Non-Relativistic Supersymmetry on Curved Three-Manifolds

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Introduction and Motivation

For certain supersymmetric quantum field theories on a (compact) manifold $\mathcal{M}$ the partition function $Z[\mathcal{M}]$ or expectation values of BPS operators $\langle \mathcal{O} \rangle$ can be determined exactly.
[Pestun 2012], [Nekrasov 2003], [Kapustin et. al. 2010]

Questions:

How robust is localization? Does it rely on Lorentzian/Euclidean symmetries?
NR supersymmetry on curved backgrounds?
Can we localize non-relativistic QFTs?

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Minimal Coupling?

Start with some flat space theory and couple minimally to some background $\mathcal{M}$ together with a Killing spinor equation

$$\partial_\mu \varepsilon = 0 \quad \longrightarrow \quad D_\mu \varepsilon = \mathcal{A}_\mu \varepsilon \, .$$

$$e.g. \qquad \mathcal{M} = S^3_\ell : \qquad \mathcal{A}_\mu = \frac{\mathrm{i}}{2\ell}\gamma_\mu$$

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This can be seen by noting that

$$\delta \mathcal{L}_{min} \sim g^{\mu\nu} \delta T_{\mu\nu} \neq 0 \,,$$

since the EM-Tensor is not a singlet but part of the supercurrent multiplet.

Three established solutions:

Quick fix: Noether procedure

Add non-minimal terms of order $1/\ell$ to the minimally coupled theory and solve iteratively

Superminimal coupling: Festuccia-Seiberg

Couple to the full supercurrent multiplet. This is done via off-shell supergravity [F-S 2012; ...]. Vanishing of the fermions gives rise to Killing spinor equations

$$\delta\Psi_\mu = (D_\mu - \mathcal{A}_\mu)\,\varepsilon = 0$$

Recycling: Dimensional reduction

Start with an established, rigid theory in D dimensions, i.e.

$$\delta\mathcal{L} = 0 \qquad + \qquad D_\mu\varepsilon = \mathcal{A}_\mu\varepsilon$$

and reduce this information to $(D-1)$ dimensions.

For us: start with rigid 4D old-minimal supergravity and perform a twisted null reduction, yielding rigid $3D$ NR susy theory plus Killing spinor equations

Note, there has been an attempt of adapting FS to NR supergravity [Knodel,Lisbao,Liu 2015]

Spinors $\chi_\pm$ furnish an irrep of $SO(2)$ rotations

$$\delta_J \chi_\pm = \frac{1}{4}\lambda_{ab}\gamma_{ab}\chi_\pm\,, \qquad\qquad (a, b = 1, 2)$$

and a reducible but in-decomposable representation of Galilean boosts

$$\delta_G \chi_+ = 0\,, \qquad\qquad \delta_G \chi_- = -\frac{\sqrt{2}}{2}\lambda_a\gamma_{a0}\chi_+\,.$$

$3D\ \mathcal{N}=2$ Bargmann Supersymmetry

Extending the Bargmann algebra (J_{ab}, H, P_a, G_a, M) with two fermionic generators $Q_\pm$

$$[J_{ab}, Q_\pm] \sim \gamma_{ab} Q_\pm\,, \qquad [G_a, Q_+] \sim \gamma_{a0} Q_-\,, \qquad [G_a, Q_-] \sim 0\,,$$

$$\{Q_+, Q_+\} \sim \gamma_0 H\,, \qquad \{Q_-, Q_-\} \sim \gamma_0 M\,, \qquad \{Q_+, Q_-\} \sim \gamma_a P_a$$

Can be obtained by construction or an Inönü-Wigner contraction of the Poincaré superalgebra.

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$\rightarrow$ Field theory representation?

Three-dimensional pseudo chiral and pseudo anti-chiral multiplets

$$(z, \pi\chi_+, \bar{\pi}\chi_-, h)\,, \qquad\qquad (\bar{z}, \bar{\pi}\chi_+, \pi\chi_-, \bar{h})$$

where $\pi = 1/2(1 - \mathrm{i}\gamma_0)$ and $\bar{\pi} = 1/2(1 + \mathrm{i}\gamma_0)$ are projections compatible with Bargmann transformations.

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$$\begin{aligned}\mathcal{L} &= \mathrm{i}m(\bar{z}\dot{z} - \dot{\bar{z}}z) - \partial_a\bar{z}\partial_a z + \bar{h}h \\ &\quad - \bar{\chi}_-\gamma_a\partial_a\chi_+ - \bar{\chi}_+\gamma_a\partial_a\chi_- + \sqrt{2}\,\bar{\chi}_+\gamma_0\dot{\chi}_+ - m\sqrt{2}\,\bar{\chi}_-\chi_-\end{aligned}$$

(note that χ_- is non-dynamical)

and is supersymmetric under $\mathcal{N} = 2$ supersymmetry with parameters (ϵ_+, ϵ_-)

$$\begin{aligned}
\delta z &= \bar{\epsilon}_+ \bar{\pi} \chi_- + \bar{\epsilon}_- \pi \chi_+ \,, \\
\delta \pi \chi_+ &= \frac{1}{2} \bar{\pi} \left(\gamma_a \partial_a z + h \right) \epsilon_+ - \frac{m\sqrt{2}}{2} \gamma_0 \pi \epsilon_- \, z \,, \\
\delta \bar{\pi} \chi_- &= \frac{1}{2} \pi \left(\gamma_a \partial_a z + h \right) \epsilon_- - \frac{\sqrt{2}}{2} \gamma_0 \bar{\pi} \epsilon_+ \, \dot{z} \,, \\
\delta h &= \bar{\epsilon}_+ \gamma_a \partial_a \pi \chi_+ + \bar{\epsilon}_- \gamma_a \partial_a \bar{\pi} \chi_- \\
&\quad - \sqrt{2}\, \bar{\epsilon}_+ \gamma_0 \pi \dot{\chi}_+ + m\sqrt{2}\, \bar{\epsilon}_- \bar{\pi} \chi_-
\end{aligned}$$

See also [Auzzi, Baiguera, Nardelli, Penati 2019]

Null-Embedding/Null-Reduction

Can embed this into $4D$ Lorentzian space w/ coordinates $(x^i, t, \mathbf{v})$ and null isometry $K = \partial_{\mathbf{v}}$

$4D$ chiral multiplet $\sim$ $3D$ pseudo chiral multiplet

$$(Z, \chi_L, H) \sim \mathrm{e}^{-\mathrm{i}\, m \mathbf{v}} (z, \pi\chi_+, \bar{\pi}\chi_-, h)$$

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$$(Z, \chi_L, H) \sim \mathrm{e}^{-\mathrm{i}\, m \mathbf{v}} (z, \pi\chi_+, \bar{\pi}\chi_-, h)$$

and

$$ds^2 = (h_{\mu\nu} + 2\tau_\mu m_\nu) dx^\mu dx^\nu - 2\tau d\mathbf{v}\,,$$

where $x^\mu = (x^i, t)$ and $h = e^a e^a$.

3D NR Rigid Supersymmetry

A Lagrangian coupled to a manifold with NC structure (τ, e^a, m) (a priori arbitrary torsion)

$$\mathcal{L} = \mathcal{L}_{min} + \mathcal{L}_{non-min} ,$$

which is supersymmetric under

$$\delta = \delta_{min} + \delta_{non-min}$$

Minimal coupling means $\partial_0 \to \tau^\mu \bar{\bar{\nabla}}_\mu$ and $\partial_a \to e_a{}^\mu \bar{\bar{\nabla}}_\mu$.

The $\mathcal{N} = 2$ supersymmetry parameters (ϵ_+, ϵ_-) satisfy appropriate Killing spinor equations.

Non-relativistic Killing Spinor Equations

Killing spinors $(\epsilon_+, \epsilon_-)(x^i, t)$ associated to conserved supercharges $(Q(\epsilon_+), Q(\epsilon_-))$ satisfy equations of the form

$$\bar{\nabla}_\mu \epsilon_+ = \mathcal{A}_\mu^{(+)} \epsilon_+ + \mathcal{A}_\mu^{(-)} \epsilon_- \,,$$
$$\bar{\nabla}_\mu \epsilon_- = \mathcal{B}_\mu^{(+)} \epsilon_+ + \mathcal{B}_\mu^{(-)} \epsilon_- \,.$$

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$$\bar{\nabla}_\mu \epsilon_- = \mathcal{B}^{(+)}_\mu \epsilon_+ + \mathcal{B}^{(-)}_\mu \epsilon_- \,.$$

Note, however: these equations do not close under Galilean boosts. Using $\delta_G \epsilon_+ = 0$, the first equation transforms to

$$0 = \mathcal{C}^{(+)}_\mu \epsilon_+ + \mathcal{C}^{(-)}_\mu \epsilon_-$$

Lesson:

Non-relativistic Killing spinor equations naturally form a set of differential and algebraic equations.

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Strategy:

Solve the algebraic Killing spinor equations first. This typically leads a trade-off between restrictions on backgrounds and the number of preserved supercharges.

Example 1: Twistless Torsional

We consider a three-manifold with coordinates $x^\mu = (t, x^i)$ with $i = 1, 2$ and NC structure

$$\tau = \mathrm{e}^{-\lambda}\, dt\,, \qquad e^a = \mathrm{e}^{\lambda}\, dx^a\,, \qquad m = \mathrm{e}^{\lambda}\, \tau\,,$$

with

$$\mathrm{e}^{\lambda} = \frac{2}{1 - x_i x^i/\ell^2} \qquad (x_i x^i < \ell^2)\,,$$

hence spatial slices are locally isomorphic to the Poincaré disc.

Note, the NC structure is twistless torsional $\tau \wedge d\tau = 0$

This background allows for two supercharges of the form

$$(\epsilon_+, 0) = \mathrm{e}^{\frac{\pi}{8}\gamma_0 - \frac{\lambda}{2}}(\epsilon_0, 0)$$

with ϵ_0 constant and arbitrary,

$$\begin{aligned} \delta_{non-min} h &= -\frac{1}{2}\tau_{0a}\bar{\epsilon}_+\gamma_a\bar{\pi}\psi_- \, , \\ \mathcal{L}_{non-min} &= -\frac{1}{12\ell^2}\left(11 - 7\mathrm{e}^{-\lambda}\right)\bar{z}z \\ &\qquad + \tau_{0a}\bar{\psi}_+\gamma_a\psi_- - \frac{\mathrm{i}}{2}\tau_{0a}\epsilon_{ab}\left(\bar{z}\bar{\nabla}_b z - z\bar{\nabla}_b\bar{z}\right) \, , \end{aligned}$$

and realizes the following rigid superalgebra

$$\{Q(\epsilon_+), Q(\epsilon_+)\} = -\mathrm{i}\frac{\sqrt{2}}{2}\mathcal{L}\left[N^+\tau^\mu\right] \, ,$$

where $N^+ = -\mathrm{i}\bar{\epsilon}_+\gamma_0\epsilon_+$.

Example 2: Torsional

Consider a three-manifold with coordinates $x^\mu = (t, \eta_1, \eta_2)$ and NC structure [Grosvenor,Hartong,Keeler,Obers 2017]

$$\begin{aligned} \tau &= dt - \frac{\ell}{2}\cos\eta_1\, d\eta_2\,, \\ e^a &= \frac{\ell}{2}\left(d\eta_1 + \sin\eta_1\, d\eta_2\right)\,, \\ m &= \frac{\ell}{4}\cos\eta_1\, d\eta_2 \end{aligned}$$

The NC structure is torsional $\tau \wedge d\tau \neq 0$ (but $\iota_\tau d\tau = 0$)

This background allows for two supercharges of the form

$$(0, \epsilon_-) = \mathrm{e}^{\frac{t}{\ell}\gamma_0}(0, \epsilon_0)$$

with ϵ_0 constant and arbitrary,

$$\delta_{non-min} = 0\,,$$

$$\mathcal{L}_{non-min} = \frac{1}{6\ell}\left(\frac{29}{\ell} - 10\,m\right)\bar{z}z + \frac{5}{\ell\sqrt{2}}\bar{\psi}_+\psi_+ - \frac{\mathrm{i}}{\ell}\left(\bar{z}\bar{\nabla}_0 z - z\bar{\nabla}_0\bar{z}\right)$$

and realizes the following rigid superalgebra

$$\{Q(\epsilon_-), Q(\epsilon_-)\} = -\mathrm{i}\frac{\sqrt{2}}{2}\delta_{U(1)}\left(N^-\right)\,,$$

where $N^- = \mathrm{i}\bar{\epsilon}_-\gamma_0\epsilon_-$.

Summary and Extensions

We have obtained first non-trivial examples of NC three-manifolds compatible with supersymmetry.

Natural extensions:

1. gauge multiplet? (SGED/SGYM)
2. new minimal sugra? conformal?
3. other dimensions?
4. euclidean? compact manifolds?
5. FS for NR off-shell sugra?
6. Susy Lifshitz?

Possible applications:

1. localization? susy indices?
2. anomaly structure?

Thank You!