Chern-Simons formulation of non- and ultra-relativistic (super)gravity theories in 2+1 spacetime dimensions

Based on Phys. Lett. B **804** (2020), 135392, JHEP **04** (2020), 051, and JHEP **02** (2020), 128

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Outline

- Non-relativistic (NR) and ultra-relativistic (UR) theories: Framework and Chern-Simons (CS) formulation of three-dimensional models
- State of art, open issues $\rightsquigarrow$ Recent results in the context of NR and UR $2+1$-dimensional CS (super)gravity theories:
  - Exotic Newtonian gravity with cosmological constant

    P. Concha, L.R., E. Rodríguez, Phys. Lett. B **804** (2020), 135392 [arXiv:1912.02836 [hep-th]]
  - Maxwellian extended Bargmann supergravity

    P. Concha, L.R., E. Rodríguez, JHEP **04** (2020), 051 [arXiv:1912.09477 [hep-th]]
  - $\mathcal{N}$-extended CS Carrollian supergravities (AdS and flat limit)

    Farhad Ali, L.R., JHEP **02** (2020), 128 [arXiv:1912.04172 [hep-th]]
- Conclusions and future developments

NR and UR symmetries

- NR symmetries $\rightarrow$ NR limit, $c \rightarrow \infty$
- UR symmetries $\rightarrow$ UR limit, $c \rightarrow 0$

Useful to describe some physical phenomena. For instance:

- NR: Gravity models $\rightarrow$ Different NR gravity theories invariant under distinct extensions of the Galilei symmetries ($c \rightarrow \infty$ limit of Poincaré); Galilean conformal symmetries $\rightarrow$ Applications in AdS/CFT; Useful to approach condensed matter systems; Newton-Cartan geometry $\rightarrow$ Lifshitz holography; NR superstrings and superbranes were studied as special points in the parameter space of M-theory; NR strings appear as a possible soluble sector within string or M-theory; etc.
- UR: Carroll symmetry introduced by Lévy-Leblond emerged as the UR contraction ($c \rightarrow 0$) of Poincaré (dual to the NR contraction); Applications in the study of tachyon condensation; Study of warped CFTs; Tensionless strings; Relations among BMS group, Carrollian physics, flat holography and fluid/gravity correspondence; etc.

NR and UR gravity theories

NR

- Simplest example: Galilei gravity theory, invariant under the (unextended) Galilei symmetries
- Newtonian gravity and Newton-Cartan gravity (frame-independent reformulation) are invariant under the Bargmann algebra (central extension of the Galilei algebra)
- Dynamical (field theoretic) realization of Newton-Cartan geometry formulated by Banerjee *et al.* $\rightarrow$ Galilean gauge theory of gravity $\rightsquigarrow$ Exact mapping between this theory and the Poicaré gauge theory of gravity

UR

- Theories of Carrollian gravity developed and analyzed by Hartong *et al.* and Bergshoeff *et al.*; In particular, (NR and) UR CS actions in $2+1$ dimensions
- Bergshoeff *et al.*: The geometry of flat and curved (AdS) Carroll space and the symmetries of a particle moving in such a space, both in the bosonic as well as in the supersymmetric case, were investigated
- Matulich *et al.*: AdS Carroll CS gravity discussed for the first time

FOCUS ON NR AND UR THREE-DIMENSIONAL CS GRAVITY THEORIES: WHY?

- $3D$ theories: Toy models to approach higher-dimensional theories
- $3D$ gravity does not allow for propagating local degrees of freedom $\rightarrow$ The solutions of the corresponding theories are consequently locally $3D$ Minkowski or (A)dS, with the symmetries of the corresponding spacetime as gauge algebras
- In particular, gravity on AdS_3 + proper boundary conditions $\rightarrow$ The asymptotic symmetries of AdS_3 yield the infinite-dimensional symmetries of a CFT_2 $\rightarrow$ Well celebrated AdS/CFT duality

- CS formulation: Possibility to write gravity in a gauge theory formulation
- CS action based on either of the gauge algebras $\mathfrak{iso}(2,1)$, $\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$ $\leftrightarrow$ Classically equivalent to the Einstein-Hilbert action in its first-order formulation with zero, positive, or negative cosmological constant, respectively

- Obs. 1: NR and UR gravity theories can be constructed either as limit of relativistic theories or by hand
- Obs. 2: In the case of a CS model, the proper construction of the action requires a **non-degenerate invariant bilinear form** (invariant tensor) $\Rightarrow$ Introduction of additional generators (together with their dual 1-form fields) is often necessary $\rightarrow$ Extensions (or expansions) of the algebraic structure on which the theory is based

Framework: Lie (super)algebras and Maurer-Cartan equations

Lie (super)algebras and dual formulation

$$[T_A, T_B\} = C_{AB}{}^C T_C$$

$$A^A(T_B) = \delta^A_B$$

A^A: Differential 1-form fields

$$\downarrow$$

Maurer-Cartan equations (vacuum)

$$dA^A + \frac{1}{2} C_{BC}{}^A A^B \wedge A^C = 0$$

$d^2 = 0 \leftrightarrow$ Jacobi identities

$$F^A \equiv dA^A + \frac{1}{2} C_{BC}{}^A A^B \wedge A^C \neq 0 \quad \text{(out of the vacuum)}$$

F^A: Curvatures, (super) field strengths

Chern-Simons construction

General expression of a CS action in $D=3$

$$I_{\text{CS}} = \frac{k}{4\pi}\int_{\mathcal{M}} \left\langle A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right\rangle = \frac{k}{4\pi}\int_{\mathcal{M}} \left\langle A\wedge F - \frac{1}{3}A\wedge A\wedge A\right\rangle$$

- k: CS level of the theory (for gravitational theories $k=1/(4G)$); $\langle\ldots\rangle$ denotes the invariant tensor (non-degenerate invariant bilinear form); The integral is over a $3D$ manifold $\mathcal{M}$
- A: Gauge connection 1-form $A=A^A T_A$; Corresponding curvature 2-form $F=F^A T_A = dA + A\wedge A$
- E.o.m.: Vanishing of the (super) field strengths

- (Super)algebra $\rightarrow$ CS action (non-degenerate invariant bilinear form)
- NR and UR CS (super)gravity actions can be constructed
  - Either **by hand** (algebraic structure providing a non-degenerate invariant bilinear form) $\rightarrow$ NR/UR limit of which relativistic theory?
  - Or through **NR and UR limits** of a (super)algebra $\rightarrow$ NR/UR CS action (well defined contraction)

State of art (purely bosonic, $D = 3$)

- **(A)dS** (Generators: J_A, P_A, $A = 0, 1, 2$; Dual 1-form fields: ω^A, V^A; Length parameter $\ell \leftrightarrow$ Cosmological constant $\Lambda \propto \pm 1/\ell^2$) $\rightarrow$ Flat limit ($\ell \rightarrow \infty$, that is $\Lambda \rightarrow 0$) $\rightarrow$ **Poincaré**
- **(A)dS**, split $A \rightarrow \{a, 0\}$, $a = 1, 2 \Rightarrow J_A \rightarrow \{K_a, J\}$, $P_A \rightarrow \{P_a, H\}$; Then rescale $K_a \rightarrow \sigma K_a$ and $H \rightarrow \sigma H$, and $\sigma \rightarrow \infty$ ($c \rightarrow 0$, UR limit) $\rightarrow$ **(A)dS Carroll**
- **Poincaré**, split $A \rightarrow \{a, 0\}$, $a = 1, 2 \Rightarrow J_a \rightarrow \{K_a, J\}$, $P_A \rightarrow \{P_a, H\}$; Then, $K_a \rightarrow \sigma K_a$ and $H \rightarrow \sigma H$, and $\sigma \rightarrow \infty$ ($c \rightarrow 0$, UR limit) $\rightarrow$ **Carroll** (flat limit of (A)dS Carroll)
- Dual limit ($c \rightarrow \infty$, NR limit): **Poincaré** $\rightarrow$ **Galilei**
- **Bargmann** algebra (centrally extended Galilean algebra) is the underlying symmetry of Newtonian gravity
- Recently: A $3D$ CS (super)gravity theory based on the **extended Newtonian algebra** (which requires to extend the so-called extended Bargmann algebra by including new generators and central charges) has been presented
- NR models that include a cosmological constant described through the **Newton-Hooke** symmetry; Flat limit $\rightarrow$ **Galilei** symmetry
- **The incorporation of a cosmological constant in the extended Newtonian gravity was an open issue**

Some open issues

- The incorporation of a cosmological constant in the (extended) Newtonian gravity was an open issue

◊ We have a solution to this open problem, finding an **action principle for Newtonian gravity including a cosmological constant**

P. Concha, L.R., E. Rodríguez, Phys. Lett. B **804** (2020), 135392 [arXiv:1912.02836 [hep-th]]

- NR construction of supergravity theories is challenging and has only been approached in $D = 3$ (it requires additional fermionic generators to construct a non-degenerate invariant bilinear form which ensures the proper construction of a CS action)

◊ We explored the NR limit of the so-called Maxwell superalgebra for $\mathcal{N} = 1$ and $\mathcal{N} = 2$, showing that a **well-defined NR Maxwellian CS supergravity action** requires additional fermionic and bosonic generators

P. Concha, L.R., E. Rodríguez, JHEP **04** (2020), 051 [arXiv:1912.09477 [hep-th]]

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- A study of Carrollian (UR) supersymmetric models in the context of supergravity was still lacking

◊ We developed and analyzed $\mathcal{N}$**-extended CS Carrollian supergravity theories in** $D = 3$ for the first time

Farhad Ali, L.R., JHEP **02** (2020), 128 [arXiv:1912.04172 [hep-th]]

Review of the extended Newtonian gravity theory

- **Extended Newtonian algebra**: Extended Bargmann generators $\{J, G_a, S, H, P_a, M\}$, $a = 1, 2$, together with $\{T_a, B_a\}$ and two central charges, Y and Z (the latter ensure non-degeneracy of the invariant tensor)
- Non-vanishing components of the invariant tensor:

$$\langle MS \rangle = \langle HZ \rangle = -\langle JY \rangle = -\beta_1 \,, \quad \langle P_a B_b \rangle = \langle G_a T_b \rangle = \beta_1 \delta_{ab} \,, \quad \langle JS \rangle = -\alpha_0 \,, \quad \langle G_a G_b \rangle = \alpha_0 \delta_{ab} \,,$$
$$\langle JM \rangle = \langle HS \rangle = -\alpha_1 \,, \quad \langle G_a P_b \rangle = \alpha_1 \delta_{ab} \,, \quad \langle SS \rangle = \langle JZ \rangle = -\beta_0 \,, \quad \langle G_a B_b \rangle = \beta_0 \delta_{ab}$$

- Gauge connection 1-form $A = \tau H + e^a P_a + \omega J + \omega^a G_a + mM + sS + t^a T_a + b^a B_a + yY + zZ$
- Corresponding 2-form curvatures:

$$R(\tau) = d\tau \,, \quad R^a\left(e^b\right) = de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c \,, \quad R(\omega) = d\omega \,, \quad R^a\left(\omega^b\right) = d\omega^a + \epsilon^{ac}\omega\omega_c \,,$$
$$R(m) = dm + \epsilon^{ac}\omega_a e_c \,, \quad R(s) = ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c \,, \quad R^a\left(t^b\right) = dt^a + \epsilon^{ac}\omega t_c + \epsilon^{ac}\tau b_c + \epsilon^{ac} s e_c + \epsilon^{ac} m\omega_c \,,$$
$$R^a\left(b^b\right) = db^a + \epsilon^{ac}\omega b_c + \epsilon^{ac} s\omega_c \,, \quad R(y) = dy - \epsilon^{ac}\omega_a t_c - \epsilon^{ac} e_a b_c \,, \quad R(z) = dz + \epsilon^{ac}\omega_a b_c$$

Three-dimensional exotic Newtonian gravity with cosmological constant

- $3D$ CS generalized extended Newtonian gravity action:

$$
\begin{aligned}
I_{\text{gEN}} \;=\; & \frac{k}{4\pi}\int \alpha_0\left[\omega_a R^a\left(\omega^b\right)-2sR\left(\omega\right)\right]+2\alpha_1\left[e_a R^a\left(\omega^b\right)-mR\left(\omega\right)-\tau R\left(s\right)\right] \\
& +\beta_0\left[b_a R^a\left(\omega^b\right)+\omega_a R^a\left(b^b\right)-2zR\left(\omega\right)-sds\right] \\
& +2\beta_1\left[e_a R^a\left(b^b\right)+t_a R^a\left(\omega^b\right)+yR\left(\omega\right)-mR\left(s\right)-\tau R\left(z\right)\right]
\end{aligned}
$$

Explore the possibility to include a cosmological constant by introducing an explicit length parameter ℓ

- Same generators of the extended Newtonian algebra, but the presence of a **cosmological constant** implies **new non-vanishing commutators** involving an explicit scale $\ell \Rightarrow$ **Exotic Newtonian algebra**:

$$
\begin{aligned}
&[J,G_a]=\epsilon_{ab}G_b\,, \quad [G_a,G_b]=-\epsilon_{ab}S\,, \quad [H,G_a]=\epsilon_{ab}P_b\,,\\
&[J,P_a]=\epsilon_{ab}P_b\,, \quad [G_a,P_b]=-\epsilon_{ab}M\,, \quad [H,B_a]=\epsilon_{ab}T_b\,,\\
&[J,B_a]=\epsilon_{ab}B_b\,, \quad [G_a,B_b]=-\epsilon_{ab}Z\,, \quad [J,T_a]=\epsilon_{ab}T_b\,,\\
&[S,G_a]=\epsilon_{ab}B_b\,, \quad [G_a,T_b]=\epsilon_{ab}Y\,, \quad [S,P_a]=\epsilon_{ab}T_b\,,\\
&[M,G_a]=\epsilon_{ab}T_b\,, \quad [P_a,B_b]=\epsilon_{ab}Y\,, \quad [H,P_a]=\frac{1}{\ell^2}\epsilon_{ab}G_b\,,\\
&[H,T_a]=\frac{1}{\ell^2}\epsilon_{ab}B_b\,, \quad [P_a,P_b]=-\frac{1}{\ell^2}\epsilon_{ab}S\,, \quad [M,P_a]=\frac{1}{\ell^2}\epsilon_{ab}B_b\,, \quad [P_a,T_b]=-\frac{1}{\ell^2}\epsilon_{ab}Z
\end{aligned}
$$

Three-dimensional exotic Newtonian gravity with cosmological constant

- The exotic Newtonian algebra admits the non-vanishing components of the invariant tensor of the extended Newtonian algebra along with

$$\langle HM\rangle = -\frac{\alpha_0}{\ell^2}\,, \quad \langle P_a P_b\rangle = \frac{\alpha_0}{\ell^2}\delta_{ab}\,, \quad \langle MM\rangle = -\langle HY\rangle = -\frac{\beta_0}{\ell^2}\,, \quad \langle P_a T_b\rangle = \frac{\beta_0}{\ell^2}\delta_{ab}$$

- The 1-form gauge connection is the same, as the field content is the same
- The exotic Newtonian 2-form curvatures are

$$R(\tau) = d\tau\,, \quad R^a\left(e^b\right) = de^a + \epsilon^{ac}\omega e_c + \epsilon^{ac}\tau\omega_c\,, \quad R(\omega) = d\omega\,, \quad \hat{R}^a\left(\omega^b\right) = d\omega^a + \epsilon^{ac}\omega\omega_c + \frac{1}{\ell^2}\epsilon^{ac}\tau e_c\,,$$

$$R(m) = dm + \epsilon^{ac}\omega_a e_c\,, \quad \hat{R}(s) = ds + \frac{1}{2}\epsilon^{ac}\omega_a\omega_c + \frac{1}{2\ell^2}\epsilon^{ac}e_a e_c\,,$$

$$R^a\left(t^b\right) = dt^a + \epsilon^{ac}\omega t_c + \epsilon^{ac}\tau b_c + \epsilon^{ac}s e_c + \epsilon^{ac}m\omega_c\,,$$

$$\hat{R}^a\left(b^b\right) = db^a + \epsilon^{ac}\omega b_c + \epsilon^{ac}s\omega_c + \frac{1}{\ell^2}\epsilon^{ac}\tau t_c + \frac{1}{\ell^2}\epsilon^{ac}m e_c\,,$$

$$R(y) = dy - \epsilon^{ac}\omega_a t_c - \epsilon^{ac}e_a b_c\,, \quad \hat{R}(z) = dz + \epsilon^{ac}\omega_a b_c + \frac{1}{\ell^2}\epsilon^{ac}e_a t_c$$

- The flat limit $\ell \to \infty$ reproduces the extended Newtonian ones

Three-dimensional exotic Newtonian gravity with cosmological constant

- 3D CS action exotic Newtonian gravity action (well-defined, non-degeneracy of the invariant bilinear form):

$$
\begin{aligned}
I_{\text{exN}} \;=\; & \frac{k}{4\pi}\int \alpha_0\left[\omega_a\hat{R}^a\left(\omega^b\right)-2sR\left(\omega\right)+\frac{1}{\ell^2}e_aR^a\left(e^b\right)-\frac{2}{\ell^2}mR\left(\tau\right)\right] \\
& +\alpha_1\left[e_a\hat{R}^a\left(\omega^b\right)+\omega_aR^a\left(e^b\right)-2mR\left(\omega\right)-2sR\left(\tau\right)\right] \\
& +\beta_0\left[b_a\hat{R}^a\left(\omega^b\right)+\omega_a\hat{R}^a\left(b^b\right)-2zR\left(\omega\right)-sds\right. \\
& \left.+\frac{2}{\ell^2}yR\left(\tau\right)-\frac{1}{\ell^2}mdm+\frac{1}{\ell^2}t_aR^a\left(e^b\right)+\frac{1}{\ell^2}e_aR^a\left(t^b\right)\right] \\
& +\beta_1\left[e_a\hat{R}^a\left(b^b\right)+b_aR^a\left(e^b\right)+t_a\hat{R}^a\left(\omega^b\right)+\omega_aR^a\left(t^b\right)+2yR\left(\omega\right)-2zR\left(\tau\right)-2mds\right]
\end{aligned}
$$

- I_{exN} is invariant under the exotic Newtonian algebra; $\ell \to \infty \Rightarrow I_{\text{gEN}}$
- $\ell \to \infty$: The α_0 and α_1 sectors can be rewritten as the Lagrangian invariant under the so-called extended Bargmann algebra
- $\ell \to \infty$ applied in the β_0 and β_1 sectors reproduces the extended Newtonian gravity Lagrangian and the corresponding exotic term
- Each independent term of I_{exN} is invariant under the gauge transformation laws $\delta A = d\lambda + [A, \lambda]$
- E.o.m.: Vanishing of the exotic Newtonian curvature 2-forms

Three-dimensional Maxwellian extended Bargmann supergravity

- Maxwell algebra: First introduced to describe Minkowski space in the presence of a constant electromagnetic field background
- In the gravity context, the Maxwell algebra and its generalizations have been useful to recover standard General Relativity from CS and Born-Infeld gravity theories
- At the supersymmetric level, the (minimal) Maxwell superalgebra describes a constant Abelian supersymmetric gauge field background in a $4D$ superspace; Generalizations have been studied
- NR version of the Maxwell CS gravity theory has only been presented recently (presence of three U(1) gauge fields required in order to establish a well-defined NR limit and to avoid degeneracy) $\rightarrow$ **Maxwellian Extended Bargmann** (MEB) algebra and CS gravity theory

- **Supersymmetric extension of the NR Maxwell CS supergravity was unknown till now**
- We have explored the NR limit of the Maxwell superalgebra for $\mathcal{N}=1$ and $\mathcal{N}=2$ $\rightsquigarrow$ A well-defined NR Maxwellian CS supergravity action requires to introduce by hand additional fermionic and bosonic generators $\rightarrow$ **MEB superalgebra** and $3D$ **CS MEB supergravity** theory

Three-dimensional Maxwellian extended Bargmann supergravity

- We first analyzed the $\mathcal{N} = 1$ case: The $\mathcal{N} = 1$ **MEB superalgebra** we obtained is **not a true supersymmetry algebra**, since the anti-commutator of two supercharges leads to a central charge transformation instead of a time and space translation $\Rightarrow$ We move on to $\mathcal{N} = 2$
- We applied the NR contraction to a $\mathcal{N} = 2$ relativistic Maxwell superalgebra spanned by the set of generators $\left\{ J_A, P_A, Z_A, \mathcal{B}, \mathcal{Z}, Q^i_\alpha, \Sigma^i_\alpha \right\}$, $A = 0, 1, 2$, $\alpha = 1, 2$, $i = 1, 2$
- **The presence of a** $\mathfrak{so}(2)$ **internal symmetry generator is crucial in order to admit a non-degenerate invariant inner product** in the **relativistic** CS theory
- **But** the NR Maxwell superalgebra obtained did **not** allow for the proper construction of a **NR CS supergravity** action (although its relativistic analogue is well-defined)

A proper NR CS supergravity action based on a supersymmetric extension of the MEB algebra requires a NR superalgebra which not only contains the MEB algebra as a subalgebra but also admits a non-degenerate invariant supertrace

- We constructed **by hand** such a supersymmetric extension of the MEB algebra by introducing six Majorana fermionic generators $\tilde{Q}^+$, $\tilde{Q}^-$, $\tilde{\Sigma}^+$, $\tilde{\Sigma}^-$, $\tilde{R}$, and $\tilde{W}$; Furthermore, we introduced six extra bosonic generators $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{B}_1$, and $\tilde{B}_2$ ($\tilde{B}_1$ and $\tilde{B}_2$ central, while the others act non-trivially on the spinor generators)

Three-dimensional Maxwellian extended Bargmann supergravity

- Full set of generators of the **MEB superalgebra**:

$$\{\tilde{J},\tilde{G}_a,\tilde{S},\tilde{H},\tilde{P}_a,\tilde{M},\tilde{Z},\tilde{Z}_a,\tilde{T},\tilde{Y}_1,\tilde{Y}_2,\tilde{U}_1,\tilde{U}_2,\tilde{B}_1,\tilde{B}_2,\tilde{Q}^+_\alpha,\tilde{Q}^-_\alpha,\tilde{R}_\alpha,\tilde{\Sigma}^+_\alpha,\tilde{\Sigma}^-_\alpha,\tilde{W}_\alpha\}$$

- **Non-degenerate invariant bilinear form**
- Gauge connection 1-form $\tilde{A}$ for the MEB superalgebra:

$$\begin{aligned}\tilde{A} &= \omega\tilde{J}+\omega^a\tilde{G}_a+\tau\tilde{H}+e\tilde{P}_a+k\tilde{Z}+k^a\tilde{Z}_a+m\tilde{M}+s\tilde{S}+t\tilde{T}\\ &\quad +y_1\tilde{Y}_1+y_2\tilde{Y}_2+b1\tilde{B}_1+b_2\tilde{B}_2+u_1\tilde{U}_1+u_2\tilde{U}_2\\ &\quad +\psi^+\tilde{Q}^++\psi^-\tilde{Q}^-+\xi^+\tilde{\Sigma}^++\xi^-\tilde{\Sigma}^-+\rho\tilde{R}+\chi\tilde{W}\end{aligned}$$

- Corresponding curvature 2-form $\tilde{F}$:

$$\begin{aligned}\tilde{F} &= R(\omega)\tilde{J}+R^a\left(\omega^b\right)\tilde{G}_a+F(\tau)\tilde{H}+F^a\left(e^b\right)\tilde{P}_a+F(k)\tilde{Z}+F^a\left(k^b\right)\tilde{Z}_a\\ &\quad +F(m)\tilde{M}+R(s)\tilde{S}+F(t)\tilde{T}+F(y_1)\tilde{Y}_1+F(y_2)\tilde{Y}_2+F(b_1)\tilde{B}_1+F(b_2)\tilde{B}_2\\ &\quad +F(u_1)\tilde{U}_1+F(u_2)\tilde{U}_2+\nabla\psi^+\tilde{Q}^++\nabla\psi^-\tilde{Q}^-+\nabla\xi^+\tilde{\Sigma}^++\nabla\xi^-\tilde{\Sigma}^-+\nabla\rho\tilde{R}+\nabla\chi\tilde{W}\end{aligned}$$

CS Maxwellian extended Bargmann supergravity action

$$
\begin{aligned}
I_{\text{MEB}} \quad = \quad & \int \Big\{ \tilde{\alpha}_0 \left[\omega_a R^a(\omega^b) - 2sR(\omega) + 2y_1 dy_2 \right] + \tilde{\alpha}_1 \left[2e_a R^a(\omega^b) - 2mR(\omega) - 2\tau R(s) + 2y_1 du_2 \right. \\
& \left. + 2u_1 dy_2 + 2\bar{\psi}^+ \nabla \rho + 2\bar{\rho} \nabla \psi^+ + 2\bar{\psi}^- \nabla \psi^- \right] + \tilde{\alpha}_2 \left[e_a R^a\left(e^b\right) + k_a R^a\left(\omega^b\right) + \omega_a R^a\left(k^b\right) \right. \\
& - 2sR(k) - 2mR(\tau) - 2tR(\omega) + 2y_1 db_2 + 2u_1 du_2 + 2y_2 db_1 + 2\bar{\psi}^- \nabla \xi^- + 2\bar{\xi}^- \nabla \psi^- \\
& \left. + 2\bar{\psi}^+ \nabla \chi + 2\bar{\chi} \nabla \psi^+ + 2\bar{\xi}^+ \nabla \rho + 2\bar{\rho} \nabla \xi^+ \right] \Big\}
\end{aligned}
$$

- For $\tilde{\alpha}_2 \neq 0$ the field equations from I_{MEB} reduce to the vanishing of the curvature 2-forms associated with the MEB superalgebra
- I_{MEB} contains the so-called extended Bargmann supergravity theory (supplemented with some additional bosonic 1-form fields) as a sub-case

Chern-Simons Carrollian supergravities

- (A)dS CS gravity theories (and algebras) in $3D$ presented, together with their flat limit, are known
- Supersymmetric models less investigated; In particular, no Carrollian CS supergravity theory
- Recently presented: Construction of the $\mathcal{N}=1$, $D=3$ CS supergravity invariant under the AdS Carroll superalgebra, the latter obtained in the literature as the UR contraction of the AdS superalgebra; Flat limit $\ell \to \infty$ analyzed $\Rightarrow$ Carrollian CS supergravity and Carroll superalgebra (UR contraction of the Poincaré superalgebra)

L.R., Phys. Lett. B **795** (2019) 331-338 [arXiv:1905.00766 [hep-th]]

- In that work, the method introduced by Concha *et al.* was adopted and improved in order to coherently perform the flat limit: A generalization of the standard Inönü-Wigner (IW) contraction, consisting in rescaling not only the generators of a Lie (super)algebra but also the arbitrary constants appearing in the components of the invariant tensor; Improvement: Consider dimensionful generators from the very beginning

Let us directly move on to the $\mathcal{N}$-extended cases

Remarks on the relativistic side

- $\mathcal{N}=2$ supersymmetric extension of Poincaré and AdS algebras not unique, can be subdivided into two classes: $(2,0)$ and $(1,1)$
- Extension to the (p,q) case $\rightarrow$ **Subtleties** arise: The (p,q) Poincaré superalgebra can be derived as an IW contraction of the (p,q) AdS superalgebra, but the Poincaré limit applied at the level of the CS action requires to consider a direct sum of an $\mathfrak{so}(p)\oplus\mathfrak{so}(q)$ algebra and the (p,q) AdS superalgebra $\mathfrak{osp}(p|2)\oplus\mathfrak{osp}(q|2)$
- The semi-direct extension of the $\mathfrak{so}(p)\oplus\mathfrak{so}(q)$ automorphism algebra by the (p,q) Poincaré superalgebra allows to produce a non-degenerate bilinear form which is used to construct a well-defined CS (p,q) Poincaré supergravity theory

This also has an effect on the UR theories

- Distinguish between $\mathcal{N}=(p,q)$, that is $\mathcal{N}=p+q$, with $p,q>0$, and $\mathcal{N}=(\mathcal{N},0)$
- **$\mathcal{N}$-extended AdS Carroll superalgebras** obtained through UR contraction applied to the direct sum of an $\mathfrak{so}(p)\oplus\mathfrak{so}(q)$ algebra and $\mathfrak{osp}(p|2)\oplus\mathfrak{osp}(q|2)$, and to an $\mathfrak{so}(\mathcal{N})$ extension of $\mathfrak{osp}(\mathcal{N}|2)\oplus\mathfrak{sp}(2)$, respectively
- $\mathcal{N}=(\mathcal{N},0)$ case more subtle: Carroll limit requires to redefine the supersymmetry generators ($\mathcal{N}$ even)
- Limit $\ell\rightarrow\infty$ also analyzed $\Rightarrow$ Flat super-Carroll

Farhad Ali, L.R., JHEP **02** (2020), 128 [arXiv:1912.04172 [hep-th]]

$\mathcal{N}$-EXTENDED CHERN-SIMONS CARROLLIAN SUPERGRAVITIES: $\mathcal{N} = (p, q)$

- $\mathfrak{osp}(p|2) \oplus \mathfrak{osp}(q|2)$

A. Achucarro and P. Townsend, Phys. Lett. B **229** (1989), 383-387

- Consider the direct sum

$$\underbrace{[\mathfrak{osp}(p|2) \oplus \mathfrak{osp}(q|2)]}_{\tilde{J}_{AB}\,,\,\tilde{P}_A\,,\,\tilde{Z}^{ij}\,,\,\tilde{Z}^{IJ}\,,\,\tilde{Q}^i_\alpha\,,\,\tilde{Q}^I_\alpha} \oplus \underbrace{[\mathfrak{so}(p) \oplus \mathfrak{so}(q)]}_{\tilde{S}^{ij}\,,\,\tilde{S}^{IJ}}$$

- $A, B = 0, 1, 2$, $\alpha = 1, 2$, $i, j = 1, \ldots, p$, $I, J = 1, \ldots, q$; $\tilde{Z}^{ij} = -\tilde{Z}^{ji}$, $\tilde{S}^{ij} = -\tilde{S}^{ji}$, $\tilde{Z}^{IJ} = -\tilde{Z}^{JI}$, $\tilde{S}^{IJ} = -\tilde{S}^{JI}$
- We consider dimensionful generators (improve the method of P. Concha *et al.* to apply the flat limit)

UR limit

1. Perform the following redefinition: $\tilde{T}^{ij} \equiv \tilde{Z}^{ij} - \ell \tilde{S}^{ij}\,, \quad \tilde{T}^{IJ} \equiv \tilde{Z}^{IJ} - \ell \tilde{S}^{IJ}$
2. Write the non-vanishing components of the invariant tensor
3. Split $A \to \{a, 0\} \Rightarrow \tilde{J}_{AB} \to \{\tilde{J}_{ab}, \tilde{J}_{a0} \equiv \tilde{K}_a\}$, $\tilde{P}_A \to \{\tilde{P}_a, \tilde{P}_0 \equiv \tilde{H}\}$
4. Rescale the generators as $\tilde{H} \to \sigma H\,,\ \tilde{K}_a \to \sigma K_a\,,\ \tilde{S}^{ij} \to \sigma S^{ij}\,,\ \tilde{S}^{IJ} \to \sigma S^{IJ}\,,\ \tilde{Q}^i_\alpha \to \sqrt{\sigma} Q^i_\alpha\,,\ \tilde{Q}^I_\alpha \to \sqrt{\sigma} Q^I_\alpha$
5. Take $\sigma \to \infty$ (UR) $\Rightarrow \mathcal{N} = (p, q)$ **AdS Carroll superalgebra**

$\mathcal{N}$-EXTENDED CHERN-SIMONS CARROLLIAN SUPERGRAVITIES: $\mathcal{N}=(p,q)$

1. Connection 1-form A:

$$A=\frac{1}{2}\omega^{ab}J_{ab}+k^aK_a+V^aP_a+hH+\frac{1}{2}t^{ij}T_{ij}+\frac{1}{2}t^{IJ}T_{IJ}+\frac{1}{2}s^{ij}S_{ij}+\frac{1}{2}s^{IJ}S_{IJ}+\psi_iQ^i+\psi_IQ^I$$

2. Related curvature 2-form F:

$$F=\frac{1}{2}\mathcal{R}^{ab}J_{ab}+\mathcal{K}^aK_a+R^aP_a+\mathcal{H}H+\frac{1}{2}\mathcal{T}^{ij}T_{ij}+\frac{1}{2}\mathcal{T}^{IJ}T_{IJ}+\frac{1}{2}\mathcal{S}^{ij}S_{ij}+\frac{1}{2}\mathcal{S}^{IJ}S_{IJ}+\nabla\psi_iQ^i+\nabla\psi_IQ^I$$

3. Rescaling and Carroll (UR) limit $\sigma\to\infty$ of the invariant tensor $\Rightarrow$ UR invariant tensor

$\mathcal{N}=(p,q)$ CS Carrollian supergravity action

$$\begin{aligned}I^{(p,q)}_{CS}=\frac{k}{4\pi}\int_{\mathcal{M}}\Bigg\{&\frac{\alpha_0}{2}\left(\omega^a{}_bR^b{}_a+\frac{2}{\ell^2}V^aR_a+2t^i{}_jdt^j{}_i+\frac{4}{3}t^i{}_jt^j{}_kt^k{}_i+2t^I{}_Jdt^J{}_I+\frac{4}{3}t^I{}_Jt^J{}_Kt^K{}_I\right)\\&+\alpha_1\left[\epsilon_{ab}R^{ab}h-2\epsilon_{ab}\mathfrak{K}^aV^b+\frac{1}{\ell^2}\epsilon_{ab}V^aV^bh-2t^i{}_j\left(ds^j{}_i+t^j{}_ks^k{}_i\right)+2t^I{}_J\left(ds^J{}_I+t^J{}_Ks^K{}_I\right)+2\bar{\psi}^i\nabla\psi^i+2\bar{\psi}^I\nabla\psi^I\right]\\&-d\left(\frac{\alpha_1}{2}\epsilon_{ab}\omega^{ab}h-\alpha_1\epsilon_{ab}k^aV^b+\alpha_1t^i{}_js^j{}_i-\alpha_1t^I{}_Js^J{}_I\right)\Bigg\}\end{aligned}$$

- $I^{(p,q)}_{CS}$ invariant by construction under the $\mathcal{N}=(p,q)$ AdS Carroll superalgebra
- For $\alpha_1\neq0$, e.o.m. reduce to the vanishing of the (p,q) super-AdS Carroll curvature 2-forms
- $\ell\to\infty$ can be applied at the superalgebra, CS action, supersymmetry transformation laws, and e.o.m.

$\mathcal{N}$-extended Chern-Simons Carrollian supergravities: $\mathcal{N}=(\mathcal{N},0)$

- Consider $[\mathfrak{osp}(\mathcal{N}|2)\oplus\mathfrak{sp}(2)]\oplus\mathfrak{so}(\mathcal{N})$:

$$\left[\tilde{J}_{AB},\tilde{J}_{CD}\right]=\eta_{BC}\tilde{J}_{AD}-\eta_{AC}\tilde{J}_{BD}-\eta_{BD}\tilde{J}_{AC}+\eta_{AD}\tilde{J}_{BC}\,,\quad \left[\tilde{J}_{AB},\tilde{P}_{C}\right]=\eta_{BC}\tilde{P}_{A}-\eta_{AC}\tilde{P}_{B}\,,\quad \left[\tilde{P}_{A},\tilde{P}_{B}\right]=\frac{1}{\ell^2}\tilde{J}_{AB}\,,$$
$$\left[\tilde{Z}^{ij},\tilde{Z}^{kl}\right]=\delta^{jk}\tilde{Z}^{il}-\delta^{ik}\tilde{Z}^{jl}-\delta^{jl}\tilde{Z}^{ik}+\delta^{il}\tilde{Z}^{jk}\,,\quad \left[\tilde{S}^{ij},\tilde{S}^{kl}\right]=-\frac{1}{\ell}\left(\delta^{jk}\tilde{S}^{il}-\delta^{ik}\tilde{S}^{jl}-\delta^{jl}\tilde{S}^{ik}+\delta^{il}\tilde{S}^{jk}\right)\,,$$
$$\left[\tilde{J}_{AB},\tilde{Q}^{i}_{\alpha}\right]=-\frac{1}{2}\left(\Gamma_{AB}\tilde{Q}^{i}\right)_{\alpha}\,,\quad \left[\tilde{P}_{A},\tilde{Q}^{i}_{\alpha}\right]=-\frac{1}{2\ell}\left(\Gamma_{A}\tilde{Q}^{i}\right)_{\alpha}\,,\quad \left[\tilde{Z}^{ij},\tilde{Q}^{k}_{\alpha}\right]=\delta^{jk}\tilde{Q}^{i}_{\alpha}-\delta^{ik}\tilde{Q}^{j}_{\alpha}\,,$$
$$\left\{\tilde{Q}^{i}_{\alpha},\tilde{Q}^{j}_{\beta}\right\}=\delta^{ij}\left[-\frac{1}{2\ell}\left(\Gamma^{AB}C\right)_{\alpha\beta}\tilde{J}_{AB}+\left(\Gamma^{A}C\right)_{\alpha\beta}\tilde{P}_{A}\right]+\frac{1}{\ell}C_{\alpha\beta}\tilde{Z}^{ij}$$

  - $A,B,\ldots=0,1,2$, $\alpha,\beta,\ldots=1,2$, $i,j,\ldots=1,\ldots,\mathcal{N}$ (we considered $\mathcal{N}=2x$, $x=1,\ldots,\mathcal{N}/2$)

- Construction **analogous** to the $\mathcal{N}=(p,q)$ case, **but** here we also have to **redefine the supersymmetry charges** as

$$\tilde{Q}^{\pm\,\lambda}_{\alpha}=\frac{1}{\sqrt{2}}\left(\tilde{Q}^{\lambda}_{\alpha}\pm\left(\Gamma_0\right)_{\alpha\beta}\tilde{Q}^{x+\lambda}_{\beta}\right)\,,\qquad \lambda,\mu,\ldots=1,\ldots,x$$

- This has an effect on the generators $\tilde{T}^{ij}$ and $\tilde{S}^{ij}$:

$$\tilde{T}^{\lambda\mu}\,,\quad \tilde{T}'^{\lambda\mu}\equiv\tilde{T}^{\lambda+x\;\mu+x}\,,\quad \tilde{U}^{\lambda\mu}\equiv\tilde{T}^{x+\lambda\;\mu}\,,\quad \tilde{U}'^{\lambda\mu}\equiv\tilde{T}^{\lambda\;x+\mu}\,,$$
$$\tilde{S}^{\lambda\mu}\,,\quad \tilde{S}'^{\lambda\mu}\equiv\tilde{S}^{\lambda+x\;\mu+x}\,,\quad \tilde{V}^{\lambda\mu}\equiv\tilde{S}^{x+\lambda\;\mu}\,,\quad \tilde{V}'^{\lambda\mu}\equiv\tilde{S}^{\lambda\;x+\mu}\,;$$

$$\tilde{T}^{\lambda\mu}=-\tilde{T}^{\mu\lambda}\,,\ \tilde{T}'^{\lambda\mu}=-\tilde{T}'^{\mu\lambda}\,,\ \tilde{U}^{\lambda\mu}=-\tilde{U}'^{\mu\lambda}\,,\ \tilde{S}^{\lambda\mu}=-\tilde{S}^{\mu\lambda}\,,\ \tilde{S}'^{\lambda\mu}=-\tilde{S}'^{\mu\lambda}\,,\ \tilde{V}^{\lambda\mu}=-\tilde{V}'^{\mu\lambda}$$

$\mathcal{N}$-EXTENDED CHERN-SIMONS CARROLLIAN SUPERGRAVITIES: $\mathcal{N} = (\mathcal{N}, 0)$

UR limit

1. Write the non-vanishing components of the invariant tensor
2. Split $A \to \{a, 0\} \Rightarrow \tilde{J}_{AB} \to \{\tilde{J}_{ab}, \tilde{J}_{a0} \equiv \tilde{K}_a\}$, $\tilde{P}_A \to \{\tilde{P}_a, \tilde{P}_0 \equiv \tilde{H}\}$
3. Rescale as $\tilde{H} \to \sigma H$, $\tilde{K}_a \to \sigma K_a$, $\tilde{S}^{\lambda\mu} \to \sigma S^{\lambda\mu}$, $\tilde{S}'^{\lambda\mu} \to \sigma S'^{\lambda\mu}$, $\tilde{V}^{\lambda\mu} \to \sigma V^{\lambda\mu}$, $\tilde{Q}^{\pm\,\lambda}_{\alpha} \to \sqrt{\sigma} Q^{\pm\,\lambda}_{\alpha}$
4. Take $\sigma \to \infty$ (UR) $\Rightarrow$ $\mathcal{N} = (\mathcal{N}, 0)$ **AdS Carroll superalgebra**

CS construction

1. Connection 1-form A:

$$A = \frac{1}{2}\omega^{ab} J_{ab} + k^a K_a + V^a P_a + hH + \frac{1}{2} t^{\lambda\mu} T_{\lambda\mu} + \frac{1}{2} t'^{\lambda\mu} T'_{\lambda\mu} + u^{\lambda\mu} U_{\lambda\mu} + \frac{1}{2} s^{\lambda\mu} S_{\lambda\mu} + \frac{1}{2} s'^{\lambda\mu} S'_{\lambda\mu} + v^{\lambda\mu} V_{\lambda\mu} + \psi^{+}_{\lambda} Q^{+\,\lambda} + \psi^{-}_{\lambda} Q^{-\,\lambda}$$

2. Related curvature 2-form F:

$$F = \frac{1}{2}\mathcal{R}^{ab} J_{ab} + \mathcal{K}^a K_a + R^a P_a + \mathcal{H} H + \frac{1}{2}\mathcal{T}^{\lambda\mu} T_{\lambda\mu} + \frac{1}{2}\mathcal{T}'^{\lambda\mu} T'_{\lambda\mu} + \mathcal{U}^{\lambda\mu} U_{\lambda\mu} + \frac{1}{2}\mathcal{S}^{\lambda\mu} S_{\lambda\mu} + \frac{1}{2}\mathcal{S}'^{\lambda\mu} S'_{\lambda\mu} + \mathcal{V}^{\lambda\mu} V_{\lambda\mu}$$
$$+ \nabla\psi^{+}_{\lambda} Q^{+\,\lambda} + \nabla\psi^{-}_{\lambda} Q^{-\,\lambda}$$

3. Rescaling and Carroll (UR) limit $\sigma \to \infty$ of the invariant tensor $\Rightarrow$ UR invariant tensor

$\mathcal{N}$-extended Chern-Simons Carrollian supergravities: $\mathcal{N} = (\mathcal{N}, 0)$

$\mathcal{N} = (\mathcal{N}, 0)$ CS Carrollian supergravity action

$$
\begin{aligned}
I_{CS}^{(\mathcal{N},0)} = \frac{k}{4\pi}\int_{\mathcal{M}} \Bigg\{ & \frac{\alpha_0}{2}\Big(\omega^a{}_b R^b{}_a + \frac{2}{\ell^2}V^a R_a + 2t^\lambda{}_\mu dt^\mu{}_\lambda + \frac{4}{3}t^\lambda{}_\mu t^\mu{}_\nu t^\nu{}_\lambda + 2t'^\lambda{}_\mu dt'^\mu{}_\lambda + \frac{4}{3}t'^\lambda{}_\mu t'^\mu{}_\nu t'^\nu{}_\lambda \\
& + 4u^\lambda{}_\mu du'^\mu{}_\lambda - 4t_{\lambda\mu}u'^\lambda{}_\nu u^{\nu\mu} - 4t'_{\lambda\mu}u^\lambda{}_\nu u'^{\nu\mu}\Big) + \alpha_1\Big[\epsilon_{ab}R^{ab}h - 2\epsilon_{ab}\mathfrak{K}^a V^b + \frac{1}{\ell^2}\epsilon_{ab}V^a V^b \\
& - 2t^\lambda{}_\mu\left(ds^\mu{}_\lambda + t^\mu{}_\nu s^\nu{}_\lambda\right) - 2t'^\lambda{}_\mu\left(ds'^\mu{}_\lambda + t'^\mu{}_\nu s'^\nu{}_\lambda\right) - 4u^\lambda{}_\mu dv'^\mu{}_\lambda - 2u'^\lambda{}_\mu u^\mu{}_\nu s^\nu{}_\lambda \\
& - 2u^\lambda{}_\mu u'^\mu{}_\nu s'^\nu{}_\lambda - 4u'^\lambda{}_\mu v^\mu{}_\nu t^\nu{}_\lambda - 4u^\lambda{}_\mu v'^\mu{}_\nu t'^\nu{}_\lambda + 2\bar{\psi}^{+\,\lambda}\nabla\psi^{+\,\lambda} + 2\bar{\psi}^{-\,\lambda}\nabla\psi^{-\,\lambda}\Big] \\
& - d\left(\frac{\alpha_1}{2}\epsilon_{ab}\omega^{ab}h - \alpha_1\epsilon_{ab}k^a V^b + \alpha_1 t^\lambda{}_\mu s^\mu{}_\lambda + \alpha_1 t'^\lambda{}_\mu s'^\mu{}_\lambda + 2\alpha_1 u^\lambda{}_\mu v'^\mu{}_\lambda\right)\Bigg\}\,,
\end{aligned}
$$

where

$$
u^{\lambda\mu} = t^{\lambda+x\ \mu} = -t^{\mu\ \lambda+x} = -u'^{\mu\lambda}\,, \quad v^{\lambda\mu} = s^{\lambda+x\ \mu} = -s^{\mu\ \lambda+x} = -v'^{\mu\lambda}
$$

- $I_{CS}^{(\mathcal{N},0)}$ invariant by construction under the $\mathcal{N} = (\mathcal{N}, 0)$ AdS Carroll superalgebra
- For $\alpha_1 \neq 0$, e.o.m. reduce to the vanishing of the $(\mathcal{N}, 0)$ super-AdS Carroll curvature 2-forms
- $\ell \to \infty$ can be applied at the superalgebra, CS action, supersymmetry transformation laws, and e.o.m.

Conclusions

Summary

- Importance of NR and UR models
- Explicit construction of the theories either by hand or through a contraction procedure
- New results in gravity both at the purely bosonic and supersymmetric level (construction of NR and UR $3D$ CS (super)gravity theories)

Possible future developments

- Extensions to higher-dimensional theories
- Matter coupling?
- Relations among different NR and UR CS theories can be found by means of mathematical tools applied at the algebraic level $\rightarrow$ Gain something on the relativistic side?
- Role of NR/UR symmetries in the study of supergravity theories on a manifold with boundary (e.g., Carrollian structures, holography)?

Conclusions

Summary

- Importance of NR and UR models
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Thank you!

Direct sum $[\mathfrak{osp}(p|2) \oplus \mathfrak{osp}(q|2)] \oplus [\mathfrak{so}(p) \oplus \mathfrak{so}(q)]$

$$\left[\tilde{J}_{AB}, \tilde{J}_{CD}\right] = \eta_{BC}\tilde{J}_{AD} - \eta_{AC}\tilde{J}_{BD} - \eta_{BD}\tilde{J}_{AC} + \eta_{AD}\tilde{J}_{BC} \,,$$

$$\left[\tilde{J}_{AB}, \tilde{P}_{C}\right] = \eta_{BC}\tilde{P}_{A} - \eta_{AC}\tilde{P}_{B} \,, \qquad \left[\tilde{P}_{A}, \tilde{P}_{B}\right] = \frac{1}{\ell^2}\tilde{J}_{AB} \,,$$

$$\left[\tilde{Z}^{ij}, \tilde{Z}^{kl}\right] = \delta^{jk}\tilde{Z}^{il} - \delta^{ik}\tilde{Z}^{jl} - \delta^{jl}\tilde{Z}^{ik} + \delta^{il}\tilde{Z}^{jk} \,, \quad \left[\tilde{Z}^{IJ}, \tilde{Z}^{KL}\right] = \delta^{JK}\tilde{Z}^{IL} - \delta^{IK}\tilde{Z}^{JL} - \delta^{JL}\tilde{Z}^{IK} + \delta^{IL}\tilde{Z}^{JK} \,,$$

$$\left[\tilde{S}^{ij}, \tilde{S}^{kl}\right] = -\frac{1}{\ell}\left(\delta^{jk}\tilde{S}^{il} - \delta^{ik}\tilde{S}^{jl} - \delta^{jl}\tilde{S}^{ik} + \delta^{il}\tilde{S}^{jk}\right) \,,$$

$$\left[\tilde{S}^{IJ}, \tilde{S}^{KL}\right] = -\frac{1}{\ell}\left(\delta^{JK}\tilde{S}^{IL} - \delta^{IK}\tilde{S}^{JL} - \delta^{JL}\tilde{S}^{IK} + \delta^{IL}\tilde{S}^{JK}\right) \,,$$

$$\left[\tilde{J}_{AB}, \tilde{Q}^{i}_{\alpha}\right] = -\frac{1}{2}\left(\Gamma_{AB}\tilde{Q}^{i}\right)_{\alpha} \,, \qquad \left[\tilde{J}_{AB}, \tilde{Q}^{I}_{\alpha}\right] = -\frac{1}{2}\left(\Gamma_{AB}\tilde{Q}^{I}\right)_{\alpha} \,,$$

$$\left[\tilde{P}_{A}, \tilde{Q}^{i}_{\alpha}\right] = -\frac{1}{2\ell}\left(\Gamma_{A}\tilde{Q}^{i}\right)_{\alpha} \,, \qquad \left[\tilde{P}_{A}, \tilde{Q}^{I}_{\alpha}\right] = \frac{1}{2\ell}\left(\Gamma_{A}\tilde{Q}^{I}\right)_{\alpha} \,,$$

$$\left[\tilde{Z}^{ij}, \tilde{Q}^{k}_{\alpha}\right] = \delta^{jk}\tilde{Q}^{i}_{\alpha} - \delta^{ik}\tilde{Q}^{j}_{\alpha} \,, \qquad \left[\tilde{Z}^{IJ}, \tilde{Q}^{K}_{\alpha}\right] = \delta^{JK}\tilde{Q}^{I}_{\alpha} - \delta^{IK}\tilde{Q}^{J}_{\alpha} \,,$$

$$\left\{\tilde{Q}^{i}_{\alpha}, \tilde{Q}^{j}_{\beta}\right\} = \delta^{ij}\left[-\frac{1}{2\ell}\left(\Gamma^{AB}C\right)_{\alpha\beta}\tilde{J}_{AB} + \left(\Gamma^{A}C\right)_{\alpha\beta}\tilde{P}_{A}\right] + \frac{1}{\ell}C_{\alpha\beta}\tilde{Z}^{ij} \,,$$

$$\left\{\tilde{Q}^{I}_{\alpha}, \tilde{Q}^{J}_{\beta}\right\} = \delta^{IJ}\left[\frac{1}{2\ell}\left(\Gamma^{AB}C\right)_{\alpha\beta}\tilde{J}_{AB} + \left(\Gamma^{A}C\right)_{\alpha\beta}\tilde{P}_{A}\right] - \frac{1}{\ell}C_{\alpha\beta}\tilde{Z}^{IJ}$$