Outline

1. Lecture 1: gravitational wave astronomy, the two-body problem, and self-force theory
2. Lecture 2: the local problem: how to deal with small bodies
3. Lecture 3: the global problem: orbital dynamics in Kerr
   - Geodesic motion in Kerr
   - Perturbed motion in Kerr
   - Transient resonances
4. Lecture 4: the global problem: black hole perturbation theory

Solving the Einstein equations globally

- solving the local problem told us how to replace the small object with a moving puncture in the field equations:

$$
\begin{aligned}
G^{(1)}_{\mu\nu}[h^{\mathcal{R}(1)}] &= -G^{(1)}_{\mu\nu}[h^{\mathcal{P}(1)}] \\
G^{(1)}_{\mu\nu}[h^{\mathcal{R}(2)}] &= -G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] - G^{(1)}_{\mu\nu}[h^{\mathcal{P}(2)}] \\
\frac{D^2 z^\mu}{d\tau^2} &= -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(g_\nu{}^\delta - h^{\mathcal{R}\,\delta}_\nu)(2h^{\mathcal{R}}_{\delta\beta;\gamma} - h^{\mathcal{R}}_{\beta\gamma;\delta})u^\beta u^\gamma
\end{aligned}
$$

  where $G^{(1)}_{\mu\nu}[h] \sim \Box h_{\mu\nu}$, $G^{(2)}_{\mu\nu}[h,h] \sim \nabla h \nabla h + h \nabla\nabla h$
- the global problem: how do we solve these equations in practice in a particular background?

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Symmetries of Kerr

Kerr metric in Boyer-Lindquist coordinates:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2$$
$$+ \left[\Delta + \frac{2Mr(r^2+a^2)}{\Sigma}\right] \sin^2\theta\, d\phi^2$$

$\Sigma := r^2 + a^2\cos^2\theta$ and $\Delta := r^2 - 2Mr + a^2$

Symmetries:

- two Killing vectors $\xi^\alpha_{(t)} = \delta^\alpha_t$ and $\xi^\alpha_{(\phi)} = \delta^\alpha_\phi$ $\qquad (\nabla_{(\alpha}\xi_{\beta)} = 0)$
- one Kiling tensor $K_{\alpha\beta}$ $\qquad (\nabla_{(\alpha}K_{\beta\gamma)} = 0)$

Geodesic motion is integrable [Carter]

- three constants of geodesic motion: $E = -u_\alpha \xi^\alpha_{(t)}$, $L_z = u_\alpha \xi^\alpha_{(\phi)}$, and the *Carter constant* $C = u^\alpha u^\beta K_{\alpha\beta}$. Also normalization $g^{\alpha\beta} u_\alpha u_\beta = -1$
- can invert these four equations to obtain $u^\alpha(r, \theta, E, L_z, C)$:

$$\Sigma^2 \left(\frac{dr}{d\tau}\right)^2 = R(r)$$

$$\Sigma^2 \left(\frac{dz}{d\tau}\right)^2 = Z(z)$$

$$\Sigma \frac{dt}{d\tau} = T_r(r) + T_z(z) + aL_z := \omega_t(r, z)$$

$$\Sigma \frac{d\phi}{d\tau} = \Phi_r(r) + \Phi_z(z) - aE := \omega_\phi(r, z)$$

  orbital inclination $z := \cos\theta$
- radial and polar motion oscillate between turning points:

$$R(r) = -(1 - E^2)(r - r_1)(r - r_2)(r - r_3)(r - r_4)$$

$$Z(z) = a^2(1 - E^2)(z^2 - z_1^2)(z^2 - z_2^2)$$

Geodesic motion is integrable [Carter]

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$$\Sigma^2\left(\frac{dz}{d\tau}\right)^2=Z(z)$$

$$\Sigma\frac{dt}{d\tau}=T_r(r)+T_z(z)+aL_z:=\omega_t(r,z)$$

$$\Sigma\frac{d\phi}{d\tau}=\Phi_r(r)+\Phi_z(z)-aE:=\omega_\phi(r,z)$$

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Decoupling the r-θ motion

$r(\tau)$ and $z(\tau)$ are immediately decoupled by adopting *Mino time* as parameter:

$$\frac{d\lambda}{d\tau} = \Sigma^{-1}$$

$$\Rightarrow \left(\frac{dr}{d\lambda}\right)^2 = R(r)$$
$$\left(\frac{dz}{d\lambda}\right)^2 = Z(z)$$
$$\frac{dt}{d\lambda} = \omega_t(r,z)$$
$$\frac{d\phi}{d\lambda} = \omega_\phi(r,z).$$

Quasi-Keplerian description [Schmidt, Drasco and Hughes]

- manifestly periodic parametrizations:

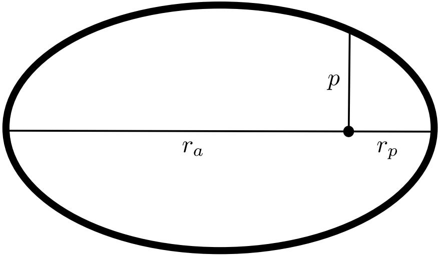
$$r(\psi_r) = \frac{pM}{1 + e\cos\psi_r}$$
$$z(\psi_z) = z_{\max}\cos\psi_z$$

with

$$\frac{d\psi_r}{d\lambda} = \omega_r(\psi_r) \quad \text{and} \quad \frac{d\psi_z}{d\lambda} = \omega_z(\psi_z)$$

- $\{p, e, z_{\max}\} \leftrightarrow \{E, L_z, Q\}$
- $\{p, e, z_{\max}\}$ describe "shape":

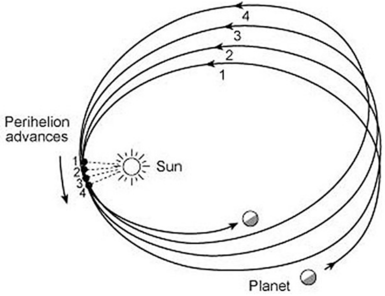
$$r_p = \frac{pM}{1+e} \quad \text{and} \quad r_a = \frac{pM}{1-e}$$



Precession of periapsis

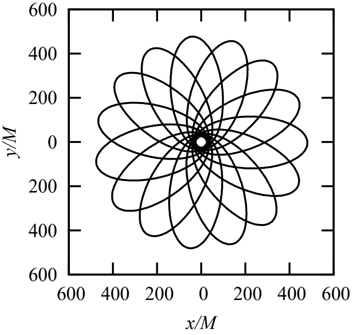
r and ϕ periods are (generically) incommensurate
$\Rightarrow$ orbit does not come back to itself

mild planar orbit



[image credit: K. R. Lang]

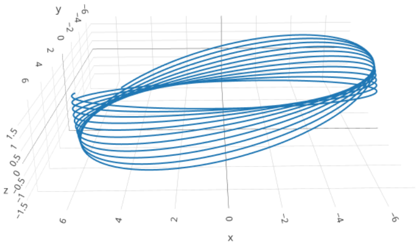
extreme planar orbit



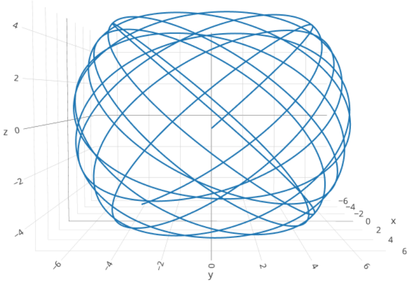
Precession of orbital plane

z and ϕ periods are (generically) incommensurate

mild spherical orbit

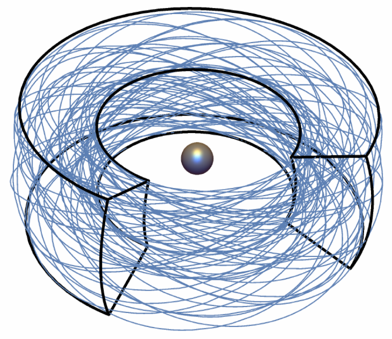


extreme spherical orbit



Orbits are generically space-filling

r and z periods are (generically) incommensurate



Generate orbits yourself:

- `http://nielswarburton.net/geodesics/interactive/Kerr_geodesic.html`
- `https://bhptoolkit.org/KerrGeodesics/`

Mino-time action angles [Schmidt, Fujita and Hikida]

- Let $\psi_\alpha = (t, \psi_r, \psi_z, \phi)$ and $J^\alpha = (p, e, z_{\max})$

$$\Rightarrow \frac{d\psi_\alpha}{d\lambda} = \omega_\alpha(\psi_r, \psi_z)$$
$$\frac{dJ^\alpha}{d\lambda} = 0$$

- Better: $(\psi_\alpha, J^\alpha) \to (q_\alpha, J^\alpha)$ such that

$$\frac{dq_\alpha}{d\lambda} = \Upsilon_\alpha(J^\beta)$$
$$\frac{dJ^\alpha}{d\lambda} = 0$$

- q_α is "averaged" ψ_α: $\Upsilon_\alpha = \langle\omega_\alpha\rangle_\lambda = \lim_{\Lambda\to\infty} \frac{1}{2\Lambda}\int_{-\Lambda}^{\Lambda} \omega_\alpha d\lambda$
- $z^\alpha(q_\beta, J^\beta)$ known analytically in terms of Jacobi elliptic functions

Boyer-Lindquist action angles [Moxon et al., Pound and Wardell]

- q_α oscillate wrt Boyer-Lindquist t. Bad for field equations.
- Better: $(q_\alpha, J^\alpha) \to (\varphi_A, J^A)$ such that

$$\frac{d\varphi_A}{dt} = \Omega_A(J^B)$$
$$\frac{dJ^A}{dt} = 0$$

- $\varphi_A = (\varphi_r, \varphi_z, \varphi_\phi)$, $J^A = (p, e, z_{\max})$
- $\Omega_A = \dfrac{\Upsilon_A}{\Upsilon_t}$

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Perturbed equations of motion in terms of action angles

- $\dfrac{D^2 z^\alpha}{d\tau^2} = \epsilon F^\alpha_{(1)} + \epsilon^2 F^\alpha_{(2)} + O(\epsilon^3)$
- If we keep fixed the relationship $(z^\alpha, u_\alpha) \to (\varphi_A, J^A)$, then

$$\frac{d\varphi_A}{dt} = \Omega^{(0)}_A(J^B) + \epsilon\Omega^{(1)}_A(J^B, \varphi_B) + O(\epsilon^2)$$
$$\frac{dJ^A}{dt} = \epsilon G^A_{(1)}(J^B, \varphi_B) + \epsilon^2 G^A_{(2)}(J^B, \varphi_B) + O(\epsilon^3)$$

Perturbed equations in terms of deformed action angles

[van de Meent and Warburton, Pound and Wardell]

- oscillations all over the place
- Better: $(\varphi_A, J^A) \to (\tilde{\varphi}_A, \tilde{J}^A)$ such that

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}^B)$$
$$\frac{d\tilde{J}^A}{dt} = \epsilon \tilde{G}^A_{(1)}(J^B) + \epsilon^2 \tilde{G}^A_{(2)}(J^B) + O(\epsilon^3)$$

- Let $\tilde{t} = \epsilon t$. Equations admit asymptotic solution

$$\tilde{\varphi}_A(\tilde{t}, \epsilon) = \frac{1}{\epsilon}\left[\tilde{\varphi}^{(0)}_A(\tilde{t}) + \epsilon \tilde{\varphi}^{(1)}_A(\tilde{t}) + O(\epsilon^2)\right]$$
$$\tilde{J}^A(\tilde{t}, \epsilon) = \tilde{J}^A_{(0)}(\tilde{t}) + \epsilon \tilde{J}^A_{(1)}(\tilde{t}) + O(\epsilon^2)$$

- 0PA and 1PA terms dictated by dissipative and conservative pieces of $F^\alpha_{(n)}$ as per Hinderer and Flanagan

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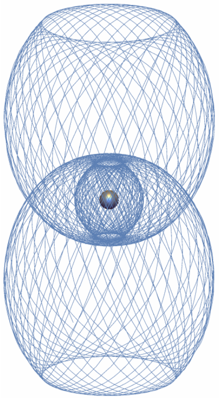
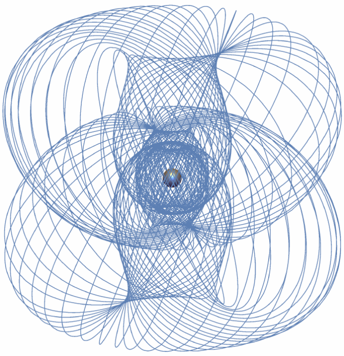
Orbital resonances

- orbital frequencies can be commensurate: e.g., rational Ω_z/Ω_r
- shape of orbit strongly depends on relative φ_r-φ_z initial phase

$$\Omega_z/\Omega_r = 3/2$$





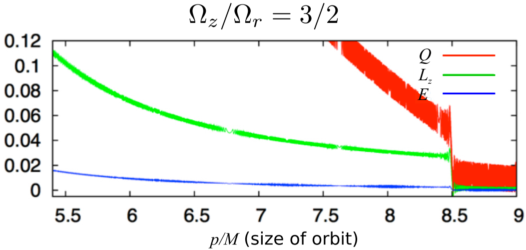
Passage through resonance [Hinderer and Flanagan]

- passage takes time $\Delta t \sim 1/\sqrt{\epsilon}$

  $\Rightarrow$ frequencies change by $\Delta\Omega_A \sim \sqrt{\epsilon}$

  $\Rightarrow$ causes cumulative shift $\Delta\varphi_A \sim 1/\sqrt{\epsilon}$ by end of inspiral
- new form of solution:

$$\tilde{\varphi}_A(\tilde{t},\epsilon) = \frac{1}{\epsilon}\left[\tilde{\varphi}_A^{(0)}(\tilde{t}) + \sqrt{\epsilon}\tilde{\varphi}_A^{(1/2)}(\tilde{t}) + \epsilon\tilde{\varphi}_A^{(1)}(\tilde{t}) + O(\epsilon^{3/2})\right]$$
$$\tilde{J}^A(\tilde{t},\epsilon) = \tilde{J}^A_{(0)}(\tilde{t}) + \sqrt{\epsilon}\tilde{J}^A_{(1/2)}(\tilde{t}) + \epsilon\tilde{J}^A_{(1)}(\tilde{t}) + O(\epsilon^2)$$



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