Non-Relativistic Gravity
(& the (opposite) Carroll expansion)

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DET FRIE FORSKNINGSRÅD
DANISH COUNCIL FOR
INDEPENDENT RESEARCH

THE VELUX FOUNDATIONS
VILLUM FONDEN VELUX FONDEN

based on work with

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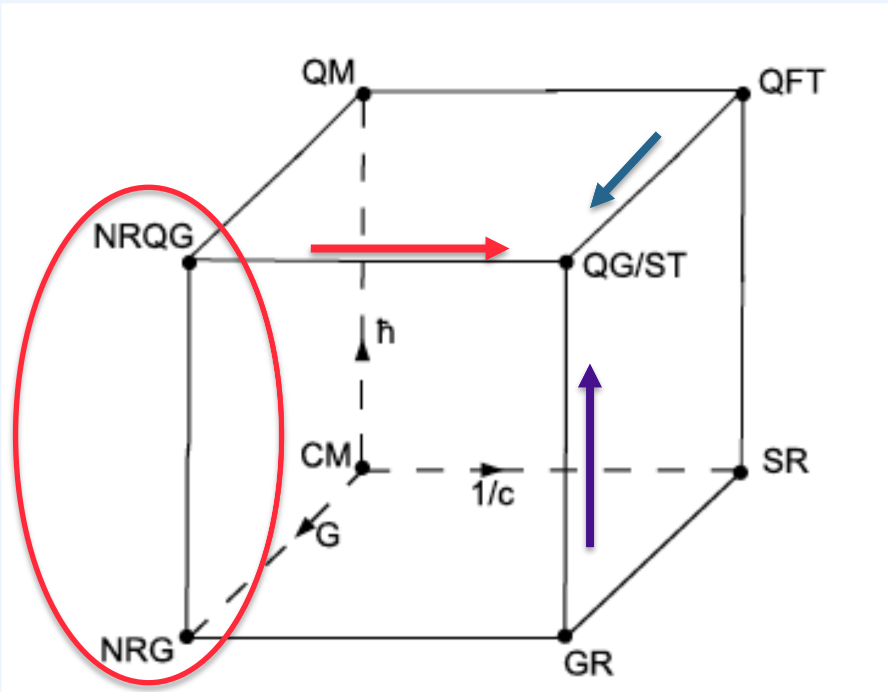
Cube of physical theories

$(\hbar, G_N, 1/c)$



a third route towards (relativistic) quantum gravity

how does this fit with string theory/holography ?

already (classical) non-relativistic gravity (NRG) is more than just Newtonian gravity

-→ uses Newton-Cartan geometry

Return of Newton-Cartan geometry

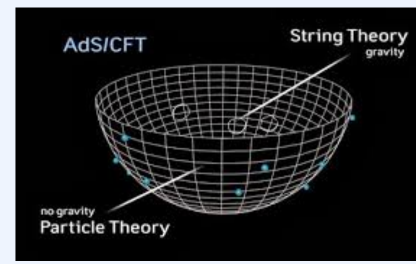
Non-relativistic gravity

(large speed of light expansion of General Relativity)

- e.g. relevant for Post-Newtonian expansion & ….

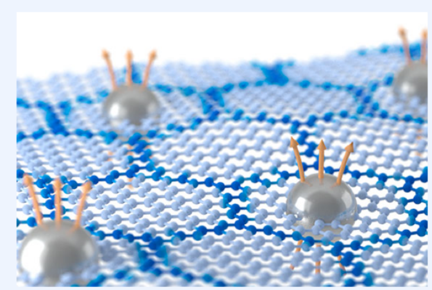
String theory and holography

- realization of quantum non-relativistic gravity
- tractable limits of holography



Applications to condensed matter/biophysical systems

- non-relativistic symmetries ubiquitous
- effective descriptions/hydrodynamics



Mathematics

Outline

- spacetime symmetries and Newton-Cartan geometry
- 1/c expansion of GR and its coupling to matter
- non-relativistic field theories and coupling to NC geometry
- small speed of light expansion (Carroll limit)

Newtonian gravity

- earth-based frame: Newton potential

$$\nabla^2 \Phi_{\mathrm{Newton}} = 4\pi G \rho_{\mathrm{mass}}$$

- freely falling frames: Galilean symmetries

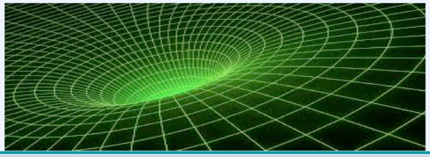
equivalence principle: inertial mass = gravitational mass

$$\vec{a} = -\nabla \Phi_N$$

no frame independent formulation → need geometry

Space-Time symmetries and Geometry

local
symmetries of space and time ←→ geometry of space and time



spacetime symmetries:

time translations, space translations, spatial rotations + …..

crucial difference → type of boosts

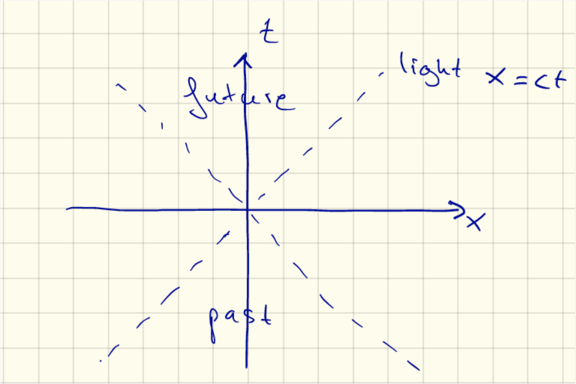
relativistic (Lorentz) $t \to \gamma(t + \vec{v}\vec{x}/c^2) \quad , \quad \vec{x} \to \gamma(\vec{x} + \vec{v}t)$

non-relativisistic (Galilean) $c \to \infty$ $t \to t \quad , \quad \vec{x} \to \vec{x} + \vec{v}t$

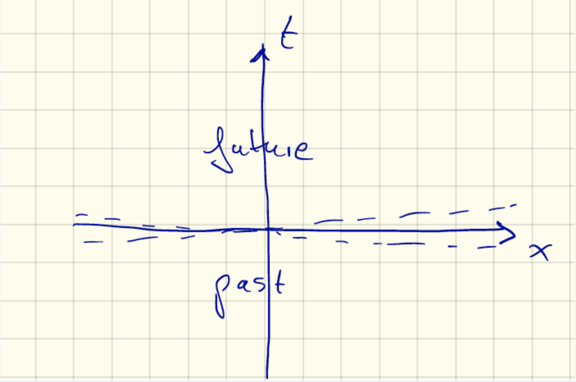
ultra-relativistic (Carroll) $c \to 0$ $t \to t + \vec{\tilde{v}}\vec{x} \quad , \quad \vec{x} \to \vec{x}$

Lightcones

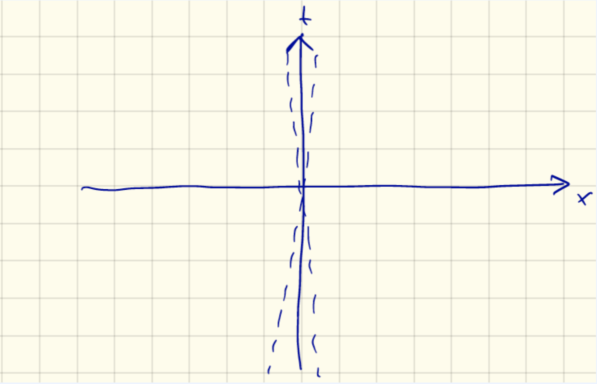
Lorentz



Galilean



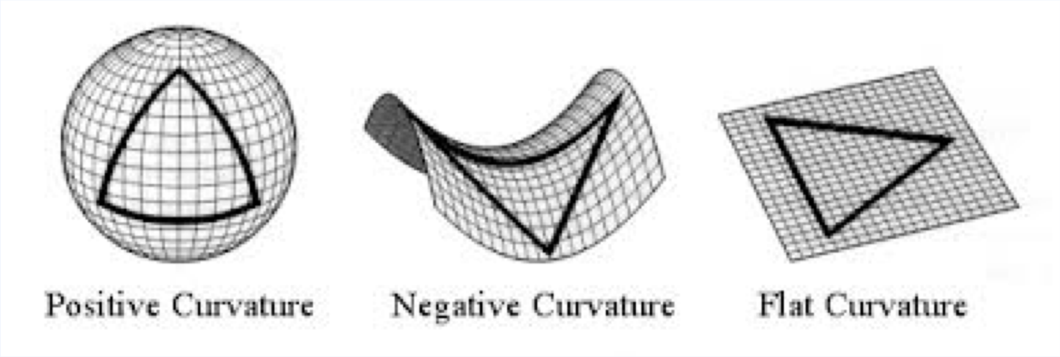
Carroll



(Pseudo)-Riemannian geometry

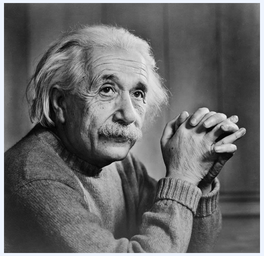
local
symmetries of space and time ←→ geometry of space and time

Einstein: Lorentz ←→ (pseudo-)Riemannian geometry



Einstein equivalence principle:
freely falling observers do not experience gravity
& laws of physics obey special relativity (local Lorentz sym.)

Einstein's General Relativity



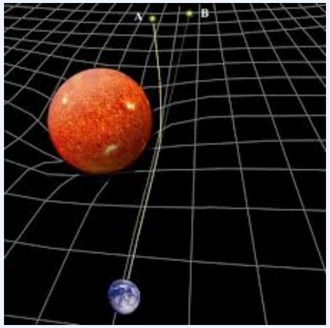
GR: gravity = dynamical pseudo-Riemannian geometry

frame-independence encoded in metric tensor:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

Einstein's equations: $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

motion of test-bodies: geodesics



- Einstein achieved two things:

- frame independence (diffeomorphism invariance)
- local Lorentz symmetry (relativity)

Newton-Cartan geometry

Cartan (1923): Newtonian gravity written in frame-independent way
using Newton-Cartan geometry

local
symmetries of space and time $\leftarrow\rightarrow$ geometry of space and time

Cartan: Galilean $\leftarrow\rightarrow$ Newton-Cartan geometry

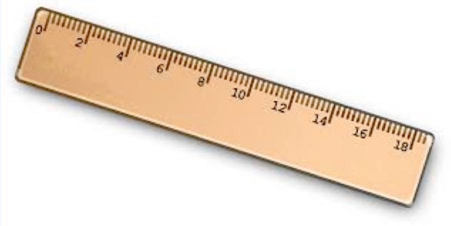
[Eisenhart,Trautman,Dautcourt,Kuenzle,Duval,Burdet,Perrin,Gibbons,Horvathy,Nicolai,Julia…] ..
[Andriga,Bergshoeff,Panda,deRoo(CQG 2011)]

Equivalence principle: freely falling observers see Galilean laws of physics

Newton-Cartan geometric datea

clock form:  τ_μ

$\tau_\mu h^{\mu\nu} = 0$

spatial metric:  $h^{\mu\nu}$

(ruler)

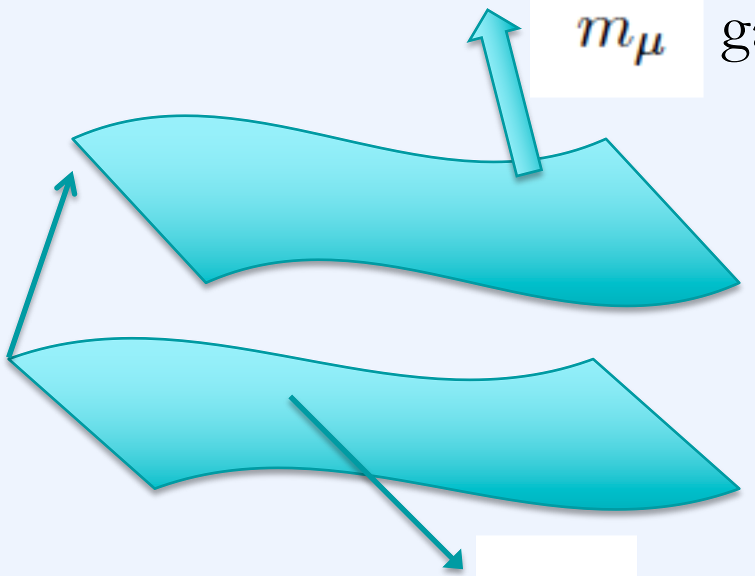
vector field: m_μ

(encodes Newtonian potential) .

- in Newtonian gravity time is absolute: $\partial_\mu \tau_\nu - \partial_\nu \tau_\mu = 0$

- geometrizes Poisson equation of Newtonian gravity:

Torsional Newton-Cartan geometry



NC = no torsion → $\tau_\mu = \partial_\mu t$ absolute time

TTNC = twistless torsion → $\tau_\mu = \mathrm{HSO}$ preferred foliation equal time slices

TNC no condition on τ_μ

Christensen,Hartong,NO,Rollier (PRD, 2013)

Poisson equation of Newtonian gravity in NC form

$$\bar{R}_{\mu\nu} = 4\pi G \rho_{\mathrm{m}} \tau_\mu \tau_\nu$$

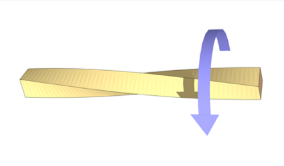
- with $d\tau = 0$ (abs. time)

recent new insights: [Hansen,Hartong,NO (PRL 2019)]

- need different version of NC geometry to find action for Newtonian gravity

- follows from (careful) large speed expansion of GR
- goes beyond Newtonian gravity

inclusion of torsion essential



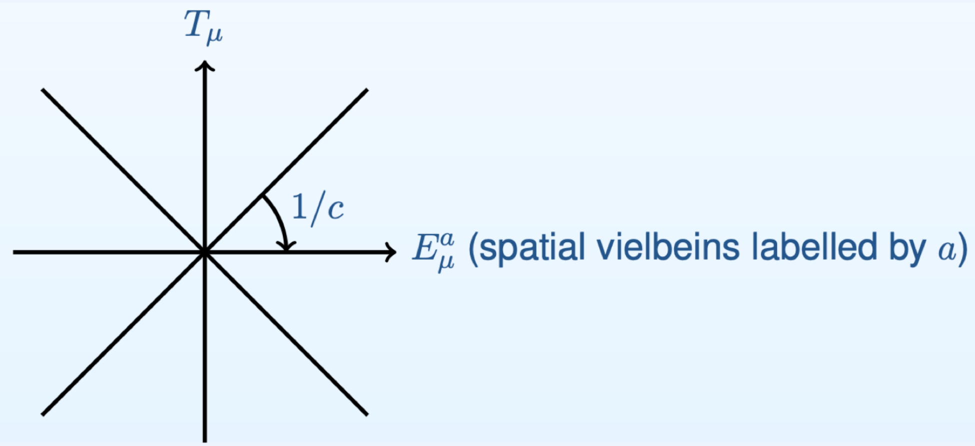
- new symmetry principle which can be used at any order in 1/c

- non-relativistic gravity from $1/c$ expansion

speed of light dependence in GR

metric: $$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}$$

T = time-like vielbein
Pi = spatial metric



large speed of light → light-cone opens up

expand in $$\sigma = 1/c^2$$

pre-non-relativistic GR

rewrite GR in terms of T and Pi:

new choice of connection:

$$C^{\rho}_{\mu\nu} = -T^{\rho}\partial_{\mu}T_{\nu} + \frac{1}{2}\Pi^{\rho\sigma}\left(\partial_{\mu}\Pi_{\nu\sigma} + \partial_{\nu}\Pi_{\mu\sigma} - \partial_{\sigma}\Pi_{\mu\nu}\right)$$

has torsion: proportional to: $T_{\mu\nu} \equiv \partial_{\mu}T_{\nu} - \partial_{\nu}T_{\mu}\,.$

analogue of metric compatibility $\overset{(C)}{\nabla}_{\mu}T_{\nu} = 0\,, \qquad \overset{(C)}{\nabla}_{\mu}\Pi^{\nu\rho} = 0\,,$

EH Lagrangian in pre-non-relativistic form

$$\tilde{\mathcal{L}} = E\left[\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}T_{\mu\rho}T_{\nu\sigma} + \sigma\Pi^{\mu\nu}\overset{(C)}{R}_{\mu\nu} - \sigma^{2}T^{\mu}T^{\nu}\overset{(C)}{R}_{\mu\nu}\right]$$

recent progress: understand this in 1st order formulation of GR

Hansen,Hartong,OlingNO(2020)

NR gravity from large c expansion of GR

metric in GR depends on speed of light c:

expand in 1/c

Dautcourt (1996)
van den Bleeken(2017)
Hansen,Hartong,NO(2018,2020)

$$g_{\mu\nu} = -c^2\tau_\mu\tau_\nu + h_{\mu\nu} - m_\mu\tau_\nu - m_\nu\tau_\mu + \frac{1}{c^2}(\cancel{B_\mu}\tau_\nu + \cancel{B_\nu}\tau_\mu - \Phi_{\mu\nu}) + O(c^{-4})$$

- LO and NLO fields define a novel version of NC geometry

LO fields:	$\tau_\mu \quad h_{\mu\nu}$
NLO fields:	$m_\mu \quad \Phi_{\mu\nu}$

- expand Einstein-Hilbert action of GR:

$$S_{\text{EH}} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$

weak NR limit of Schwarzschild

Schw with factors of c reinstated

$$ds^2 = -c^2\left(1 - \frac{2Gm}{c^2 r}\right) dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

weak limit: m independent of c

$$\begin{aligned} \tau_\mu dx^\mu &= dt\,, \qquad h_{\mu\nu} dx^\mu dx^\nu = dr^2 + r^2 d\Omega_{S^2} \\ m_\mu dx^\mu &= -\frac{Gm}{r} dt\,, \qquad \Phi_{\mu\nu} dx^\mu dx^\nu = \frac{2Gm}{r} dr^2 \end{aligned}$$

point mass in flat space with Newtonian potential: Phi = -Gm/r

absolute time: tau is exact

strong NR limit of Schwarzschild

m = c^2 M; M independent of m (VdB, 2017)

$$\tau_\mu dx^\mu = \sqrt{1 - \frac{2GM}{r}} dt\,, \qquad h_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

$$m_\mu dx^\mu = 0 = \Phi_{\mu\nu} dx^\mu dx^\nu$$

this strong expansion of Schw is not captured by Newtonian gravity:

- still described by NC geometry

different approx. of GR as compared to Post-Newtonian expansion (strong field)

tau no longer exact but: hypersurface orthogonal $\tau \wedge d\tau = 0$

strong limit captures gravitational time dilaton:
clocks tick faster/slower depending on position on constant time slice

Action of non-relativistic gravity (NRG)

expand Einstein-Hilbert action of GR

leading order → EOMs imply causality

next-to-leading order: bifurcation in either → absolute time / → hypersurface orthogonality
(depends on the matter sources)

- NNLO gives NRG action:

$$\mathcal{L} = -\frac{1}{16\pi G} e \Big[\hat{v}^\mu \hat{v}^\nu \bar{R}_{\mu\nu} - \tilde{\Phi} h^{\mu\nu} \bar{R}_{\mu\nu} - \Phi_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} \left(\bar{R}_{\rho\sigma} - a_\rho a_\sigma - \bar{\nabla}_\rho a_\sigma \right) + \frac{1}{2} \Phi_{\mu\nu} h^{\mu\nu} \left[h^{\rho\sigma} \bar{R}_{\rho\sigma} - 2 e^{-1} \partial_\rho \left(e h^{\rho\sigma} a_\sigma \right) \right] \Big]$$

- for the first time an action principle for Newtonian gravity !
- goes beyond by allowing for strong gravity (gravitational time dilation)

Coupling of matter to NRG

coupling of matter: perform 1/c expansion of relativistic matter
e.g. point particles, scalar/vector fields, fluids, …

- simplest case: non-relativistic point particle

$$-m\int \mathrm{d}\lambda\, \delta(x - x(\lambda))\tau_\mu \dot{x}^\mu$$

EOMs of the total action give:

- time = absolute
- Newton's Poisson equation

Strong gravity in NRG

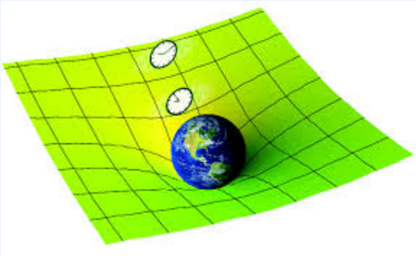
Strong gravity regime:
close to compact object with Schwarzshild radius R_s

warping of time → spacetime with torsion

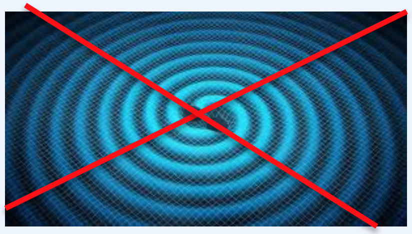
$$\tau_t = -\left(1 - \frac{R_s}{r}\right)$$

NR geodesics pass 3 classical tests of GR:

- precession perihelium
- bending of light
- gravitational redshift



but: no gravitational waves



Punchlines

new version of TNC (type II) is what the large c expansion of GR tells us to do !

What does it achieve ?

- while Cartan's original geometry can geometrize EOMS, it cannot be used to define the theory off-shell
  → needed for the action (analogue of EH action for NRG)

- the NRG action can then simply be obtained by doing the (right) expansion of GR (see below)

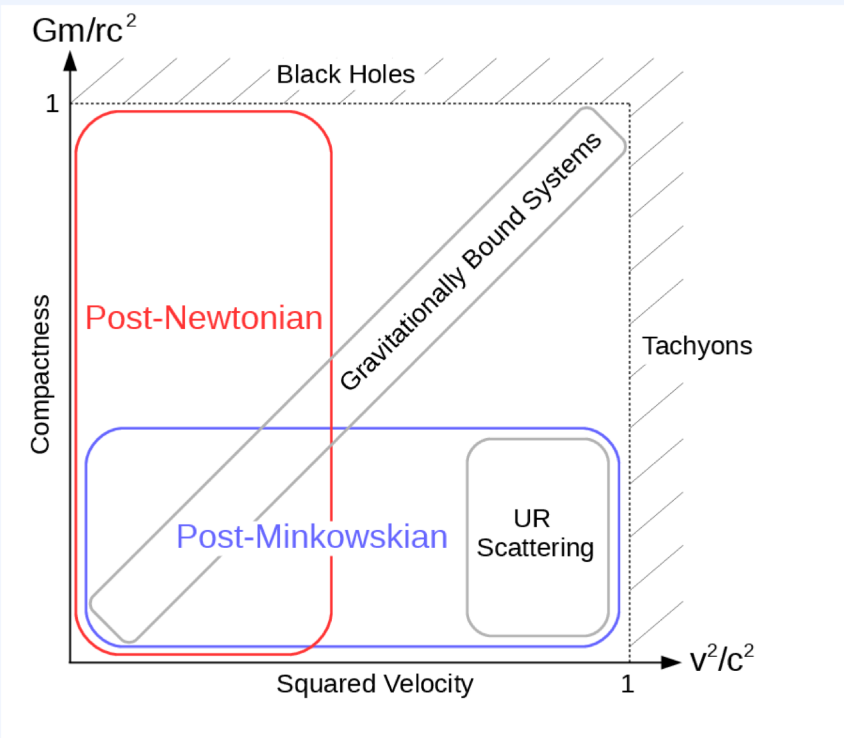
- what replaces Poincare invariance ?
  → new symmetry algebra that follows from Poincare from a well-defined procedure (Lie algebra expansion of Poincare)

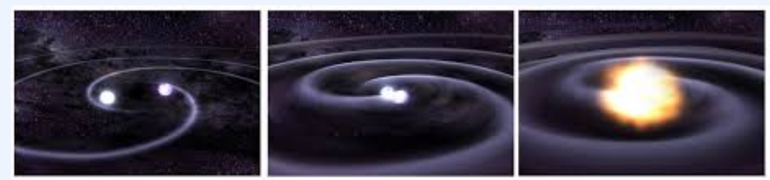
(principle which can be used to geometrize any Post-Newtonian order)

Post-Newtonian expansion

covariant treatment of PN physics

- application to (early phase of) binary inspirals ?



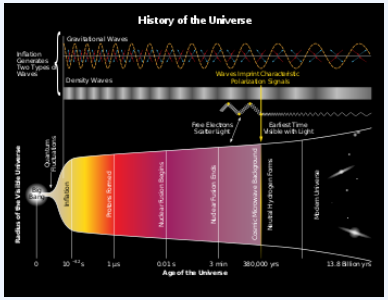


Post-Minkowskian:
resum effects in v/c at give order in G

Post-Newtonian:
use NRG to resum effects of G at given order in 1/c ?

Further properties of NRG

- cosmology: FRW solutions



- Newton-Schroedinger theory:

Coupling of non-relativistic field (electron/neutron) to NRG

- well-defined framework to treat PN corrections
- possible useful starting point to further analyze QM effects (gravitationally induced quantum interference with neutron beams)

- non-relativistic geometry in the Lab

Field theory on curved spacetime

Coupling to a background metric is powerful tool in relativistic theories (QED, QCD ,..)

→ putting the field theory on a curved spacetime

$$\delta S_{\mathrm{rel.matter}} \sim \int d^4x \, T_{\mu\nu} \, \delta g^{\mu\nu}$$

- responses to varying metric gives energy-momentum tensor

- can find Ward identities as consequence of symmetries

- organizing principle for effective theories/hydrodynamics

Newton-Cartan geometry in the Lab

non-relativistic systems ubiquitous condensed matter/biophysical systems

- Newton-Cartan geometry as a tool in effective theories
  e.g. in fractional quantum Hall effect: displays non-relativistic symmetries
  (Son, 2013)

Newton-Cartan geometry is correct background spacetime

$$\delta S_{\text{non-rel matter}} \sim \int d^3x [E^\mu \delta\tau_\mu + S^{\mu\nu}\delta h_{\mu\nu} + T^\mu \delta m_\mu]$$

energy current   spatial stress & momentum current   mass current

- universal predictions
- first-principle method to write down possible terms in effective action

Schroedinger field coupled to Newton-Cartan

simplest case: complex non-relativistic scalar field (Schroedinger field)

$$S_{\text{Schr.}} \sim \int d^4x [i\psi^\dagger \partial_t \psi - \frac{1}{2m} \nabla\psi\nabla\psi^\dagger] + \text{h.c.}$$

covariant coupling to background Newton-Cartan geometry:

$$S_{\text{Schr.}}^{\text{NC}} \sim \int d^4x e \left[i(\tau^\mu - h^{\mu\nu} m_\nu)\psi^\dagger \partial_\mu \psi - \frac{1}{2m} h^{\mu\nu} \partial_\mu \psi^\dagger \partial_\nu \psi \right] . + \text{h.c}$$

- enters Son's description of FQHE & interactions to EM field

- opposite case: Carrollian expansion of GR

Hansen,NO,Oling,Søgaard (2112.12684)

What about small speed of light limit ?

Carroll limit:

zero speed of light contraction of Poincare

running (boosting) without moving:

Red Queens race of L. Carroll's Through the looking glass Levy-Leblond(1965)

→ expand relativistic theories around c=0:

what do we get ? what can we use it for ?

(Carroll symmetry as organizing principle for perturbative expansion around c=0)

terminology: this is not ultra-relativistic limit ($v/c \rightarrow 1$)

rather ultra-local limit

Intro

Carroll group is kinematical group → Carrollian manifolds

(timelike and spacelike) vielbeine transforming under local Carroll boosts

Bekaert,Morand,2015/Hartong,2015/Figueroa-O'Farrill,Prohazka,2018

example: null hypersurfaces

e.g. null infinity of assymptotically flat spacetime

Duvall,Gibbons,Horvathy,2014

Examples

- black hole membrane paradagim
  [Donnay,Marteau,2019],[Penna,2018]

- (3D) flat space holography
  [Bagchi,Detournay,Fareghbal,Simon,2012],[Hartong,2015],
  [Ciambella,Marteau,Petkou,Petropoulos,Siampos,2018]

- tensionless limits of strings
  [Bagchi,2013]

- limits of GR
  [Henneaux,1979],[Bergshoeff,Gomis,Rollier,ter Veldhuis,2017],[Hansen et al, 2021]

- inflationary cosmology
  [de Boer,Hartong,Obers,Sybesma,Vandoren,2021]

- setups with: effective speed of light (characteristic speed) << velocity

Carroll geometry

- PUR (pre-"ultra-relativistic") expansion of pseudo-Riem. geometry:

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \Pi_{\mu\nu}, \qquad g^{\mu\nu} = -\frac{1}{c^2} V^\mu V^\nu + \Pi^{\mu\nu}.$$

- $c \to 0$ expansion:

NLO fields

$$V^\mu = v^\mu + c^2 M^\mu + \mathcal{O}(c^4), \qquad T_\mu = \tau_\mu + \mathcal{O}(c^2)$$

$$\Pi^{\mu\nu} = h^{\mu\nu} + c^2 \Phi^{\mu\nu} + \mathcal{O}(c^4), \qquad \Pi_{\mu\nu} = h_{\mu\nu} + \mathcal{O}(c^2),$$

LO fields: Carrollian geometry.

$$\tau_\mu v^\mu = -1, \quad \tau_\mu h^{\mu\nu} = 0, \quad h_{\mu\nu} v^\nu = 0, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}.$$

$$h^{\mu\nu} = \delta^{ab} e^\mu_a e^\nu_b.$$

- transformation of Carrollian geometry fields under Carrol boosts:

$$\delta\tau_\mu = \lambda_a e^a_\mu \qquad \delta h^{\mu\nu} = v^\mu e^\nu_a \lambda^a + v^\nu e^\mu_a \lambda^a \qquad \delta M^\mu = \lambda^a e^\mu_a$$

$h_{\mu\nu}$, v^μ are left invariant.

Preferred connection and EH action

- Convenient PUR connection:

$$\tilde{C}^{\rho}_{\mu\nu} = -V^{\rho}\partial_{(\mu}T_{\nu)} - V^{\rho}T_{(\mu}\mathcal{L}_V T_{\nu)} + \frac{1}{2}\Pi^{\rho\lambda}\left[\partial_\mu \Pi_{\nu\lambda} + \partial_\nu \Pi_{\lambda\mu} - \partial_\lambda \Pi_{\mu\nu}\right] - \Pi^{\rho\lambda}T_\nu \mathcal{K}_{\mu\lambda},$$

$$\mathcal{K}_{\mu\nu} = -\tfrac{1}{2}\mathcal{L}_V \Pi_{\mu\nu}.$$

extrinsic curvature.

- yields the Carrollian $(c \to 0)$ (minimal torsion) connection:

$$\tilde{\Gamma}^{\rho}_{\mu\nu} = \tilde{C}^{\rho}_{\mu\nu}\Big|_{c=0} = -v^{\rho}\partial_{(\mu}\tau_{\nu)} - v^{\rho}\tau_{(\mu}\mathcal{L}_v \tau_{\nu)} + \frac{1}{2}h^{\rho\lambda}\left[\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}\right] - h^{\rho\lambda}\tau_\nu K_{\mu\lambda},$$

$$\tilde{\nabla}_\mu v^\mu = 0, \qquad \tilde{\nabla}_\rho h_{\mu\nu}.$$

'metric' compatibility.

- Einstein-Hilbert in the PUR variables:

$$\boxed{R \approx \frac{1}{c^2}\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^2\right) + \Pi^{\mu\nu}\overset{(\tilde{C})}{R}_{\mu\nu} + \frac{c}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma}\left(dT\right)_{\mu\rho}\left(dT\right)_{\nu\sigma},}$$

LO and NLO action

- expand R for c→0 ; LO action :

$$\boxed{\overset{(2)}{\mathcal{L}}_{\text{LO}} = \frac{e}{16\pi G_N}\left[K^{\mu\nu}K_{\mu\nu} - K^2\right],}$$

Henneaux,1979

$$\delta\overset{(2)}{\mathcal{L}}_{\text{LO}} = \frac{e}{8\pi G_N}\left[\overset{(2)}{G}{}^{v}_{\mu}\delta v^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\delta h^{\mu\nu}\right],$$

$$\begin{aligned}\overset{(2)}{G}{}^{v}_{\mu} &= -\frac{1}{2}\tau_\mu(K^{\rho\sigma}K_{\rho\sigma} - K^2) + h^{\gamma\lambda}\tilde{\nabla}_\lambda(K_{\mu\gamma} - Kh_{\mu\gamma}),\\ \overset{(2)}{G}{}^{h}_{\mu\nu} &= -\frac{1}{2}h_{\mu\nu}(K^{\rho\sigma}K_{\rho\sigma} - K^2) + K(K_{\mu\nu} - Kh_{\mu\nu}) - v^\rho\tilde{\nabla}_\rho(K_{\mu\nu} - Kh_{\mu\nu}).\end{aligned}$$

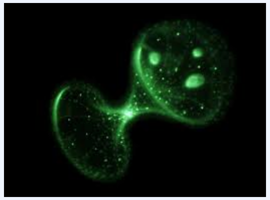
- NLO action :

$$\boxed{\overset{(4)}{\mathcal{L}}_{\text{NLO}} = \frac{e}{8\pi G_N}\left[\frac{1}{2}h^{\mu\nu}\tilde{R}_{\mu\nu} + \overset{(2)}{G}{}^{v}_{\mu}M^{\mu} + \frac{1}{2}\overset{(2)}{G}{}^{h}_{\mu\nu}\Phi^{\mu\nu}\right].}$$

see also: Bergshoeff et al,2017

Outlook

non-relativistic geometry has opened up to apply the
power of geometry in many new directions in physics:



- new approaches in wide variety of problems:

- quantum gravity / string theory / holography

- classical gravity / astrophysics

- [also: hydrodynamics / CMT / soft matter]

Carroll geometry likewise in:

limits of GR &solutions/cosmology/flat space holography/string theory
Carrollian fluids&dark energy

The End

Thank you for your attention !