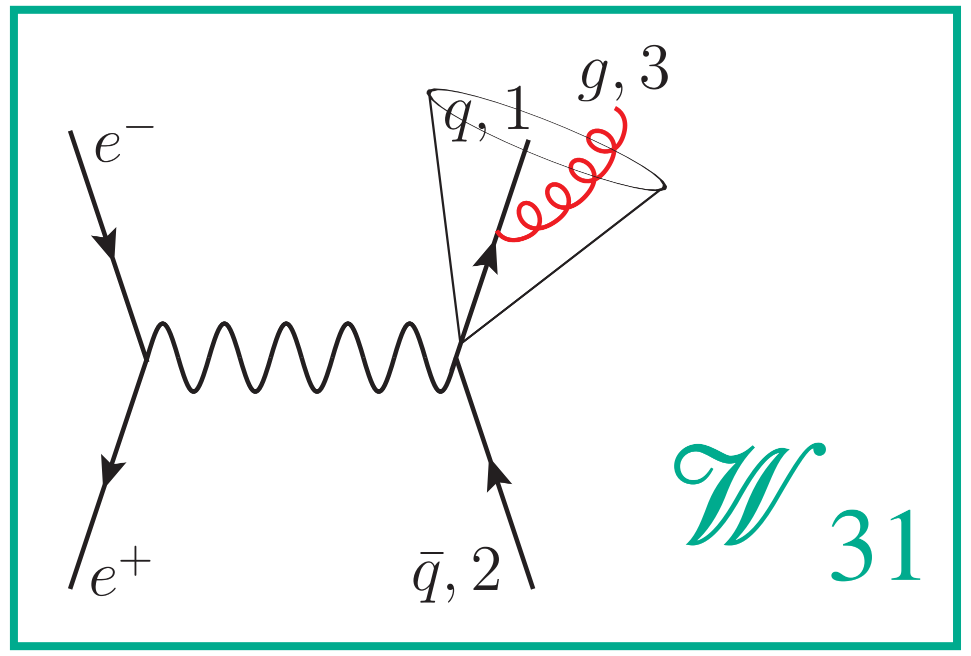
$\int$  $\mathcal{W}_{ij}\, d\Phi_{n+1} = \int \Big[$  $\mathcal{W}_{ij} -$  $\Big] d\Phi_{n+1} + \int$  $d\Phi_{n+1}$

Ingredients of the subtraction



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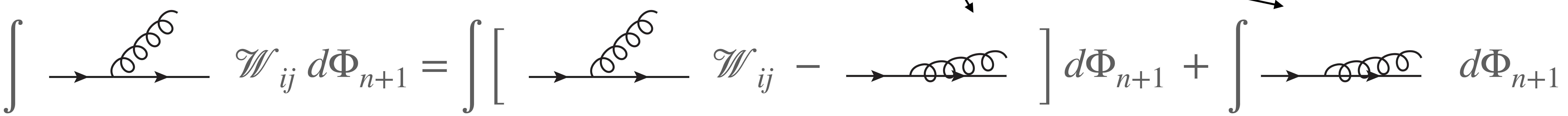

Featuring **optimised remapping** for integration $\{k_{n+1}\} \rightarrow \{k_n\}^{(abc)}$

(abc) **according to the invariants** appearing in the kernel

$$\overline{\mathbf{S}}_i\, R\left(\{k\}\right) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) \longrightarrow$$ Different mapping for each contribution

$$\overline{\mathbf{C}}_{ij}\, R\left(\{k\}\right) \propto \frac{1}{s_{ij}} P^{\mu\nu}_{ij} B_{\mu\nu}\left(\{k\}^{(ijr)}\right) \longrightarrow$$ Single mapping

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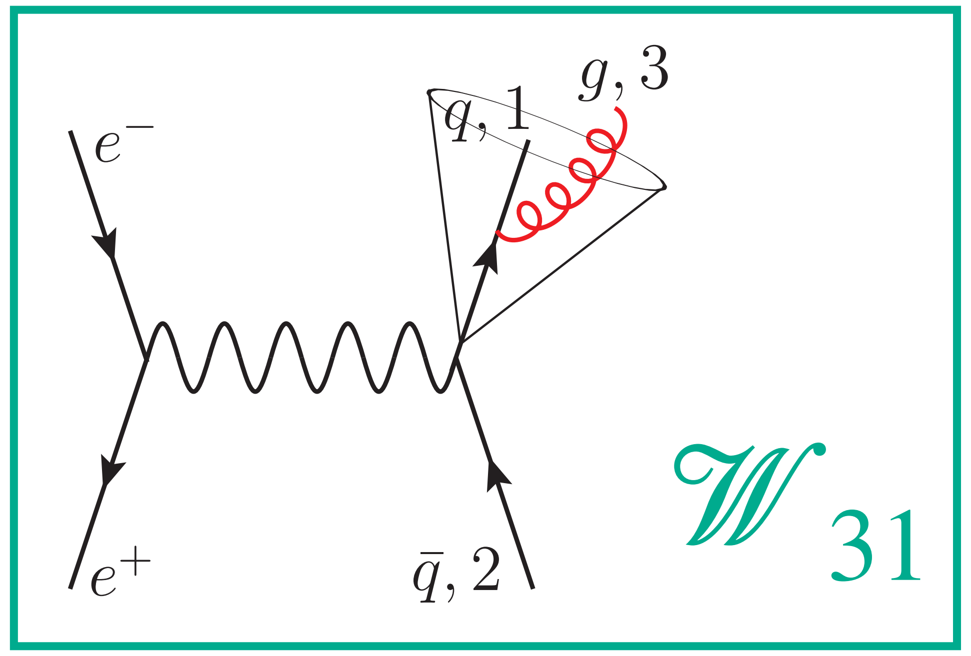

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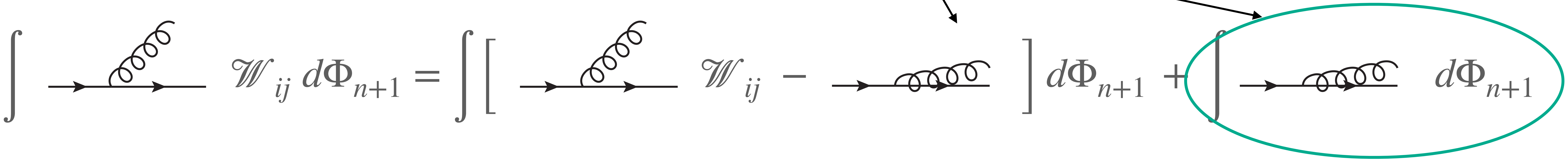

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Integration of NLO soft kernel

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\text{rad}}^{(abc)} = d\Phi_n^{(abc)} \times d\Phi_{\text{rad}}\left(s_{bc}^{(abc)}; y, z, \phi\right)$$

Catani-Seymour parameters: $s_{ab} = y s_{bc}^{(abc)}\,,\quad s_{ac} = z(1-y) s_{bc}^{(abc)}\,,\quad s_{bc} = (1-z)(1-y) s_{bc}^{(abc)}$

Radiative phase space: $d\Phi_{\text{rad}}^{(abc)} \propto \left(s_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon}$

Kernel to integrate: $\overline{\mathbf{S}}_i\, R\left(\{k\}\right) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right)$

$$I^s \propto \sum_{c,d\neq i} \int d\Phi_{\text{rad}}^{(icd)} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) = \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon} \frac{1-z}{z} B_{cd}\left(\{k\}^{(icd)}\right)$$

$$= \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} B_{cd}\left(\{k\}^{(icd)}\right)$$

General remarks:

1. Different parametrisation for the soft and for the hard-collinear counterterm
2. Each contribution to the soft is parametrised differently to simplify the integration

Sector functions at NLO in the analytic sector subtraction

Sector functions $\mathcal{W}_{ij}$:

1) Select the minimum number of singularities

$$\mathbf{S}_i \mathcal{W}_{ab} = 0\,, \qquad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = 0\,, \qquad \forall a, b \notin \{i, j\}\,.$$

2) Sum properties

$$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1 \qquad \mathbf{S}_i \sum_{j\neq i} \mathcal{W}_{ij} = 1\,, \qquad \mathbf{C}_{ij} \sum_{a,b\in\{ij\}} \mathcal{W}_{ab} = 1\,.$$

3) Explicit form

$$CM : q^\mu = (\sqrt{s}, \vec{0})\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad \omega_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,, \qquad \mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k,l\neq k} \sigma_{kl}}\,, \qquad \sigma_{ij} = \frac{1}{e_i\, \omega_{ij}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c\neq a} 1/\omega_{ac}}\,, \qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = \left(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}\right) \frac{e_b}{e_a + e_b}$$

Sector functions at NLO in the analytic sector subtraction

Sum over sectors before integration

Sector functions **sum rules** ⟶ Summing over sectors $\overline{K}$ becomes **independent** of $\mathscr{W}$

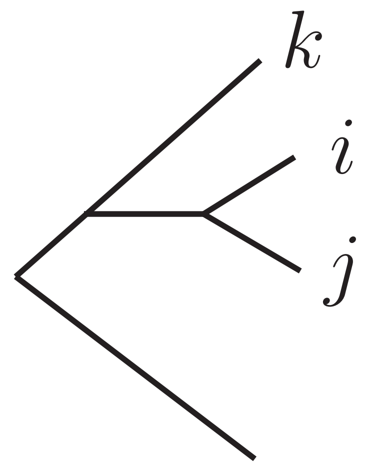
$$\overline{K} = \sum_{i,j} \overline{K}_{ij} \propto \overline{\mathbf{S}}_i R \Big[\overbrace{\sum_j \overline{\mathbf{S}}_i \mathscr{W}_{ij}}^{=1} \Big] + \overline{\mathbf{C}}_{ij} R \Big[\overbrace{\overline{\mathbf{C}}_{ij} \left(\mathscr{W}_{ij} + \mathscr{W}_{ji} \right)}^{=1} \Big] - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R \Big[\overbrace{\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \mathscr{W}_{ij}}^{=1} \Big]$$

$$\Longrightarrow \overline{K} = \sum_i \overline{\mathbf{S}}_i R + \sum_{i,j \neq i} \overline{\mathbf{C}}_{ij} \left(1 - \overline{\mathbf{S}}_i \right) R$$

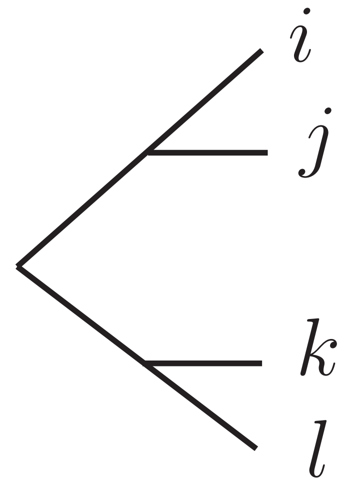
Remarks:

1. The integrated counterterm has to **match the poles of V**, which is **not split** into sectors
2. The sector functions would have made the **integration** much **more involved**

Sector functions at NNLO



$\mathcal{W}_{ijjk}\,, \qquad i \neq j \neq k$



$\mathcal{W}_{ijkj}\,, \qquad i \neq j \neq k \neq l$

Singularities selected: $\mathcal{W}_{abcd} \quad \begin{cases} a, c & \to \textit{soft} \\ ab, cd & \to \textit{collinear} \end{cases}$

Possible realisation of the desired properties:

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}\,, \qquad \sigma = \sum_{a, b \neq a} \sum_{\substack{c \neq a \\ d \neq a, c}} \sigma_{abcd} \quad \Longrightarrow \quad \sum_{a, b \neq a} \sum_{\substack{c \neq a \\ d \neq a, c}} \mathcal{W}_{abcd} = 1\,, \qquad \sigma_{abcd} = \frac{1}{(e_a w_{ab})^\alpha} \frac{1}{(e_c + \delta_{bc} e_a) w_{cd}}\,, \quad \alpha > 1$$

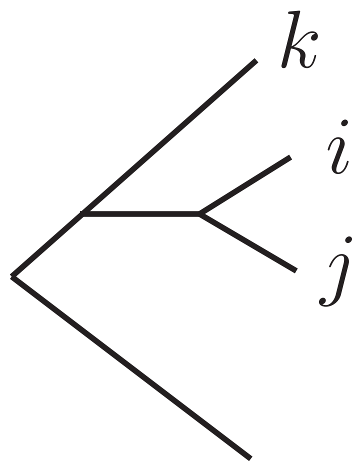
Limits selected by the topologies:

$$\mathcal{W}_{ijjk} \; : \; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkl} \; : \; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl}, \quad \mathbf{SC}_{kij}$$

NNLO sector functions factorise into products of NLO-type sector function under single-unresolved limits.

Identification of the counterterm in a given topology



$$\mathscr{W}_{ijjk} \qquad \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\underbrace{\left(1-\mathbf{S}_i\right)\left(1-\mathbf{C}_{ij}\right)}_{1-\mathbf{L}_{ij}^{(1)}} \quad \underbrace{\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\left(1-\mathbf{SC}_{ijk}\right)}_{1-\mathbf{L}_{ijjk}^{(2)}} RR\,\mathscr{W}_{ijjk} = \text{finite}$$

According to how the partons become unresolved we define:

$$K_{ijjk}^{(1)} \qquad K_{ijjk}^{(2)} \qquad K_{ijjk}^{(12)}$$

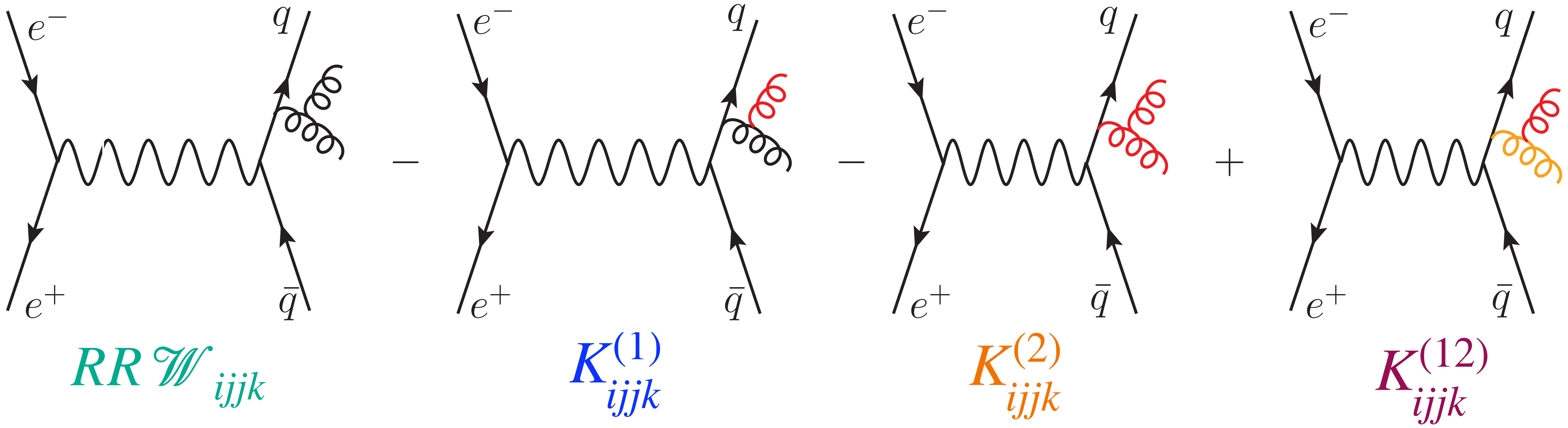
$$K_{ijjk}^{(\mathbf{1})} = \left[\mathbf{S}_i + \mathbf{C}_{ij}\left(1-\mathbf{S}_i\right)\right] RR\,\mathscr{W}_{ijjk}$$

$$K_{ijjk}^{(\mathbf{2})} = \left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\left(1-\mathbf{S}_{ij}\right) + \mathbf{SC}_{ijk}\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\right] RR\,\mathscr{W}_{ijjk}$$

$$K_{ijjk}^{(\mathbf{12})} = \left\{\left[\mathbf{S}_i + \mathbf{C}_{ij}\left(1-\mathbf{S}_i\right)\right]\left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\left(1-\mathbf{S}_{ij}\right) + \mathbf{SC}_{ijk}\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\right]\right\} RR\,\mathscr{W}_{ijjk}$$

The double real: main problems and solutions

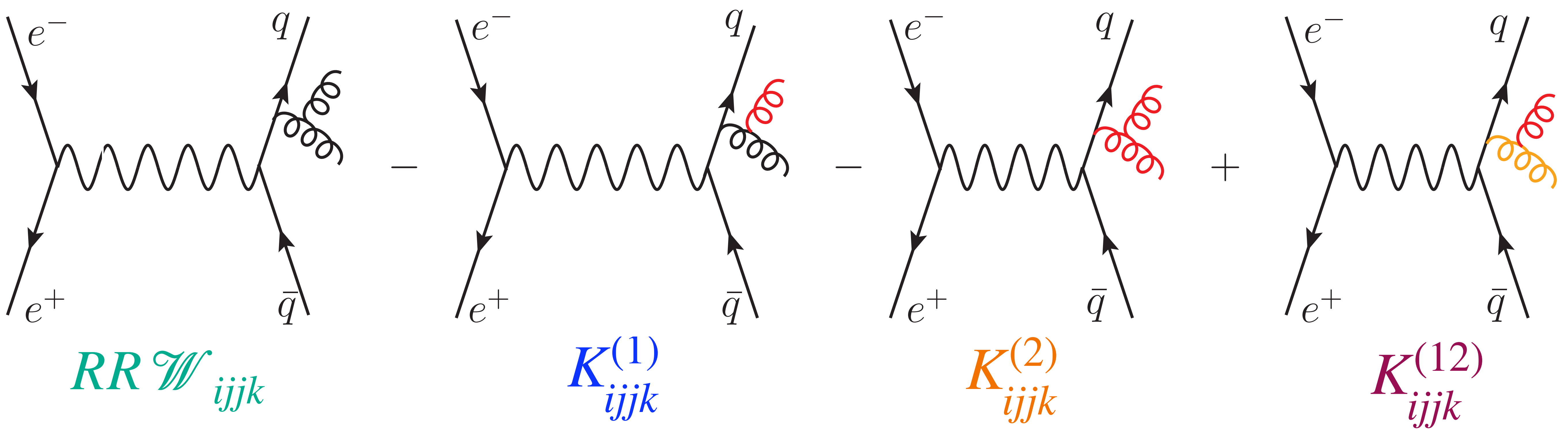
Transparent physical interpretation



$$\int d\Phi_{n+2}\Big[\, RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \big(\,K^{(2)}\; - K^{(12)}\big)\delta_n\,\Big]$$

The double real: main problems and solutions

Transparent physical interpretation



$$\int d\Phi_{n+2}\Big[\, RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \big(\, K^{(2)} \; - K^{(12)}\big)\delta_n \,\Big]$$

Next efforts:

1. **choose** an optimised **mapping**
2. **integrate** the counterterm over the **appropriate unresolved phase space**

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)}\,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}\,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad},1}\, K^{(12)}\,,$$

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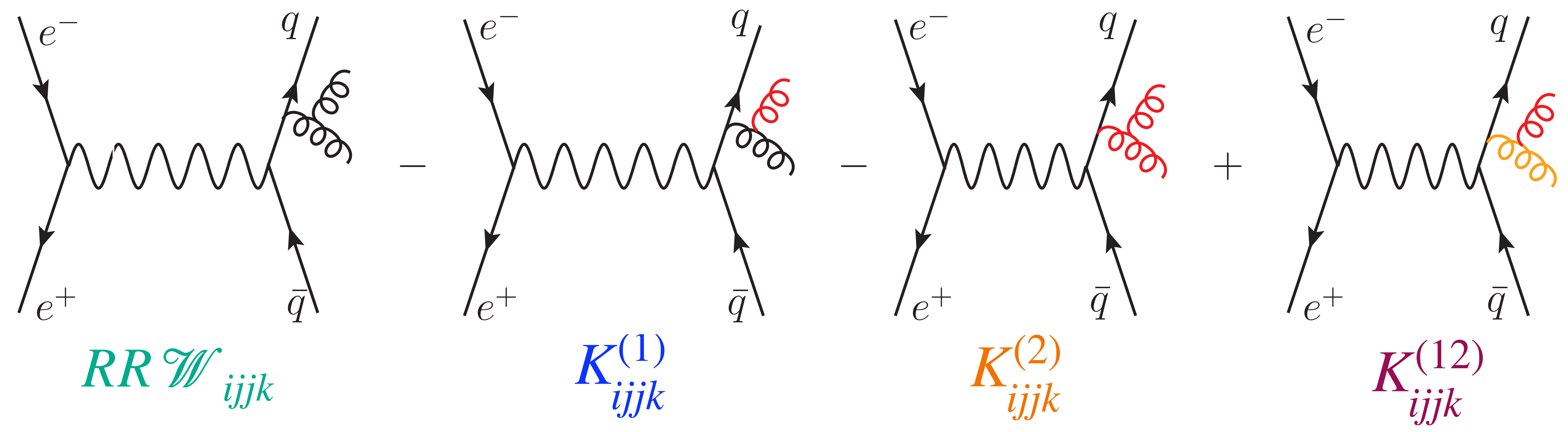

$$\int d\Phi_{n+2}\left[\, RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \left(K^{(2)} - K^{(12)}\right)\delta_n \,\right]$$

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↓

NNLO complexity: highly non trivial!

Singular structure of the RR

Under fundamental limits, the RR factorise into: (universal kernel) x (lower multiplicity matrix elements)

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}}\, P_{ij}^{\mu\nu}\big(s_{ir}, s_{jr}\big)\, P_{kl}^{\rho\sigma}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not{i}\not{j}\not{k}\not{l}}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P_{jk}^{\mu\nu}\, I_{cd}^{(i)}\, B_{\mu\nu}^{cd}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{jk}\big)$$

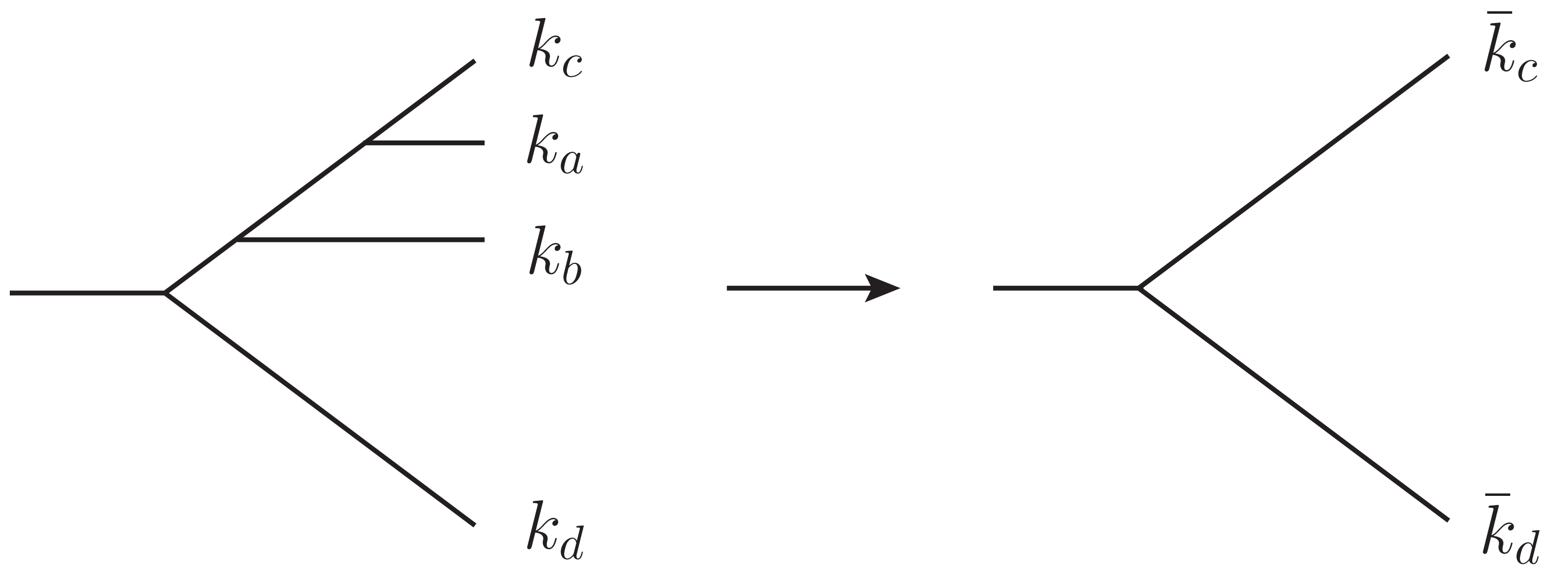
Born-level kinematics does not satisfy the mass-shell condition and momentum conservation

→ Momentum mapping needed!

NNLO momentum mapping

Two different kind of mapping to treat different kernels and simplify the integration

1. One-step mapping



$$\left\{\bar{k}_n^{(abcd)}\right\} = \left\{k_{\not{a}\not{b}\not{c}\not{d}}, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\right\}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\text{rad},2}\left(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x'\right)$$

$$\int d\Phi_{\text{rad},2} \propto \left(\bar{s}_{cd}^{(abcd)}\right)^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[w'(1-w')\right]^{-1/2-\epsilon} \left[y'(1-y')^2\, z'(1-z')\, y^2(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')\, y(1-y)$$

2. Two-step mapping

$$\left\{\bar{k}_n^{(acd,bef)}\right\} = \left\{\bar{k}^{(acd)}_{\not{a}\not{b}\not{e}\not{f}}, \bar{k}_e^{(acd,bef)}, \bar{k}_f^{(acd,bef)}\right\}$$

$$d\Phi_{n+2} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\text{rad},2}^{(acd,bef)} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\text{rad},1}\left(\bar{s}_{ef}^{(acd,bef)}; y, z, \phi\right) \cdot d\Phi_{\text{rad},1}\left(\bar{s}_{cd}^{(acd)}; y', z', \phi'\right)$$

$$d\Phi_{\text{rad},2}^{(acd,bef)} \propto \left(\bar{s}_{cd}^{(acd,bef)} \bar{s}_{ef}^{(acd,bef)}\right)^{1-\epsilon} \int_0^\pi d\phi'\, (\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2\, z'(1-z')\, y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')(1-y)$$

Triple-collinear singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{ijk}}, k_{ijk}\big)$$

$$P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2}\right] + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}}\left[\frac{2 z_i z_j z_{ik}(1 - 2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i(1 + 2z_i) - 4\right] + \frac{3(1-\epsilon)}{4}\right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2}\right)\frac{1}{s_{ik}}\right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k}\frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

How the results look like:

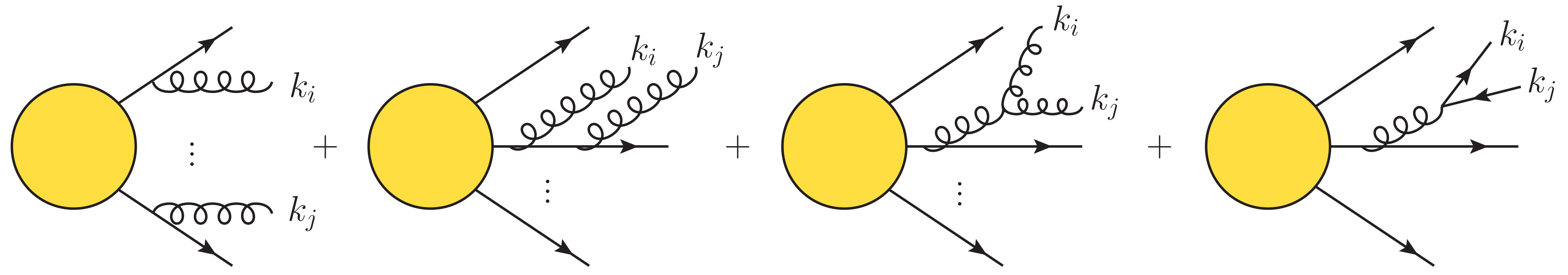
$$\int d\Phi_{n+2}\, \overline{\mathbf{C}}_{ijk}\, RR \;=\; \int d\Phi_n\big(\bar{k}^{(ijrk)}\big)\, J_{cc}\big(\bar{s}_{kr}^{ijkr}\big)\, B\big(\bar{k}^{(ijrk)}\big)$$

$$J_{cc}^{(3g)}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[\frac{15}{\epsilon^4} + \frac{63}{\epsilon^3} + \left(\frac{853}{3} - 22\,\pi^2\right)\frac{1}{\epsilon^2} + \left(\frac{10900}{9} - \frac{275}{3}\,\pi^2 - 376\,\zeta_3\right)\frac{1}{\epsilon} + \frac{180739}{36} - \frac{3736}{9}\,\pi^2 - 1555\,\zeta_3 + \frac{41}{10}\,\pi^4 + \mathcal{O}(\epsilon)\right]$$

Integration over the double phase space: example

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$



We are free to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel itself

$$\overline{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c\,\neq\, i,j \\ d\,\neq\, i,j,c}} \left[\sum_{\substack{e\,\neq\, i,j,c,d \\ f\,\neq\, i,j,c,d}} I_{cd}^{(i)}\, \bar{I}_{ef}^{(j)(icd)} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e\,\neq\, i,j,c,d} I_{cd}^{(i)}\, \bar{I}_{ed}^{(j)(icd)} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big) \right.$$
$$\left. + 2\, I_{cd}^{(i)}\, I_{cd}^{(j)} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I_{cd}^{(ij)} - \frac{1}{2}\, I_{cc}^{(ij)} - \frac{1}{2}\, I_{dd}^{(ij)}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) \right]$$

The PS parametrisation follows the mapping structure

$$I^{(2)}_{\mathrm{SS},cdef} = \int d\Phi_{\mathrm{rad},2}\, I_{cd}^{(i)}\, \bar{I}_{ef}^{(j),(icd)} = \int d\overline{\Phi}^{(icd,jef)}_{\mathrm{rad}}\, \bar{I}_{ef}^{(j),(icd)} \int d\Phi^{(icd)}_{\mathrm{rad}}\, I_{cd}^{(i)} = \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{cd}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)} \frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{ef}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\, \Gamma(2-3\epsilon)}$$

Some of the double-soft kernel structures feature a NLOxNLO complexity -> integration exact in ϵ

The most difficult part arises from the pure NNLO current.

Integration over the double phase space: example

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi_{n+2}^{(ijcd)}\,I_{cd}^{(ij)}\,B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

Mapping: $\{\bar{k}\}^{(ijcd)}$.

Catani-Seymour parameters y', z', y, z:

$$s_{ij} = y'\,y\,\bar{s}_{cd}^{(ijcd)}\,, \qquad s_{ic} = z'(1-y')\,y\,\bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{cd} = (1-y')(1-y)(1-z)\,\bar{s}_{cd}^{(ijcd)} \qquad s_{jc} = (1-y')(1-z')\,y\,\bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{id} = (1-y)\left[y'(1-z')(1-z)+z'z-2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right]\bar{s}_{cd}^{(ijcd)}\,,$$

$$s_{jd} = (1-y)\left[y'z'(1-z)+(1-z')z+2(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right]\bar{s}_{cd}^{(ijcd)}\,.$$

Use partial fractioning to isolate complicated denominators $\quad \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id} + s_{jd}}\left(\frac{1}{s_{id}} + \frac{1}{s_{jd}}\right)$

Use symmetries of the 4-partons of the phase space *[De Ridder, Gehrmann, Heinrich 0311276]* $\quad \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id} + s_{jd}}\left(\frac{1}{s_{id}} + \frac{1}{s_{jd}}\right) \xrightarrow{k_i \leftrightarrow k_j} \frac{1}{s_{id}\,s_{jd}} = \frac{1}{s_{id} + s_{jd}}\,\frac{2}{s_{jd}}$

Parametrise the PS using Catani-Seymour parameters

$$\int d\Phi_{\mathrm{rad},2}^{(ijcd)} = 2^{-4\epsilon}\,N^2(\epsilon)\,\left(\bar{s}_{cd}^{(ijcd)}\right)^{2-2\epsilon}\int_0^1 dx'\int_0^1 dy'\int_0^1 dz'\int_0^1 dx\,\left[x(1-x)\right]^{-1/2-\epsilon}\int_0^1 dy\int_0^1 dz\,\left[x'(1-x')\right]^{-1/2-\epsilon}\left[y'(1-y')^2\,z'(1-z')\,y^2(1-y)^2\,z(1-z)\right]^{-\epsilon}(1-y')\,y(1-y)$$

Integration over the double phase space: example

$$\int d\Phi_{n+2}\,\overline{\mathbf{S}}_{ij}\,RR(\{k\}) \propto \int d\Phi_{n+2}^{(ijcd)}\,I_{cd}^{(ij)}\,B_{cd}\Big(\{\bar{k}^{(ijcd)}\}\Big)$$

$$I_{cd}^{(gg)(ij)} = \frac{(1-\epsilon)(s_{ic}\,s_{jd} + s_{id}\,s_{jc}) - 2s_{ij}\,s_{cd}}{s_{ij}^2\,(s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\,\frac{s_{ic}\,s_{jd} + s_{id}\,s_{jc} - s_{ij}\,s_{cd}}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}}\left[1 - \frac{1}{2}\,\frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\int d\Phi_{n+2}^{(ijcd)}\,\frac{s_{ij}\,s_{cd}^2}{s_{ij}\,s_{ic}\,s_{id}\,s_{jd}\,s_{jc}} \propto \int_0^1 \frac{dx'\,dy'\,dz'\,dx\,dy\,dz\,(z-1)^2(1-y)^{1-2\epsilon}y^{-2\epsilon-1}(1-y')^{1-2\epsilon}y'^{-\epsilon}[(1-z)z]^{-\epsilon}[(1-z')z']^{-\epsilon-1}}{[x(1-x)x'(1-x')]^{\epsilon+1/2}(y'(z-1)-z)\left(y'z'(1-z)+(1-z')z+2(2x'-1)\sqrt{y'(z-1)z(z'-1)z'}\right)}$$

- Integrate over x $\rightarrow$ simple Beta functions
- Integrate over y $\rightarrow$ simple Beta function
- Integrate over x' $\rightarrow$ Master Integral $I_{x'}$ $\rightarrow$ Hypergeometric and Theta functions
- Integrate over z' $\rightarrow$ partial fractioning $\frac{I_{x'}}{[z'(1-z')]^{1+\epsilon}} = \frac{I_{x'}}{[z'(1-z')]^{\epsilon}}\left[\frac{1}{z} + \frac{1}{1-z}\right]$
  $\rightarrow$ Master Integral $I_{x'z'} + J_{x'z'}$ $\rightarrow$ Hypergeometric functions
- Integrate over z $\rightarrow$ Integral representation of Hyp. $\rightarrow$ auxiliary t variable
- Integrate over y' $\rightarrow$ poles extraction

Subtraction pattern for Local Analytic Sector

Subtraction pattern at NNLO

$$\frac{d\sigma^{\mathrm{NNLO}}}{dX} = \int d\Phi_n \left[\underbrace{VV_n + I^{(2)} + I^{(RV)}}_{\text{finite in d=4 and in } \Phi_n} \right] \delta_n$$

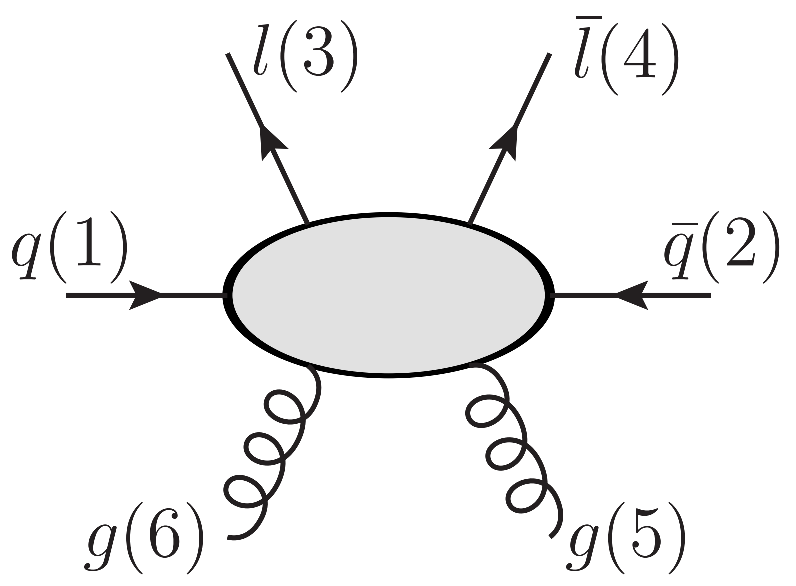
$$+ \int d\Phi_{n+1} \left[\underbrace{\underbrace{\left(RV_{n+1} + I^{(1)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_{n+1} - \underbrace{\left(\overline{K}^{(RV)} + I^{(12)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_n}_{\text{finite in d=4 and in } \Phi_{n+1}} \right]$$

$$+ \int d\Phi_{n+2} \left[\underbrace{RR_{n+2}\, \delta_{n+2} - \overline{K}^{(1)} \delta_{n+1} - \left(\overline{K}^{(2)} - \overline{K}^{(12)}\right) \delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \right]$$

Phase space partitions

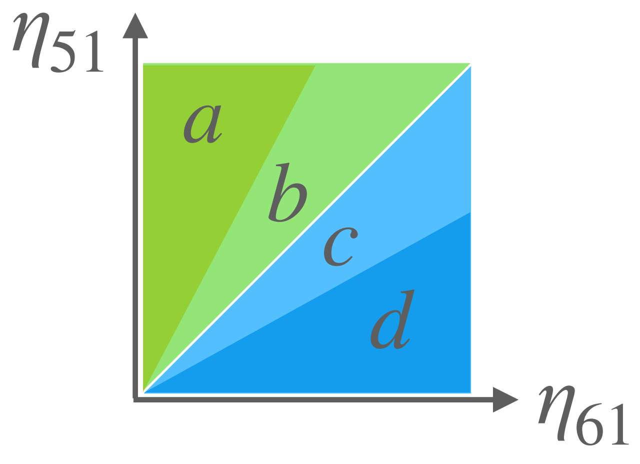
Partition functions as defined in nested-soft-collinear subtraction can be easily adapted to both mixed QCDxEQ and QCD processes

Examples: $q\bar{q} \to Z \to e^- e^+ g\, g$ *[Caola, Melnikov, Röntsch 1702.01352]*

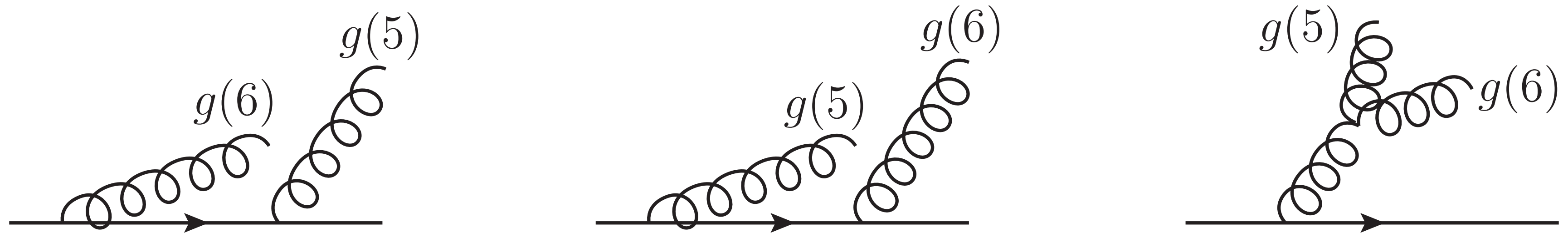


$$1 = \sum_i \omega^i\,, \qquad i \in \{(51{,}61), (52{,}62), (51{,}62), (52{,}61)\}$$

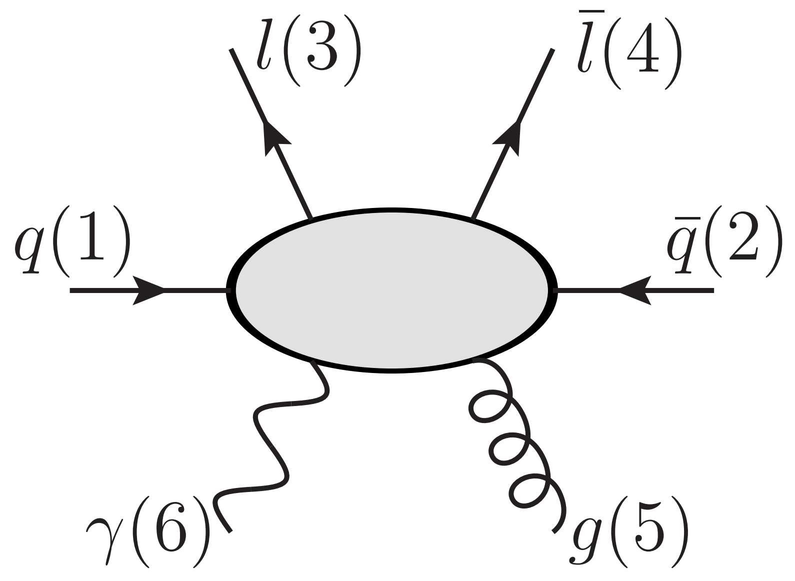
$$\omega^{51,61} = \frac{\rho_{25}\,\rho_{26}}{d_5\,d_6}\left(1 + \frac{\rho_{15}}{d_{5621}} + \frac{\rho_{16}}{d_{5612}}\right) \qquad \omega^{52,62} = \frac{\rho_{15}\,\rho_{16}}{d_5\,d_6}\left(1 + \frac{\rho_{25}}{d_{5621}} + \frac{\rho_{26}}{d_{5612}}\right) \qquad \omega^{51,62} = \frac{\rho_{25}\,\rho_{16}\,\rho_{56}}{d_5\,d_6\,d_{5612}} \qquad \omega^{52,61} = \frac{\rho_{15}\,\rho_{26}\,\rho_{56}}{d_5\,d_6\,d_{5621}}$$



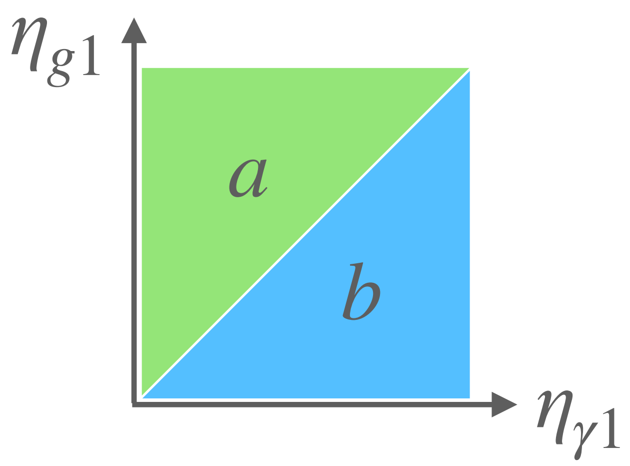
$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right) = \theta_a + \theta_b + \theta_c + \theta_d$$



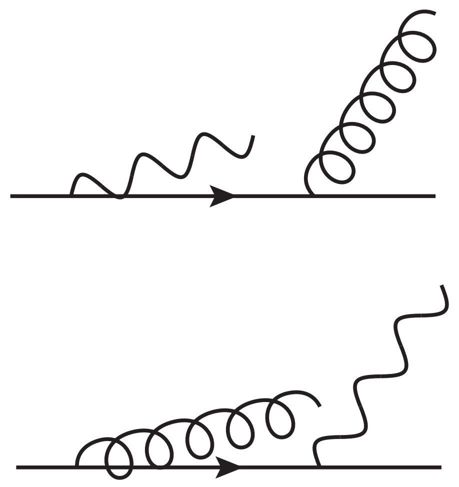
Examples: $q\bar{q} \to Z \to e^- e^+ g\,\gamma$ *[Buccioni, Caola, Chawdhry, Devoto, Melnikov, Röntsch, C.S., In preparation]*



$$1 = \sum_{i=1}^{4} \sum_{j=1}^{2} \omega^{\gamma i, g j}\,, \qquad \omega^{\gamma i, g j} = \eta_{gk}\,\frac{1/\eta_{i\gamma}}{\sum_{m=1}^{4} 1/\eta_{m\gamma}}\,, \qquad \eta_{ab} = \frac{1 - \cos\vartheta_{ab}}{2}$$



$$1 = \theta_a + \theta_b = \theta(\eta_{g1} - \eta_{\gamma 1}) + \theta(\eta_{\gamma 1} - \eta_{g1})$$



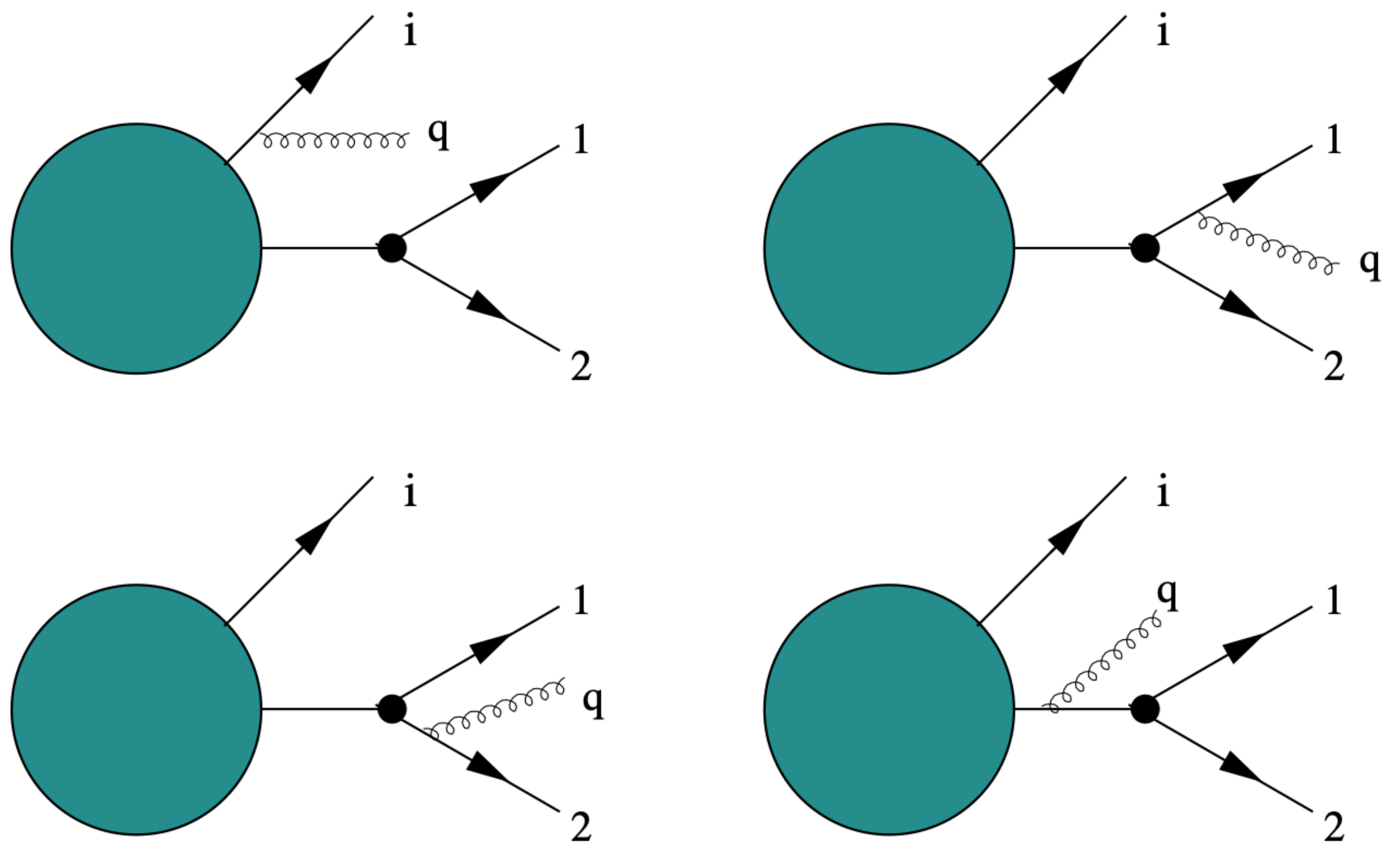
TC: 1 || 5 || 6
DC: 1 || 6

TC: 1 || 5 || 6
DC: 1 || 5

Colour coherence and disentangled soft-collinear singularities

Parton q is soft and partons $1,2$ are collinear *[Catani, Grazzini 9908523]*

$$\left|\mathscr{M}_{g,a_1,a_2,\ldots,a_n}(q,p_1,p_2,\ldots,p_n)\right|^2 \simeq -\frac{2}{s_{12}}(4\pi\mu^{2\epsilon}\alpha_s)^2 \left\langle \mathscr{M}_{a,a_3,\ldots,a_n}(p,p_3,\ldots,p_n)\right|\hat{\mathbf{P}}_{a_1a_2}\left[\mathbf{J}^{\dagger}_{(12)\mu}(q)\,\mathbf{J}^{\mu}_{(12)}(q)\right]\left|\mathscr{M}_{a,a_3,\ldots,a_n}(p,p_3,\ldots,p_n)\right\rangle$$



Mother parton: $a \to a_1 + a_2$

Altarelli-Parisi splitting functions: spin correlations

Soft current: colour correlations

$$\mathbf{J}^{\dagger}_{(12)\mu}(q)\,\mathbf{J}^{\mu}_{(12)}(q) \simeq \sum_{i,j=3}^{n}\mathbf{T}_i\cdot\mathbf{T}_j\,\mathscr{S}_{ij}(q) + 2\sum_{i=3}^{n}\mathbf{T}_i\cdot\mathbf{T}_{(12)}\,\mathscr{S}_{i\,(12)}(q)$$

$$\mathbf{T}_{(12)} = \mathbf{T}_1 + \mathbf{T}_2 \qquad \mathscr{S}_{ij}(q) = \frac{2\,s_{ij}}{s_{iq}\,s_{jq}} \qquad \mathscr{S}_{i\,(12)}(q) = \frac{2\,(s_{i1}+s_{i2})}{s_{iq}\,(s_{1q}+s_{2q})}$$

The soft-collinear limit at $\mathscr{O}\left(\alpha_s^2\right)$ is fully described in a factorised way, where the factors are the soft eikonal function and the Altarelli-Parisi splitting functions that control IR limits at $\mathscr{O}\left(\alpha_s\right)$.

This simplification, which is due to colour coherence, was not performed in FKS.