Differential BCJ relations for AdS boundary correlators

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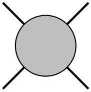
Based on 2106.10822 with Pranav Diwakar, Aidan Herderschee and Radu Roiban

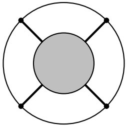
Introduction

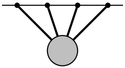
- Since its discovery, color-kinematics duality and double copy have been receiving renewed interests and wider applications in flat space amplitudes [Talks by Carrasco, Cheung, Elvang, Keeler, Paranjape, Puhm, Vazquez-Holm]
- On more generic background spacetime,

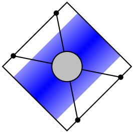
scattering amplitudes $\longrightarrow$ boundary correlators

Adamo, Casali, Ilderton,
Mason, Nekovar, Sharma
Alday, Behan, Ferrero, Zhou
Albayrak, Kharel, Meltzer
Armstrong, Farrow, Gomez, Jusinskas,
Lipstein, Mei, McFadden
Eberhardt, Komatsu, Mizera
Roehrig, Skinner









- Among others, anti-de Sitter (AdS) background is of particular interests:
  - Maximally symmetric: $\mathrm{SO}(d+1,1)$ for AdS_{d+1}
  - AdS/CFT correspondence Maldacena, 9711200
- **This work:** we study non-trivial differential relations among position space (color-ordered) AdS boundary correlators, as well as their connection to C-K duality and double copy

Main goal

We propose a C-K dual for color-dressed boundary correlators,

$$\mathcal{A} = \sum_{\text{cubic } g} C_g \left(\prod_{I \in g} \frac{1}{D_I^2} \right) \hat{N}_g D_{\underbrace{d,d,\ldots,d}_{n}}$$

$= D_{d,d,\ldots,d}$

contact Witten diagram;
AdS analog of $\delta(p)$

where the "kinematic numerators" satisfy the operator Jacobi identity

$$\hat{N}_s + \hat{N}_t + \hat{N}_u = 0$$

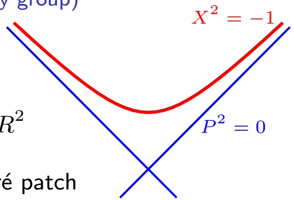
Consequently, the color-ordered boundary correlators satisfy the differential BCJ relations

$D_{ij}^2 = D_i \cdot D_j$ and D_i is a conformal generator

$$0 = D_{12}^2 A(1,2,\ldots,n) + \sum_{j=3}^{n-1} \left(D_{12}^2 + \sum_{k=3}^{j} D_{2j}^2 \right) A(1,3,\ldots,j,2,j+1,\ldots,n)$$

NLSM: both the C-K dual form and BCJ relations verified up to six points for arbitrary d
YM: differential BCJ relations verified at four points for $d = 4$

Embedding space (linearize the action of the symmetry group)



- AdS_{d+1} can be realized as a hypersurface in $\mathbb{R}^{d+1,1}$

$$(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -R^2$$

- Consider bulk point $X^A = (X^{\mathsf{a}}, X^d, X^{d+1})$ in Poincaré patch

$$X^{\mathsf{a}} = \frac{R}{z} x^{\mathsf{a}}\,, \quad X^d = \frac{R}{z}\frac{1 - x^2 - z^2}{2}\,, \quad X^{d+1} = \frac{R}{z}\frac{1 + x^2 + z^2}{2}$$

- The AdS metric

$$ds^2 = -dR^2 + \frac{R^2}{z^2}(dz^2 + dx_{\mathsf{a}} dx^{\mathsf{a}}) = -dR^2 + R^2 ds^2_{\mathrm{AdS}}$$

- Points on the conformal boundary $z \to 0$

$$P^A = \left(x^{\mathsf{a}}, \frac{1 - x^2}{2}, \frac{1 + x^2}{2}\right)$$

- Polarization vector for spinning (bosonic) boundary states

$$Z^A = \left(\epsilon^{\mathsf{a}}, -\epsilon \cdot x, \epsilon \cdot x\right) \implies P \cdot Z = 0 \text{ and } Z^2 = \epsilon^2 = 0$$

Propagators

Penedones, 1011.1485
Balitsky, 1102.0577
Fitzpatrick, Kaplan, Penedones, Raju, van Rees, 1107.1499
Paulos, 1107.1504
Costa, Goncalves, Penedones, 1404.5625

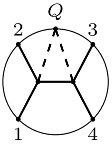
Scalar equation of motion

$$\left[\nabla^2_{\mathrm{AdS}} - \Delta(\Delta - d)\right]\phi = J$$

Bulk-boundary (embedding space)

$$\underset{\text{scalar}}{E_\Delta(P, X)} = \frac{\mathcal{N}_\Delta}{(-2P \cdot X)^\Delta} \qquad \underset{\text{vector}}{E^{MA}_\Delta(P, X)} = \frac{\mathcal{N}_{\Delta,1}\left(\eta^{MA} - \frac{X^M P^A}{P \cdot X}\right)}{(-2P \cdot X)^\Delta}$$

Bulk-bulk (split representation)



$$\left[\frac{G_\Delta(X, Y)}{G^{AB}_\Delta(X, Y)}\right] = \int_{-i\infty}^{i\infty} \frac{dc}{2\pi i} \frac{1}{(\Delta - d/2)^2 - c^2} \left[\frac{\Omega_c(X, Y)}{\Omega^{AB}_c(X, Y)}\right]$$

$$\left[\frac{\Omega_c(X, Y)}{\Omega^{AB}_c(X, Y)}\right] = -2c^2 \int_{\partial \mathrm{AdS}} dQ \left[\frac{E_{d/2+c}(Q, X) E_{d/2-c}(Q, Y)}{\eta_{MN} E^{MA}_{d/2+c}(Q, X) E^{NB}_{d/2-c}(Q, Y)}\right]$$

Boundary correlators in embedding space

- A boundary correlator $\mathcal{A}$ is a homogeneous function in P_i

$$\mathcal{A}(\lambda P_i, Z_i) = \lambda^{-\Delta_i}\mathcal{A}(P_i, Z_i)$$

Δ_i: conformal weight
l_i: spin

and a multilinear function in Z_i

$$\mathcal{A}(P_i, Z_i) = Z_i^{M_1} Z_i^{M_2} \dots Z_i^{M_{l_i}} \mathcal{A}_{M_1 M_2 \dots M_{l_i}}(P_i)$$

- For a boundary state with spin l_i, the tensor $\mathcal{A}_{M_1 M_2 \dots M_{l_i}}(P_i)$ is
  - symmetric and traceless
  - transverse to the boundary: $P_i^{M_1}\mathcal{A}_{M_1 M_2 \dots M_{l_i}}(P_i) = 0$

Costa, Penedones, Poland, Rychkov, 1107.3554
Costa, Goncalves, Penedones, 1404.5625

Conformal generators

- Conformal generators in the embedding space

$$D_i^{AB} = P_i^A \frac{\partial}{\partial P_{i,B}} - P_i^B \frac{\partial}{\partial P_{i,A}} + Z_i^A \frac{\partial}{\partial Z_{i,B}} - Z_i^B \frac{\partial}{\partial Z_{i,A}}$$

- The conformal invariance of $\mathcal{A}$ leads to the conformal Ward identity

$$\sum_{i=1}^{n} D_i^{AB} \mathcal{A} = 0$$

- For convenience, we define

$$D_{ij}^2 \equiv D_i \cdot D_j \equiv \eta_{AC} \eta_{BD} D_i^{AB} D_j^{CD} \,, \qquad D_i^2 \equiv D_i \cdot D_i$$
$$D_I^2 \equiv \frac{1}{2} \Big(\sum_{i \in I} D_i \Big) \cdot \Big(\sum_{i \in I} D_i \Big)$$

The analogy to flat space amplitudes

Eberhardt, Komatsu, Mizera, 2007.06574

Flat space amplitudes

Momentum conservation

$$\sum_{i=1}^{n} p_i = 0 \;\Rightarrow\; s_I = s_{\bar{I}}$$

$$(2\pi)^d \delta^d\Big(\sum_i p_i\Big) = \int \mathrm{d}^d x \prod_i e^{i p_i x}$$

Massless condition

$$p_i^2 = 0$$

Linearized gauge invariance

$$A(p_i, \epsilon_i)\Big|_{\epsilon_i \to p_i} = 0$$

AdS boundary correlators

Conformal Ward identity

$$\sum_{i=1}^{n} D_i^{AB} \cong 0 \;\Rightarrow\; D_I^2 \cong D_{\bar{I}}^2$$

$$D_{\Delta_1,\Delta_2,\ldots} = \int_{\mathrm{AdS}} dX \prod_i \frac{1}{(-2P_i \cdot X)^{\Delta_i}}$$

Massless condition (current conservation*)

$$D_i^2 \cong 0$$

Transversality condition

$$\mathcal{A}(P_i, Z_i)\Big|_{Z_i \to P_i} = 0$$

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Flat space amplitudes

Momentum conservation

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AdS boundary correlators

Conformal Ward

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$= D_{d,d,\ldots,d}$

contact Witten diagram;
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$$D_{\Delta_1,\Delta_2,\ldots} = \int_{\mathrm{AdS}} dX \prod_i \frac{1}{(-2P_i\cdot X)^{\Delta_i}}$$

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$$\mathcal{A}(P_i, Z_i)\Big|_{Z_i \to P_i} = 0$$

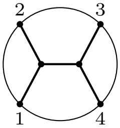
Non-commutativity

Eberhardt, Komatsu, Mizera, 2007.06574

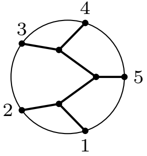
- In general, D_I^2 and D_J^2 do not commute. For example,

$$[D_{ik}^2, D_{jk}^2] = [D_i \cdot D_k, D_j \cdot D_k] = -4\eta_{BC}\eta_{DE}\eta_{FA} D_i^{AB} D_j^{CD} D_k^{EF}$$

- $[D_I^2, D_J^2] = 0$ if $I \cap J = \emptyset$ or one contains another
- If D_I^2 and D_J^2 are associated to the internal edges of a Witten (Feynman) diagram, they always commute.
- The inverse operators are bulk-bulk propagators:



$$= \frac{1}{D_{12}^2} D_{d,d,d,d}$$

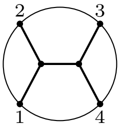


$$= \frac{1}{D_{12}^2 D_{34}^2} D_{d,d,d,d,d}$$

(Here we suppress the normalization $\mathcal{N}_{\Delta_i}$ for each boundary state)

The derivation

Eberhardt, Komatsu, Mizera, 2007.06574

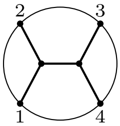


$$= \int_{\mathrm{AdS}} dX_1 dX_2 \frac{1}{(-2P_1 \cdot X_1)^d} \frac{1}{(-2P_2 \cdot X_1)^d} G(X_1, X_2) \frac{1}{(-2P_3 \cdot X_2)^d} \frac{1}{(-2P_4 \cdot X_2)^d}$$

First use the fact that the bulk-bulk propagator is the inverse of the Laplacian

$$G(X_1, X_2) = \frac{1}{\Box_{X_1}} \delta^{d+1}(X_1 - X_2) \qquad \Box_X \equiv \frac{1}{2} D_X^2$$

and then the integration-by-parts

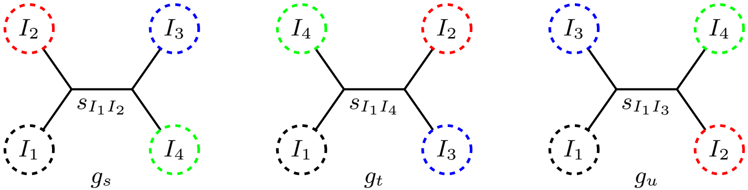


$$= \int_{\mathrm{AdS}} dX \frac{1}{(-2P_3 \cdot X)^d} \frac{1}{(-2P_4 \cdot X)^d} \frac{1}{\Box_X} \left[\frac{1}{(-2P_1 \cdot X)^d} \frac{1}{(-2P_2 \cdot X)^d} \right]$$

$$= \frac{1}{D_{12}^2} D_{d,d,d,d}$$

The last identity is obtained by noticing that the action of $\Box_X$ is the same as the action of D_{12}^2 on the two bulk-boundary propagators in the bracket

BCJ form in flat space



$$\mathcal{A}^{\text{flat}} = \sum_{\text{cubic } g} \frac{C_g N_g}{\prod_{I \in g} s_I} \delta^d\Big(\sum_{i=1}^{n} p_i\Big)$$

Color-kinematics duality:

Bern, Carrasco, Johansson, 0805.3993

$$C_{g_s} + C_{g_t} + C_{g_u} = 0$$
$$\Updownarrow$$
$$N_{g_s} + N_{g_t} + N_{g_u} = 0$$

$$\mathcal{A} = \sum_{\text{cubic } g} C_g \frac{1}{\prod_{I \in g} D_I^2} \hat{N}_g D_{\underbrace{d,d,\ldots,d}_{n}}$$

Proposal for color-kinematics duality:

$$\hat{N}_{g_s} + \hat{N}_{g_t} + \hat{N}_{g_u} = 0$$

as an operator relation

(cf. celestial amplitudes; talk by Puhm)

Supporting evidence from YM+CS in AdS_3 and NLSM (six points and arbitrary d)

Roehrig, Skinner, 2007.07234 Diwakar, Herderschee, Roiban, **FT**, 2106.10822

Differential (fundamental) BCJ relations

- The BCJ form leads to differential BCJ relations among AdS boundary correlators
- The derivation is identical to the one for flat space amplitudes

$$A(1,2,3,4) = \Big(\frac{1}{D_{12}^2}\hat{N}_s - \frac{1}{D_{23}^2}\hat{N}_t\Big) D_{d,d,d,d}$$
$$A(1,3,2,4) = \Big(\frac{1}{D_{23}^2}\hat{N}_t - \frac{1}{D_{13}^2}\hat{N}_u\Big) D_{d,d,d,d}$$

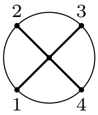
$\hat{N}_s + \hat{N}_t + \hat{N}_u = 0$
conformal Ward identity $\Longrightarrow D_{12}^2 A(1,2,3,4) = D_{13}^2 A(1,3,2,4)$

- More generally, we can go to the DDM basis Del Duca, Dixon, Maltoni, 9910563

$$A(\alpha) = \sum_{\beta \in S_{n-2}} \hat{m}(\alpha|\beta)\hat{N}(\beta) D_{d,d,\dots,d}$$

and the differential (fundamental) BCJ relations are a consequence of the bi-adjoint scalar correlator $\hat{m}(\alpha|\beta)$ having null vectors on the support of CWI

NLSM correlators at four points



The color-ordered correlators are

$$A(1,2,3,4) = \frac{\mathcal{N}_\Delta^4}{4} D_{13}^2 D_{\Delta,\Delta,\Delta,\Delta}$$
$$A(1,3,2,4) = \frac{\mathcal{N}_\Delta^4}{4} D_{12}^2 D_{\Delta,\Delta,\Delta,\Delta}$$

They do satisfy the differential BCJ relation

$$D_{12}^2 A(1,2,3,4) - D_{13}^2 A(1,3,2,4) = \frac{\mathcal{N}_\Delta^4}{4} [D_{12}^2, D_{13}^2] D_{\Delta,\Delta,\Delta,\Delta} = 0$$

Note that although $[D_{ik}^2, D_{jk}^2] \neq 0$, it annihilates the D-function

$$[D_{ik}^2, D_{jk}^2] D_{\Delta,\Delta,\Delta,\ldots} = 0$$

Differential BCJ representation

It is also straightforward to put the color-dressed correlator into the BCJ form

$$\mathcal{A}^{a_1 a_2 a_3 a_4} = \frac{\mathcal{N}_\Delta^4}{4}\left(C_s \frac{1}{D_{12}^2}\hat{N}_s + C_t \frac{1}{D_{23}^2}\hat{N}_t + C_u \frac{1}{D_{13}^2}\hat{N}_u\right) D_{\Delta,\Delta,\Delta,\Delta}$$

where $\hat{N}_s + \hat{N}_t + \hat{N}_u = 0$

$$\hat{N}_s = D_{12}^2 D_{13}^2 \qquad \hat{N}_t = -[D_{12}^2, D_{13}^2] \qquad \hat{N}_u = -D_{13}^2 D_{12}^2$$

Can we obtain the numerators from the formalism in 2108.02276 (talk by Cheung)?

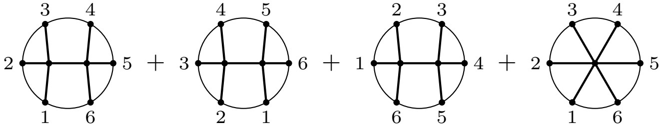
BCJ numerators in the flat space amplitude

$$N_s = s_{12}s_{13} \quad N_t = 0 \quad N_u = -s_{12}s_{13}$$

Du, Fu, 1606.05846
Carrasco, Mafra, Schlotterer, 1612.06446

NLSM correlator at six points

$\mathcal{A}_6^{a_1 a_2 \cdots a_6} =$



Color-ordered correlator (massless case $\Delta = d$)

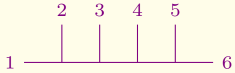
$$A(1,2,3,4,5,6) = -\frac{\mathcal{N}_d^6}{96}\left[\frac{3}{D_{123}^2} D_{13}^2 D_{46}^2 - D_{135}^2 + \text{cyclic}\right] D_{d,d,d,d,d,d}$$

Differential BCJ relations

$$0 = D_{12}^2 A(1,2,3,4,5,6) + (D_{12}^2 + D_{23}^2) A(1,3,2,4,5,6)$$
$$+ (D_{12}^2 + D_{23}^2 + D_{24}^2) A(1,3,4,2,5,6) + (D_{12}^2 + D_{23}^2 + D_{24}^2 + D_{25}^2) A(1,3,4,5,2,6)$$

DDM basis numerators (through the replacement $s_{ij} \to D_{ij}^2$)

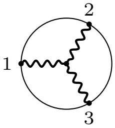
$\hat{N}\Bigg($



$\Bigg) = -\frac{\mathcal{N}_d^6}{16} D_{12}^2 (D_{13}^2 + D_{23}^2) D_{56}^2 (D_{45}^2 + D_{46}^2)$

Yang-Mills correlator at three points

Kharel, Siopsis, 1308.2515



$$\mathcal{A}^{a_1 a_2 a_3}_{\Delta_1 \Delta_2 \Delta_3} = -f^{a_1 a_2 a_3} Z_{1,M_1} Z_{2,M_2} Z_{3,M_3} \int_{\mathrm{AdS}} dX$$
$$\times \left[E^{M_1 A_1}_{\Delta_1}(P_1, X) \eta_{A_2 A_3} \left(\partial_{A_1} E^{M_2 A_2}_{\Delta_2}(P_2, X) E^{M_3 A_3}_{\Delta_3}(P_3, X) - (2 \leftrightarrow 3) \right) + \text{cyclic} \right]$$

Strategy:

- Write E^{MA}_{Δ} in terms of the scalar bulk-boundary propagator

$$E^{MA}_{\Delta}(P, X) = \frac{\Delta}{\Delta - 1} \mathcal{D}^{MA} E_{\Delta}(P, X) \qquad \mathcal{D}^{MA}_{\Delta} = \eta^{MA} + \frac{1}{\Delta} P^{A} \frac{\partial}{\partial P_M}$$

- Integrate over the bulk point X to obtain

$$\mathcal{A}^{a_1 a_2 a_3}_{\Delta_1 \Delta_2 \Delta_3} = -f^{a_1 a_2 a_3} \left(\prod_{i=1}^{3} Z_{i,M_i} \mathcal{D}^{M_i A_i}_{\Delta_i} \right) \mathcal{P}^{\Delta_1 \Delta_2 \Delta_3}_{A_1 A_2 A_3}(P_1, P_2, P_3)$$

where the tensor $\mathcal{P}$ is a linear combination of three-point D-functions

Yang-Mills correlator at three points

The three-point D-function has a closed formula ($\delta_{ij} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}$)

$$D_{\Delta_1,\Delta_2,\Delta_3} = \frac{\pi^{d/2}\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2})}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \frac{\Gamma(\delta_{12})\Gamma(\delta_{23})\Gamma(\delta_{13})}{P_{12}^{\delta_{12}}P_{23}^{\delta_{23}}P_{13}^{\delta_{13}}}$$

The massless boundary correlator ($\Delta_i = d-1$)

$$\mathcal{A}_3^{a_1a_2a_3} = -f^{a_1a_2a_3}\frac{d\,\Gamma(d-2)}{8\pi^d(d-2)}\frac{N_3}{(P_{12}P_{23}P_{13})^{d/2}}$$

$$N_3 = (4\Lambda_1 - V_1V_2V_3) - \frac{6}{d}\Lambda_1$$

Using differential operators ($\mathcal{E}^{AB} = P^AZ^B - P^BZ^A$)

$$\mathcal{A}_3^{a_1a_2a_3} = f^{a_1a_2a_3}\frac{\Gamma(d-2)}{16\pi^d(d-2)}\hat{N}_3\frac{1}{P_{12}^{d/2}P_{23}^{d/2}P_{13}^{d/2}}$$

Eberhardt, Komatsu, Mizera
QCD meets gravity 2020

$$\hat{N}_3 = \Big[(\mathcal{E}_1\cdot\mathcal{E}_2)(\mathcal{E}_3\cdot D_1) + \text{cyclic}(1,2,3)\Big] - 6(d-2)\text{Tr}(\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3)$$

Yang-Mills correlator at three points

The three-point D-function has a closed formu

$$D_{\Delta_1,\Delta_2,\Delta_3} = \frac{\pi^{d/2}\Gamma(\frac{\Delta_1+\Delta_2+\Delta_3}{2}}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(}$$

$$V_{i,jk} = \frac{(P_j \cdot Z_i)(P_i \cdot P_k) - (P_k \cdot Z_i)(P_i \cdot P_j)}{P_j \cdot P_k} \equiv V_i$$

$$H_{ij} = -2[(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(Z_j \cdot P_i)]$$

$$\Lambda_1 = V_1 V_2 V_3 + \frac{1}{2}(V_1 H_{23} + \text{cyclic})$$

The massless boundary correlator ($\Delta_i = d-1$)

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$$N_3 = (4\Lambda_1 - V_1 V_2 V_3) - \frac{6}{d}\Lambda_1$$

Using differential operators ($\mathcal{E}^{AB} = P^A Z^B - P^B Z^A$)

$$\mathcal{A}_3^{a_1 a_2 a_3} = f^{a_1 a_2 a_3} \frac{\Gamma(d-2)}{16\pi^d (d-2)} \hat{N}_3 \frac{1}{P_{12}^{d/2} P_{23}^{d/2} P_{13}^{d/2}}$$

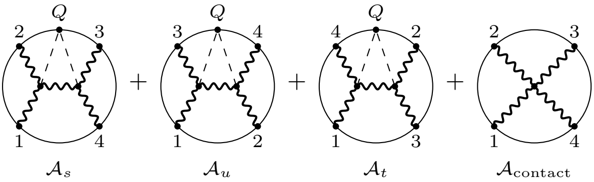
Eberhardt, Komatsu, Mizera
QCD meets gravity 2020

$$\hat{N}_3 = \Big[(\mathcal{E}_1 \cdot \mathcal{E}_2)(\mathcal{E}_3 \cdot D_1) + \text{cyclic}(1,2,3)\Big] - 6(d-2)\text{Tr}(\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3)$$

Yang-Mills correlator at four points

Kharel, Siopsis, 1308.2515

$$\mathcal{A}^{a_1 a_2 a_3 a_4} =$$



- Write the vector bulk-boundary propagators in terms of the scalar ones

$$\mathcal{A}_s^{a_1 a_2 a_3 a_4} = g^2 f^{a_1 a_2 x} f^{a_3 a_4 x} \left[\prod_{i=1}^{4} Z_{i,M_i} \mathcal{D}_{d-1}^{M_i A_i} \right] \mathcal{P}^s_{A_1 A_2 A_3 A_4} \ ,$$

- Use the split representation of the vector bulk-bulk propagator
- Trade the integral over Q with the one over the Melin space variables δ_{ij}. Then the contour integral over c can be worked out analytically

Yang-Mills correlator at four points

- Use inverse Melin transformations to write $\mathcal{P}_{A_1A_2A_3A_4}$ as a linear combination of the four-point D-functions

$$\mathcal{P}^{\mathrm{s}}_{A_1A_2A_3A_4}\Big|_{d=4} = \frac{3}{2\pi^8}\left[\frac{P_{13}\mathcal{R}_{A_1A_2A_3A_4}}{P_{12}}D_{3,2,4,3} - \frac{P_{14}\mathcal{R}^{(3\leftrightarrow 4)}_{A_1A_2A_3A_4}}{P_{12}}D_{3,2,3,4} + \frac{P_{13}\mathcal{R}_{A_1A_2A_3A_4}}{P_{12}^2}D_{2,1,4,3} - \frac{P_{14}\mathcal{R}^{(3\leftrightarrow 4)}_{A_1A_2A_3A_4}}{P_{12}^2}D_{2,1,3,4}\right]$$

  where $\mathcal{R}_{A_1A_2A_3A_4}$ is a tensor of $P_{i,A}$ and η_{AB}
- The action of $\mathcal{D}^{MA}_{\Delta}$ on $\mathcal{P}$ can still be expressed in terms of D-functions

$$\frac{\partial D_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}}{\partial P_{1,A}} = \frac{4\Delta_1}{\Delta_1+\Delta_2+\Delta_3+\Delta_4-d}\Big(\Delta_2 P_2^A D_{\Delta_1+1,\Delta_2+1,\Delta_3,\Delta_4} + \Delta_3 P_3^A D_{\Delta_1+1,\Delta_2,\Delta_3+1,\Delta_4} + \Delta_4 P_4^A D_{\Delta_1+1,\Delta_2,\Delta_3,\Delta_4+1}\Big)$$

- IBP relations among the D-functions can make the weights more uniform

BCJ relations

The four-point boundary correlator is a linear combination of D-functions

$$A(1,2,3,4) = \sum_{\Delta_i} C_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(P_i, Z_i) D_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}$$

The differential BCJ relation

$$D_{12}^2 A(1,2,3,4) = D_{13}^2 A(1,3,2,4)$$

holds at $d = 4$ for the four-point color-ordered Yang-Mills correlators with massless boundary states. Note that the Z dependence in the conformal generator is crucial.

Einstein gravity

Einstein-Hilbert Lagrangian in the embedding space (from the Gauss' equation for extrinsic curvature)

$$e^{-1}\mathcal{L} = \hat{R}_{\text{AdS}} - 2\Lambda = R - 2\sigma R_{AB}X^A X^B + \sigma(K^2 - K_{AB}K^{AB}) - 2\Lambda$$

$\sigma = -1 \qquad \Lambda = -\frac{1}{2}d(d-1)$

$K_{AB} = \nabla_A X_B - \sigma X_A X^C \nabla_C X_B$

Massless three-point correlator ($\Delta_i = d$)

$$\mathcal{M}_3 = \frac{d^2\,\Gamma(d)}{16\pi^d(d+1)^3}\frac{M_3}{(P_{12}P_{13}P_{23})^{1+d/2}}$$

$$M_3 = f_1\Lambda_1^2 + f_2\Lambda_1 V_1V_2V_3 + f_3(V_1V_2V_3)^2 + f_4H_{12}H_{23}H_{31} + f_5(V_1V_2H_{13}H_{23} + \text{cyclic})$$

$$f_1 = 16 - \frac{16}{d} - \frac{8}{d^2} \qquad f_2 = -8 - \frac{8}{d} + \frac{24}{d^2} + \frac{16}{d^3}$$

$$f_3 = 1 + \frac{4}{d} - \frac{4}{d^2} - \frac{16}{d^3} \qquad f_4 = \frac{8}{d} \qquad f_5 = \frac{4}{d^2} + \frac{8}{d^3}$$

Zhiboedov, 1206.6370
Arutyunov, Frolov, 9901121

note that $N_3 = (4\Lambda_1 - V_1V_2V_3) - \frac{6}{d}\Lambda_1$

Double copy in large dimensions

- The double copy of position-space correlators holds in the $d \to \infty$ limit

$$\lim_{d\to\infty} M_3 = \lim_{d\to\infty} (N_3)^2$$

Eberhardt, Komatsu, Mizera
QCD meets gravity 2020

$$\lim_{d\to\infty} \frac{M_3}{(P_{12}P_{23}P_{13})^{d/2+1}} = \frac{1}{4} \lim_{d\to\infty} \hat{N}_3 \hat{N}_3 \frac{1}{(P_{12}P_{23}P_{13})^{d/2+1}}$$

- It also trivially holds in $d = 2$ since the only independent structure in gauge and gravity correlators is $V_1 V_2 V_3$ and $(V_1 V_2 V_3)^2$ respectively

Attempts using other formalisms

- Melin space: super-conformal primaries
- Momentum space: flat space limit

Alday, Behan, Ferrero, Zhou, 2103.15830
Zhou, 2106.07651

Farrow, Lipstein, McFadden, 1812.11129
Lipstein, McFadden, 1912.10046
Albayrak, Kharel, Meltzer, 2012.10460

Summary

- We propose a form of color-kinematics duality in the position-space AdS boundary correlators.
- The C-K duality implies differential BCJ relations among color-ordered boundary correlators.
- For NLSM, we find the explicit BCJ representation of the "kinematic numerators" up to six points, and prove the differential BCJ relations
- For Yang-Mills, we compute the color-ordered boundary correlators at four points, and check that the differential BCJ relations hold in $d = 4$.

Future directions:

- Find the operator form BCJ numerators for Yang-Mills correlators
- Have a better understanding on the connection between color-kinematics duality and double copy
- Heterotic double copy? [Talk by Elvang]
- Is SUSY a secret sauce for double copy to work?
- Similar construction on dS background Gomez, Jusinskas, Lipstein, 2106.11903

Thanks for listening!