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Feynman integrals beyond modular forms

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based on work in collaboration with
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and
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Amplitudes 2021
20 August 2021

Feynman integrals

- Feynman integrals are the building blocks for multi-loop scattering amplitudes.
  - ➡ Important both for collider and gravitational wave phenomenology.

- They exhibit a rich mathematical structure. [See talks by S. Weinzierl, F. Brown, R. Britto, …]
  - ➡ Connections to algebraic geometry, modular forms,…

- The understanding of the case of polylogarithms and elliptic curves is now relatively well advanced. [See talks by S. Weinzierl & C. Zhang]
  - ➡ The spaces of special functions are (relatively) well understood.

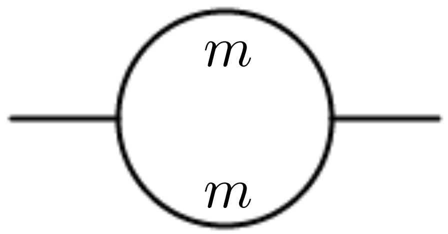
Feynman integrals

- We know that elliptic curves are not the end of the story!
  - ➡ Also Calabi-Yau varieties appear.

    [See talk by A. Klemm]

    [Brown, Schnetz; Bloch, Kerr, Vanhove; Bourjaily, He, McLeod, von Hippel, Wilhelm; Bourjaily, McLeod, von Hippel, Wilhelm; Bourjaily, McLeod, Vergu, Volk, von Hippel; Klemm, Nega, Safari; Bönisch, Fischbach, Klemm, Nega; …]
- This case is still poorly explored and understood.
- Goals of this talk:
  - ➡ Study the simplest examples of an infinite family of l-loop integrals associated to Calabi-Yau $(l-1)$-folds.
  - ➡ Show how the geometric concepts generalise from the known elliptic cases to Calabi-Yau cases.
  - ➡ Present an all-loop generalisation of known results for $l \leq 3$.

Banana integrals in 2D

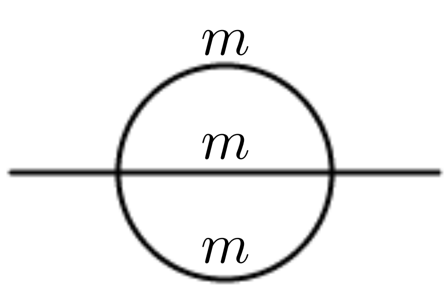


Riemann sphere

$$z = \frac{m^2}{p^2}$$

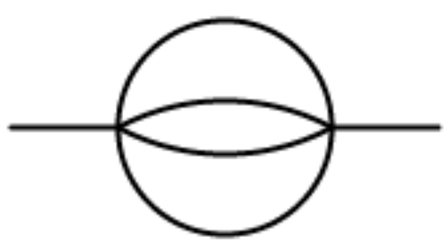
$$\sim \frac{z}{\sqrt{1-4z}} \int_0^z \frac{dz'}{z'\sqrt{1-4z'}}$$

$$\sim \frac{z}{\sqrt{1-4z}} \log \frac{1-\sqrt{1-4z}}{1+\sqrt{1-4z}}$$



Elliptic curve

$$\sim h_1(\tau) \int_{i\infty}^{\tau} \frac{\mathrm{d}\tau'}{2\pi i} h_3(\tau')\, \tau'$$



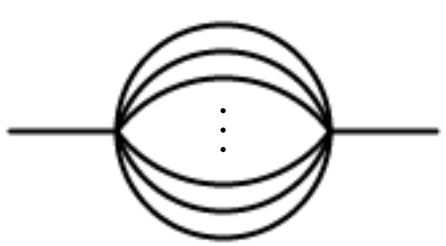
K3 surface
~ "elliptic x elliptic"
(only here!)

$$\sim h_2(\tau) \int_{i\infty}^{\tau} \frac{\mathrm{d}\tau'}{2\pi i} h_4(\tau')\, \tau'^2$$

h_n = 'modular form of weight n for $\Gamma_1(6)$'

[See talk by S. Weinzierl]

⋮



Calabi-Yau
$(l-1)$-fold

?

Modular forms

- $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a,b,c,d \in \mathbb{Z} \text{ and } \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1 \right\}$

  $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$

- Modular function for Γ : meromorphic function $f : \mathbb{H} \to \mathbb{C}$ s.t.

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$$

- Modular of weight k for Γ: holomorphic function $h_k : \mathbb{H} \to \mathbb{C}$ s.t.

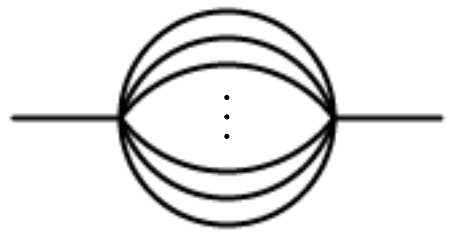
$$h_k\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\, h_k(\tau)$$

[+ holomorphicity at the cusps]

[See talk by S. Weinzierl]

Differential equations

- Master integrals for l-loop equal-mass bananas ($D = 2 - 2\epsilon$) :

$J_{l,1}(z;\epsilon) \sim$  $\quad J_{l,k}(z;\epsilon) \sim \partial_z^{k-1} J_{l,1}(z;\epsilon) \qquad 2 \leq k \leq l$

- The vector of MIs satisfies a system of differential equations:

$$\partial_z \underline{J}_l(z;\epsilon) = \mathbf{B}_l(z;\epsilon)\, \underline{J}_l(z;\epsilon) + \underline{N}_l(z,\epsilon)$$

Rational matrix

Inhomogeneity from integrals with fewer propagators

$$\mathbf{B}_l(z;\epsilon) = \mathbf{B}_{l,0}(z) + \sum_{k=1}^{l} \mathbf{B}_{l,k}(z)\epsilon^k$$

$$\underline{N}_l(z,\epsilon) = \left(0,\ldots,0,(-1)^{l+1}(l+1)!\,\frac{z}{z^l\prod_{k\in\Delta^{(l)}}(1-kz)}\frac{\Gamma(1+\epsilon)^l}{\Gamma(1+l\epsilon)}\right)^T$$

Differential equations

- Step 1: Solve the homogeneous system at $\epsilon = 0$:

$$\partial_z \underline{J}_l^{\Gamma}(z) = \mathbf{B}_{l,0}(z)\underline{J}_l^{\Gamma}(z) \qquad \longleftrightarrow \qquad \mathcal{L}_l J_{l,1}^{\Gamma} = 0$$

➡ Solution space spanned by maximal cuts:

[Tancredi, Primo; Frellesvig, Papadopoulos; Bosma, Larsen, Zhang]

$$\mathbf{W}_l(z) := \left(\underline{J}_l^{\Gamma_1}(z), \ldots, \underline{J}_l^{\Gamma_l}(z)\right)$$

- Step 2: Change basis: $\underline{J}_l(z,\epsilon) = \mathbf{W}_l(z)\underline{L}_l(z,\epsilon)$

$$\partial_z \underline{L}_l(z,\epsilon) = \widetilde{\mathbf{B}}_l(z,\epsilon)\underline{L}_l(z,\epsilon) + \underline{\widetilde{N}}_l(z,\epsilon)$$

$$\widetilde{\mathbf{B}}_l(z,\epsilon) = \mathbf{W}_l(z)^{-1}[\mathbf{B}_l(z,\epsilon) - \mathbf{B}_{l,0}(z)]\mathbf{W}_l(z) = \mathcal{O}(\epsilon)$$

$$\underline{\widetilde{N}}_l(z,\epsilon) = \mathbf{W}_l(z)^{-1}\underline{N}_l(z,\epsilon)$$

Polynomial of degree $l-1$ and l in the entries of $\mathbf{W}_l(z)$.

Differential equations

- Step 3: The new system can easily be solved order by order in ϵ .
  - ➡ Example: $\epsilon = 0$

$$\underline{L}_l(z,0) = \underline{L}_l(0,0) + \int_0^z dz'\, \underline{\widetilde{N}}_l(z',0)$$

Integrand involves polynomials of degree $l-1$ in the entries of $\mathbf{W}_l(z)$.

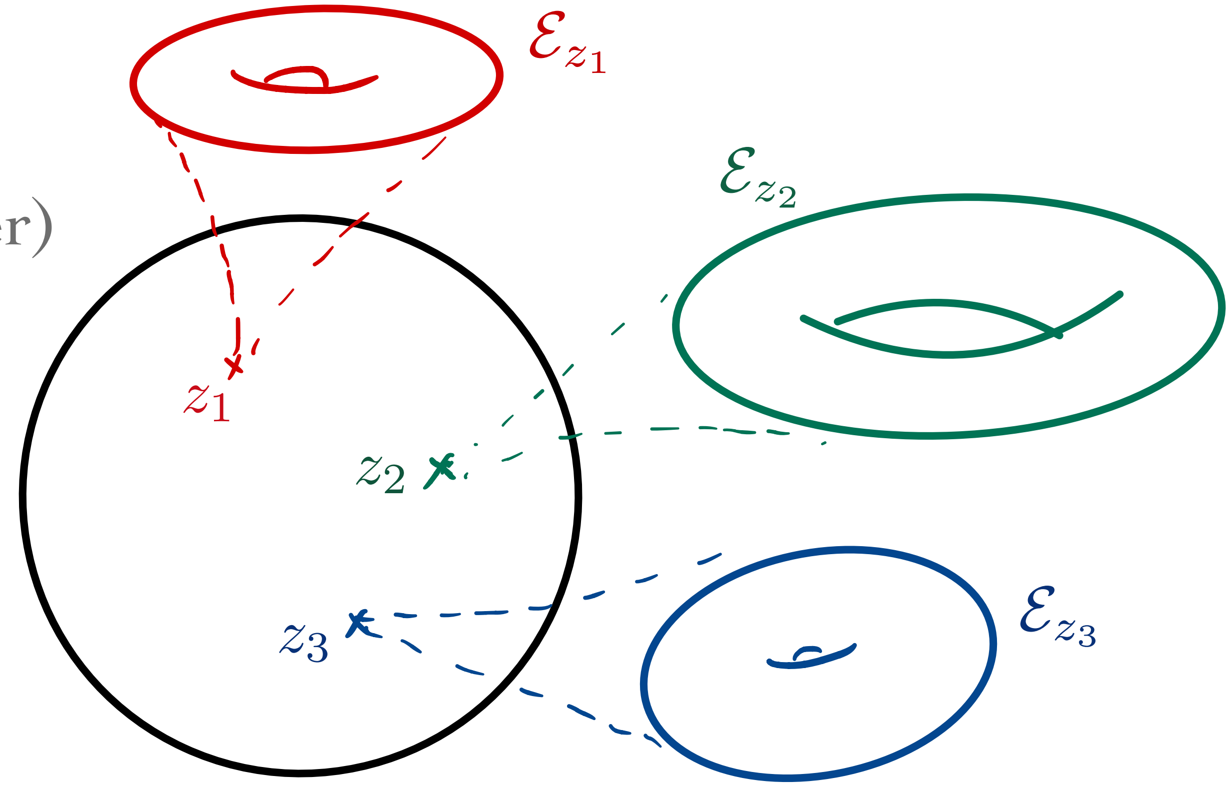
$$= \underline{L}_l(0,0) + \int_0^z dz'\, \mathbf{W}_l(z')^{-1} \underline{N}_l(z',0)$$

- Questions:
  - ➡ What are the entries of $\mathbf{W}_l(z)$?
  - ➡ What are their properties, relations, etc.?
  - ➡ Why do we get modular forms for 2 (and 3) loops?
  - ➡ What about the integrals we get?

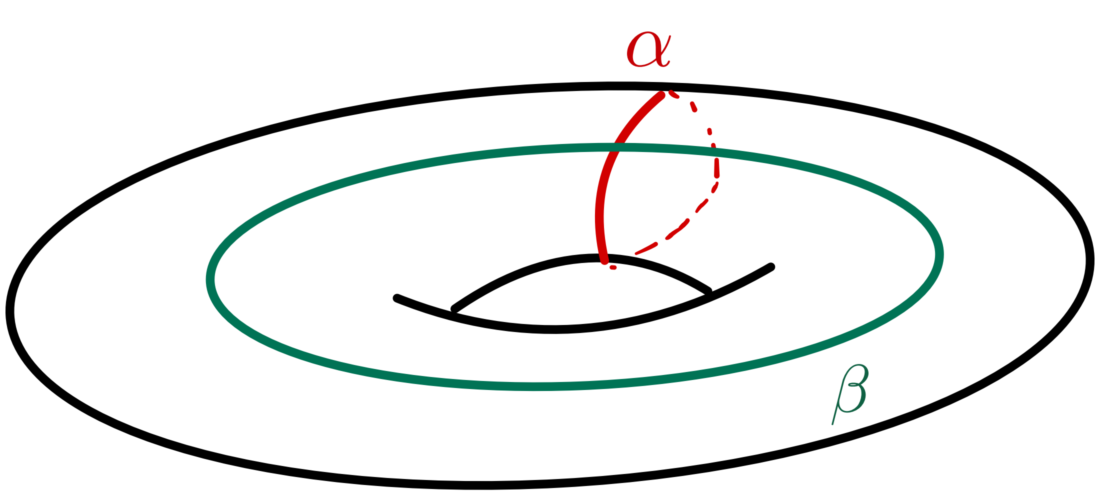
2-loop banana integrals
and
elliptic curves

Elliptic curves

- The 2-loop (equal-mass) banana family is associated to a (1-parameter) family of elliptic curves
  - ➡ For every $z \in \mathbb{CP}^1$ there is a (possibly degenerate) elliptic curve $\mathcal{E}_z$.



- Elliptic curve: Riemann surface of genus 1 = Torus



➡ There is a unique holomorphic differential 1-form $\frac{\mathrm{d}x}{y}$.

➡ Characterised by its periods:

$$\Pi_\alpha = \int_\alpha \frac{\mathrm{d}x}{y} \qquad \Pi_\beta = \int_\beta \frac{\mathrm{d}x}{y}$$

Picard-Fuchs equation

- Here: Periods will be functions of z:
  - ➡ z-dependence governed by Picard-Fuchs equation ($\theta = z\partial_z$):

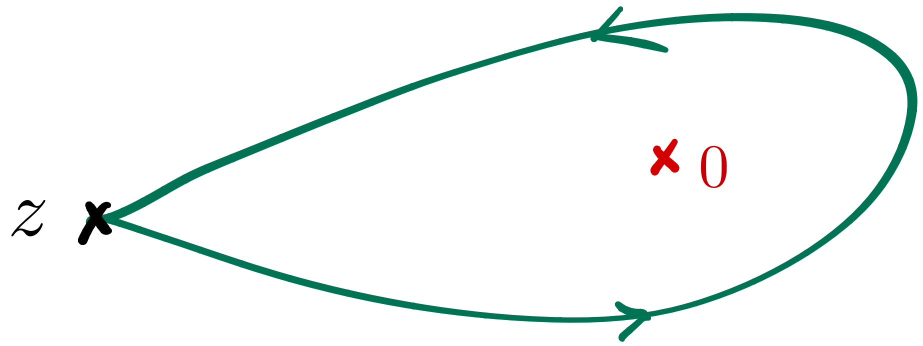
$$\mathcal{L}_2 \Pi_\gamma(z) = 0 \qquad \mathcal{L}_2 = (1-z)(1-9z)\theta^2 - 2(1-5z)\theta + (1-3z)$$

  - ➡ Same differential operator that annihilates maximal cuts!
  - ➡ Maximal cuts at two-loop can be associated to the periods of the family of elliptic curves.
  - ➡ Singular points of $\mathcal{L}_2$: $0, 1/9, 1, \infty$.
- We get multi-valued functions on $X_2 = \mathbb{CP}^1 \setminus \{0, 1/9, 1, \infty\}$.

$$\Pi_\alpha(z) = 2\pi i\, z + \mathcal{O}(z^2) \qquad \Pi_\beta(z) = \Pi_\alpha(z)\frac{\log z}{2\pi i} + c_2\, z^2 + \mathcal{O}(z^3)$$

Monodromy group

- Analytic continuation around 0:



$$\begin{pmatrix}\Pi_\beta(z)\\ \Pi_\alpha(z)\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Pi_\beta(z)\\ \Pi_\alpha(z)\end{pmatrix}$$

- We can compute such a matrix for every small loop around a singular point (N.B. $\mathbf{T}_0\mathbf{T}_{1/9}\mathbf{T}_1\mathbf{T}_\infty = 1$) :

$$\mathbf{T}_0 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad \mathbf{T}_1 = \begin{pmatrix}1 & 0\\ -6 & 1\end{pmatrix} \qquad \mathbf{T}_{1/9} = \begin{pmatrix}7 & 2\\ -18 & -5\end{pmatrix}$$

➡ These matrices generate the monodromy group of $\mathcal{L}_2$:

$$\Gamma_1(6) = \left\{ \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : a, d = 1 \bmod 6, c = 0 \bmod 6 \right\}$$

Monodromy group

- Define: $\tau = \dfrac{\Pi_\beta(z)}{\Pi_\alpha(z)}$ $\quad q = e^{2\pi i \tau} = z + 4z^2 + \mathcal{O}(z^3)$

  ➡ Can be inverted: $z(\tau) = q - 4q^2 + \mathcal{O}(q^3)$

- Action of the monodromy group:

$$\tau \longrightarrow \frac{a\Pi_\beta(z) + b\Pi_\alpha(z)}{c\Pi_\beta(z) + d\Pi_\alpha(z)} = \frac{a\tau + b}{c\tau + d}$$

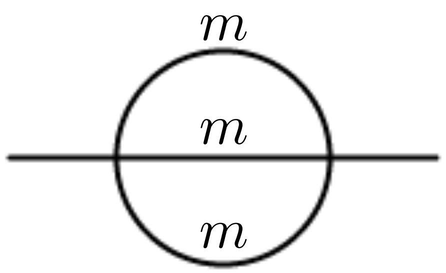
$$z(\tau) \longrightarrow z\left(\frac{a\tau + b}{c\tau + d}\right) = z(\tau)$$

Modular function for $\Gamma_1(6)$

$$h_1(\tau) = \Pi_\alpha(z(\tau)) \longrightarrow (c\tau + d)\, h_1(\tau)$$

Modular form of weight 1 for $\Gamma_1(6)$

2-loop banana integral



$$\sim h_1(\tau) \int_{i\infty}^{\tau} \frac{d\tau'}{2\pi i} h_3(\tau')\,\tau' \sim \underline{\Pi}(z)^T \mathbf{\Sigma}_2 \int_{\infty}^{z} \frac{dz'}{z'^2} \underline{\Pi}(z')$$

$= \Pi_\alpha(z)$ $\qquad$ $\sim \Pi_\alpha(z)^3$

$$\underline{\Pi}(z) = (\Pi_\alpha(z), \Pi_\beta(z))^T$$

=maximal cut of the sunrise

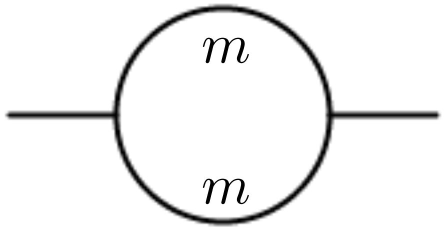
$$d\tau = \frac{z\,dz}{64(1-z)(1-9z)\,\Pi_\alpha(z)^2}$$

$$\mathbf{\Sigma}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

➡ Matches structure from differential equations:

$$\underline{J}_l(z,0) \sim \mathbf{W}_l(z) \int_0^z dz'\, \mathbf{W}_l(z)^{-1}\, \underline{N}_l(z,0)$$

➡ Also valid at 1-loop (and also 3-loop, but this is special…)



$$\sim \frac{z}{\sqrt{1-4z}} \int_0^z \frac{dz'}{z'\sqrt{1-4z'}}$$

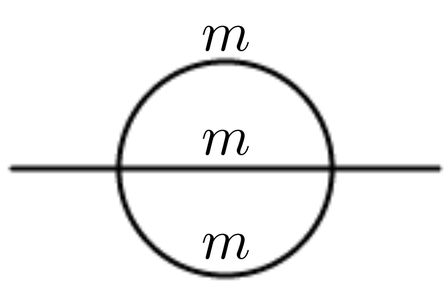
Maximal cut; Monodromy group = $\mathbb{Z}_2$

Summary: 2 loops

- Maximal cuts of $J_{l,1} \longleftrightarrow$ periods of the elliptic curve.
- Maximal cuts are annihilated by Picard-Fuchs operator $\mathcal{L}_2$.
  - ➡ Monodromy group: $\Gamma_1(6)$
- Close to $z = 0$: 1 holomorphic and 1 single-log solution:

$$\Pi_\alpha(z) = 2\pi i\, z + \mathcal{O}(z^2) \qquad \Pi_\beta(z) = \Pi_\alpha(z)\frac{\log z}{2\pi i} + c_2\, z^2 + \mathcal{O}(z^3)$$

- General structure of the full Feynman integral:



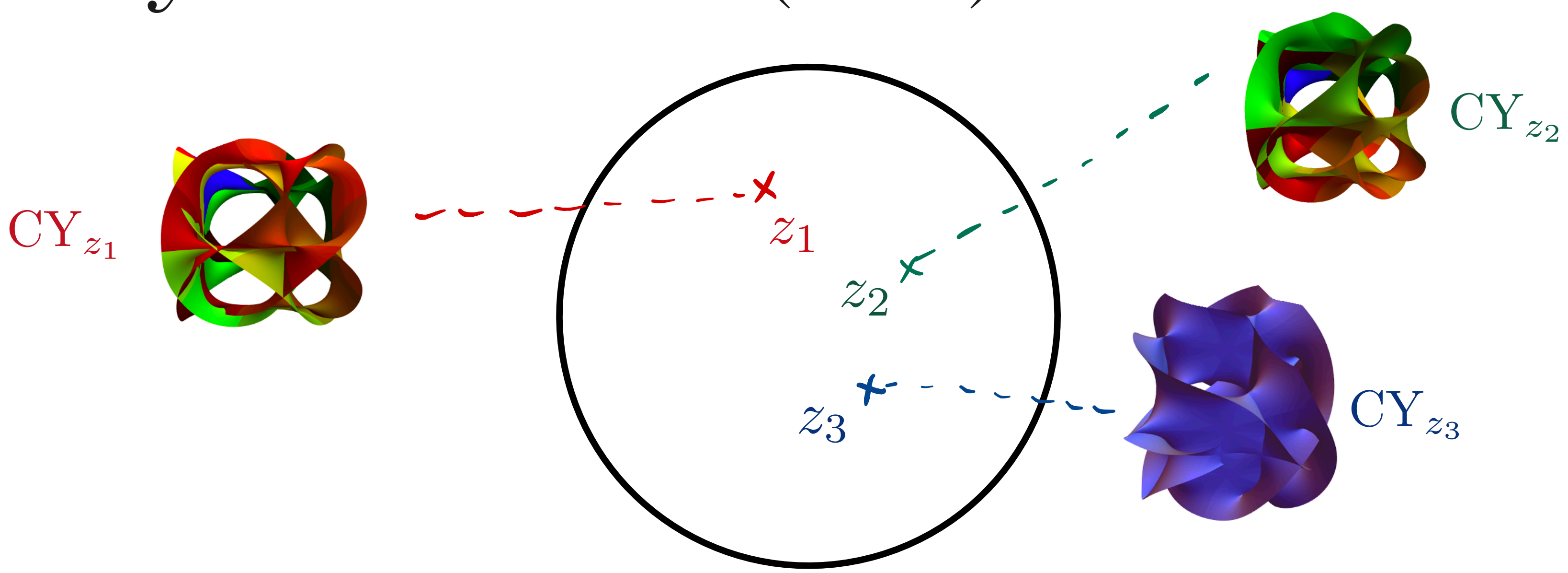
$$\sim \underline{\Pi}(z)^T \mathbf{\Sigma}_2 \int_\infty^z \frac{\mathrm{d}z'}{z'^2}\, \underline{\Pi}(z')$$

- For 2 loops: Period is a modular form.

Higher-loop banana integrals and Calabi-Yau varieties

Calabi-Yau varieties

- The l-loop (equal-mass) banana family is characterised by a (1-parameter) family of Calabi-Yau $(l-1)$-folds:



- Calabi-Yau n-fold: n-dim. Kähler manifold with a unique holomorphic differential n-form $\Omega(z)$.

➡ Characterised by its periods: $\underline{\Pi}(z) = (\Pi_1(z), \ldots, \Pi_r(z))^T$

$$\Pi_i(z) = \int_{\Gamma_i} \Omega(z) \qquad \Gamma_i = \text{Basis of independent cycles.}$$

➡ For the equal-mass banana: $r = l = \#\text{MIs}$

Picard-Fuchs equation

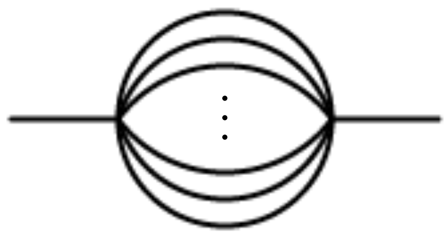
- The periods provide a basis for the maximal cuts in 2 dimensions.
- They are annihilated by the PF-operator $\mathcal{L}_l$ of degree l .
  - ➡ They can be constructed explicitly for all l . [Bönisch, Fischbach Klemm, Nega, Safari]
- Set of singular points: $$\Delta^{(l)} := \bigcup_{j=0}^{\lceil \frac{l-1}{2} \rceil} \left\{ (l+1-2j)^2 \right\}$$
- Structure of the solutions: close to $z=0$:

$$\Pi_{l,1}(z) = \mathcal{O}(z) \qquad \Pi_{l,k}(z) \sim \Pi_{l,1}(z) \, \frac{1}{(k-1)!} \log^{k-1} z + \mathcal{O}(z^2)$$

  - ➡ Point of maximal unipotent monodromy (MUM-point).
  - ➡ Not true for other singular points!
  - ➡ (Expected to be) genuinely new transcendental functions.
  - ➡ Can be evaluated in a fast and efficient way for all z .

General expectation

- General expected form:



$$\sim \underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_0^z \frac{\mathrm{d}z'}{z'^2} \underline{\Pi}_l(z')$$

Maximal cuts

Linear in the periods

➡ $\mathbf{\Sigma}_l$ is uniquely determined from the geometry (must be invariant under action of the monodromy group):

$$\mathbf{\Sigma}_l = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}$$

[e.g., $\mathbf{M}^T \mathbf{\Sigma}_2 \mathbf{M} = \mathbf{\Sigma}_2$ for all $\mathbf{M} \in \mathrm{SL}_2(\mathbb{Z})$]

- But it does not yet match what we (seem to) get from the differential equations….

$$\underline{J}_l(z,0) \sim \mathbf{W}_l(z) \int_0^z \mathrm{d}z' \, \mathbf{W}_l(z)^{-1} \underline{N}_l(z,0)$$

Maximal cuts

Polynomial of degree $l-1$ in the periods

Quadratic relations

- Are there relations among the periods? [See talk by A. Klemm]

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \Omega(z) = 0!$$

➡ This is a quadratic relation among maximal cuts!

- Are there more of these relations?

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \partial_z \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \partial_z \Omega(z) = 0$$

➡ This is a quadratic relation among maximal cuts of different MIs.

- What is their origin? How can we find these relations?

Griffiths transversality

- The cohomology group $H^n_{dR}(M)$ comes with a Hodge filtration $(n = l-1)$.

$$F^{l-1} \subseteq F^{l-2} \subseteq F^{l-3} \subseteq \cdots \subseteq F^1 \subseteq F^0 = H^n_{dR}(M)$$

Ω $\quad \partial_z \Omega$ $\quad \partial_z^2 \Omega$ $\quad \partial_z^{l-2} \Omega$ $\quad \partial_z^{l-1} \Omega$

$$F^p = \bigoplus_{p \leq q \leq l-1} H^{q,p-q} \qquad H^{p,q} = \text{cohomology class of } (p,q)\text{-forms}$$

e.g. $dz_1 \wedge \cdots \wedge dz_p \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$

- Griffiths transversality: $\partial_z F^p \subseteq F^{p-1}$

➡ cf. for the elliptic case: $\partial_z \int_\alpha \frac{dx}{y} = A \int_\alpha \frac{dx}{y} + B \int_\alpha \frac{x\,dx}{y}$

$\frac{dx}{y} \in F^1$, $\frac{dx}{y} \in F^1$, $\frac{x\,dx}{y} \in F^0$

Quadratic relations

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \Omega(z) = 0$$

'$(n,0) \wedge (n,0)$'

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \partial_z \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \partial_z \Omega(z) = 0$$

'$(n,0) \wedge (n-1,1)$'

$$\vdots$$

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \partial_z^{l-2} \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \partial_z^{l-2} \Omega(z) = 0$$

'$(n,0) \wedge (1,n-1)$'

$$\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \partial_z^{l-1} \underline{\Pi}_l(z) = \int_M \Omega(z) \wedge \partial_z^{l-1} \Omega(z) = C_l(z)$$

'$(n,0) \wedge (0,n)$'

$$C_l(z) = \frac{1}{z^{l-3} \prod_{k \in \Delta^{(l)}} (1 - kz)}$$

Quadratic relations

- We obtain a matrix of quadratic relations among maximal cuts!

$$\mathbf{Z}_l(z) = \begin{pmatrix} \underline{\Pi}_l(z)^T\,\mathbf{\Sigma}_l\,\underline{\Pi}_l(z) & \cdots & \underline{\Pi}_l(z)^T\,\mathbf{\Sigma}_l\,\partial_z^{l-1}\underline{\Pi}_l(z) \\ \vdots & \ddots & \vdots \\ \partial_z^{l-1}\underline{\Pi}_l(z)^T\,\mathbf{\Sigma}_l\,\underline{\Pi}_l(z) & \cdots & \partial_z^{l-1}\underline{\Pi}_l(z)^T\,\mathbf{\Sigma}_l\,\partial_z^{l-1}\underline{\Pi}_l(z) \end{pmatrix}$$

$$= \mathbf{W}_l(z)\mathbf{\Sigma}_l\mathbf{W}_l(z)^T$$

➡ All entries are explicitly calculable rational functions, e.g.,

$$\mathbf{Z}_3(z)^{-1} = \begin{pmatrix} \frac{1}{z^2} - \frac{8}{z} & -10+64z & 1-20z+64z^2 \\ -10+64z & -1+20z-64z^2 & 0 \\ 1-20z+64z^2 & 0 & 0 \end{pmatrix}$$

➡ We have searched for more quadratic relations, but we could not find any (for generic z).

➡ We conjecture that these are the only quadratic relations.

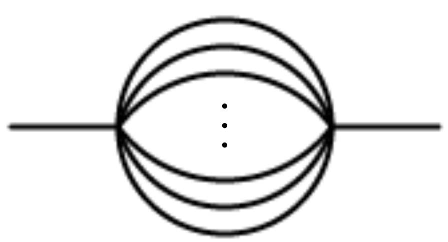
Banana integrals

- It is now easy to show that

$$\mathbf{W}_l(z)^{-1} = \mathbf{\Sigma}_l \mathbf{W}_l(z)^T \mathbf{Z}_l(z)^{-1}$$

➡ The inverse is <u>linear in the periods</u>!

- Putting it all together, solution of the differential equation takes the form:



$$= \underline{\Pi}_l(z)^T \underline{L}_l^{(0)} + (l+1)!\,\underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_{\vec{1}_0}^{z} \frac{\mathrm{d}w}{w^2} \underline{\Pi}_l(w)$$

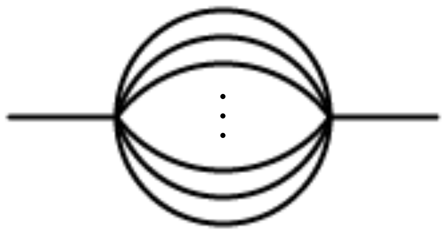
[Bönisch, CD, Fischbach Klemm, Nega]

➡ Compact formula valid for arbitrary loops!

➡ We still need the initial condition (related to $z \to 0$):

$$\underline{L}_l^{(0)} = (L_{l,1}^{(0)}, \ldots, L_{l,l}^{(0)})^T$$

Banana integrals



$$= \underline{\Pi}_l(z)^T \underline{L}_l^{(0)} + (l+1)!\, \underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_{\vec{1}_0}^{z} \frac{\mathrm{d}w}{w^2} \underline{\Pi}_l(w)$$

- The initial condition can be obtained as a generating series using techniques from mirror symmetry: [Bönisch, Fischbach Klemm, Nega, Safari]

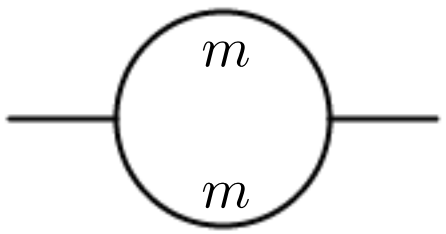
$$L^{(0)}_{l,k} = (l+1)!\, r_k \qquad \sum_{k=0}^{\infty} r_k\, x^k = -\frac{\Gamma(1-x)}{\Gamma(1+x)}$$

- Compact analytic formula for arbitrary number of loops!

  ➡ Integrals that appear in it are the natural generalisation of the integrals for $l \leq 3$.

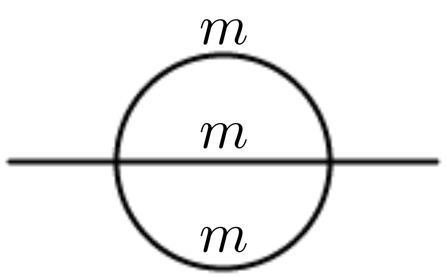
  ➡ Ingredients are of geometric origin (periods, intersection pairing, initial condition from mirror symmetry).

Banana integrals in 2D



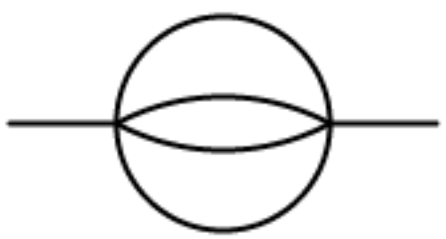
Riemann sphere

$$\sim \frac{z}{\sqrt{1-4z}} \int_0^z \frac{dz'}{z'\sqrt{1-4z'}}$$



Elliptic curve

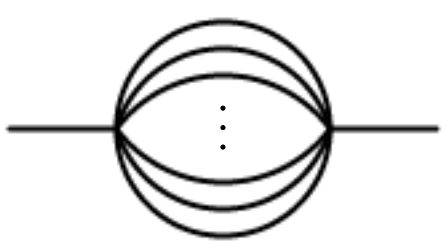
$$\sim \; h_1(\tau) \int_{i\infty}^{\tau} \frac{\mathrm{d}\tau'}{2\pi i}\, h_3(\tau')\, \tau'$$



K3 surface
~ "elliptic x elliptic"
(only here!)

$$\sim \; h_2(\tau) \int_{i\infty}^{\tau} \frac{\mathrm{d}\tau'}{2\pi i}\, h_4(\tau')\, \tau'^2$$

⋮



Calabi-Yau
$(l-1)$-fold

$$\sim \underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_{\vec{1}_0}^{z} \frac{\mathrm{d}w}{w^2} \underline{\Pi}_l(w)$$

3-loop banana integral

- The three-loop case is special!
- The geometry is a 1-parameter family of K3 surfaces (=CY 2-folds).
- The Picard-Fuchs operator $\mathcal{L}_3$ has degree 3, but it is the symmetric square of $\mathcal{L}_2$ (after a suitable change of variables).

[Verrill; Bloch, Kerr, Vanhove; Amadeo, Primo]

➡ Solution of $\mathcal{L}_3$ are products of 2 solutions of $\mathcal{L}_2$:

$$(\Pi_{3,1}, \Pi_{3,2}, \Pi_{3,3}) \sim (\Pi_{2,1}^2, \Pi_{2,1}\Pi_{2,2}, \Pi_{2,2}^2)$$

Modular forms

➡ We get the same class of functions as at 2 loops!

- This is only true at 3 loops in 2D for equal masses!

➡ One can check this explicitly for $l = 4, 5, \ldots$

Interesting questions

- Question 1: What about $D = 2 - 2\epsilon$?
  - ➡ We can get the differential equations and the initial conditions for arbitrary loops. [Bönisch, CD, Fischbach Klemm, Nega]
  - ➡ We can write down similar all-loop formulas for higher terms in the expansion.
  - ➡ We get iterated integrals involving Calabi-Yau periods, and their derivatives.

$$\partial_z \underline{L}_l(z,\epsilon) = \widetilde{\mathbf{B}}_l(z,\epsilon)\underline{L}_l(z,\epsilon) + \underline{\widetilde{N}}_l(z,\epsilon)$$

$$\underline{\widetilde{N}}_l(z,\epsilon) = \mathbf{W}_l(z)^{-1}\underline{N}_l(z,\epsilon)$$

linear in periods

$$\widetilde{\mathbf{B}}_l(z,\epsilon) = \mathbf{W}_l(z)^{-1}[\mathbf{B}_l(z,\epsilon) - \mathbf{B}_{l,0}(z)]\mathbf{W}_l(z) = \mathcal{O}(\epsilon)$$

bilinear in periods

Interesting questions

- The resulting integrals may even be structurally different.
  Example:
  - ➡ At 2 loops we get iterated integrals of Eisenstein series for $\Gamma_1(6)$, to all orders in ϵ. [Adams, Weinzierl]
  - ➡ At 3 loops in 2D we get iterated integrals of Eisenstein series for $\Gamma_1(6)$. [Bloch, Kerr, Vanhove; Brödel, CD, Dulat, Penante, Tancredi]
  - ➡ At 3 loops we get iterated integrals of meromorphic modular forms for $\Gamma_1(6)$, to all order in ϵ. [Brödel, CD, Matthes, to appear]
- Iterated integrals of meromorphic modular forms for $SL_2(\mathbb{Z})$ were very recently (2021!) introduced by Nils Matthes.
  - ➡ The theory can be generalised to subgroups $\Gamma \subseteq SL_2(\mathbb{Z})$ (where Γ has genus 0) [Brödel, CD, Matthes, to appear]

Interesting questions

- Question 2: Is there a unique Calabi-Yau variety attached to a Feynman integral, and what is it?
  - ➡ There are different families of Calabi-Yau varieties that describe the banana integrals: [Bönisch, Fischbach Klemm, Nega, Safari]

$$M_{l-1}^{\mathrm{HS}} = \{\underline{x} \in \mathbb{P}^l | \mathcal{F}(1,\underline{z};\underline{x}) = 0\}$$

F-polynomial from Feynman parameter integral

$$M_{l-1}^{\mathrm{CI}} = \left\{ \left(w_1^{(i)} : w_2^{(i)}\right) \in \mathbb{P}^1_{(i)}, \forall i \middle| P_1 := \sum_{i=1}^{l+1} a^{(i)} w_1^{(i)} + b^{(i)} w_2^{(i)} = \sum_{i=1}^{l+1} c^{(i)} w_1^{(i)} + d^{(i)} w_2^{(i)} =: P_2 = 0 \right\}$$

  - ➡ For $l = 2$ they are the same, but for higher l they are distinct (e.g., they have distinct Euler characteristic).
  - ➡ But they do define the same Calabi-Yau motive!
  - ➡ Calabi-Yau motive ~ linear subspace of a cohomology group (+other conditions).

Interesting questions

- Question 3: Which Feynman integrals can be attached to families of Calabi-Yau varieties/motives? All of them?

  ➡ There are examples of Feynman integrals whose maximal cuts give rise to Riemann surfaces of genus $g > 1$.

  [Huang, Zhang; Hauenstein, Huang, Mehta, Zhang]

  ➡ They cannot be Calabi-Yau varieties! (because they have $g > 1$ holomorphic differentials)

  ➡ There are examples of Calabi-Yau motives that describe Riemann surfaces of higher genus.

- Interesting question for the future: Are the examples of higher-genus curves from Feynman integrals associated to Calabi-Yau motives?

Conclusions

- After elliptic curves Calabi-Yau varieties are the next frontier.
- We have shown how results from the elliptic case generalise nicely to all loops for equal-mass banana integrals.
  - ➡ Compact analytic formula valid for arbitrary number of loops!
- Interesting questions for the future:
  - ➡ What about other integrals (e.g., with more scales)?
  - ➡ Which integrals are associated to Calabi-Yau motives?
  - ➡ What are the properties of the (iterated) integrals of Calabi-Yau periods?