Post-Galilean symmetries

Axel Kleinschmidt (Albert Einstein Institute, Potsdam)

Non-relativistic gravity event, Nordita, 24 June 2020

with Joaquim Gomis and Jakob Palmkvist [JHEP 09 (2019) 109 1907.00410]

with Joaquim Gomis, Jakob Palmkvist and Patricio Salgado-Rebolledo [PRL 124 (2020) 8, 081602 1910.13560] [JHEP 02 (2020) 009 1912.07564]

with Joaquim Gomis, Diederik Roest and Patricio Salgado-Rebolledo [2006.11102]

Overall picture

Minkowski space-time
Poincaré symmetry

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \dots$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

$$[P_a, P_b] = 0$$

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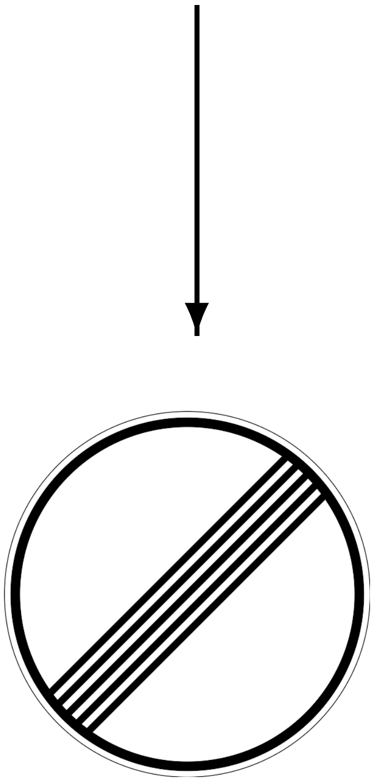

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Newtonian ‘space+time’
Galilei symmetry

$$[J_{ij}, J_{kl}] = \delta_{jk} J_{il} + \dots$$

$$[G_i, G_j] = 0$$

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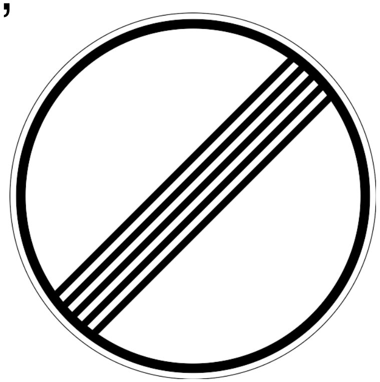
$$[H, G_i] = P_i$$

$$J_{0i} \to G_i$$

$$P_0 \to H$$



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$$1/c \to 0$$

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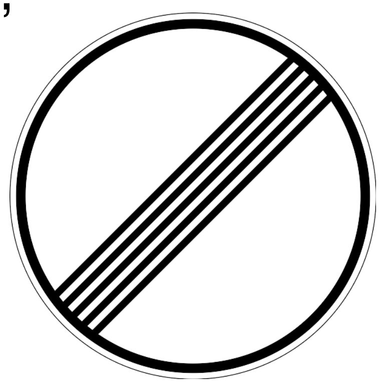

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$$[\nabla_m, \nabla_n] V^p = R^p{}_{qmn} V^q$$

e.g. AdS

$$[P_a, P_b] = R^{-2} J_{ab}$$

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weak fields $g_{mn} = \eta_{mn} + \kappa h_{mn}$
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<u>This talk</u>

explore relations and ‘in-betweens’

small velocities

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Outline

- kinematical background for systematic $1/c$ corrections
  - infinite- and finite-dimensional symmetries
  - generalised Minkowski space
- particle dynamics
- systematic $1/R$ corrections and almost flat geodesics

From Poincaré to Galilei

Poincaré algebra $\mathfrak{iso}(1, D-1)$ (-+. . . + signature)

$$[J_{ab}, J_{cd}] = 2\eta_{c[b}J_{a]d} - 2\eta_{d[b}J_{a]c}$$

$$[J_{ab}, P_c] = 2\eta_{c[b}P_{a]}\,, \qquad [P_a, P_b] = 0$$

With ($i, j, \ldots$ spatial)

$$\tilde{J}_{ij} = J_{ij}\,, \quad \tilde{G}_i = c^{-1}J_{0i}\,, \quad \tilde{P}_i = P_i\,, \quad \tilde{H} = cP_0$$

get for $c \to \infty$ Galilei algebra

$$\left[\tilde{J}_{ij}, \tilde{J}_{kl}\right] = 4\delta_{k][j}\tilde{J}_{i][l}\,, \quad \left[\tilde{J}_{ij}, \tilde{G}_k\right] = 2\delta_{k[j}\tilde{G}_{i]}\,, \quad \left[\tilde{G}_i, \tilde{G}_j\right] = 0$$

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From Galilei to Poincaré

Making c finite again? Start from Galilei algebra

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Marked places differ from Poincaré: law of velocity addition

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Marked places differ from Poincaré: law of velocity addition

To go towards Poincaré put non-zero terms there. Be liberal

$$\left[\tilde{G}_{i}, \tilde{G}_{j}\right] = \tilde{S}_{ij}\,,$$

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Bargmann (central)

($\tilde{S}_{ij}$ and $\tilde{A}_{ij}$ antisymmetric; $\tilde{B}_{ij}$ symmetric traceless.)

Most general set of new generators allowed by Lie algebra cohomology. (Keep P_a commuting.)

New generators commute with Galilei generators (but $\tilde{J}_{ij}$).

From Galilei to Poincaré (II)

Extended algebra

$$\left[\tilde{G}_i, \tilde{G}_j\right] = \tilde{S}_{ij}\,, \qquad \left[\tilde{G}_i, \tilde{P}_j\right] = \delta_{ij}\tilde{N} + \tilde{A}_{ij} + \tilde{B}_{ij}$$

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But: Thinking of

- $\tilde{G}_i$ as the boost at order $1/c$: $\tilde{G}_i \sim c^{-1} J_{0i}$ (better v/c)
- $\tilde{S}_{ij}$ as the first $1/c^2$ correction to the Galilean addition of velocities: $\tilde{S}_{ij} \sim c^{-2} J_{ij}$

$\Rightarrow [\tilde{S}_{ij}, \tilde{G}_k] = 0 + \mathcal{O}(1/c^3)$ is the correct answer approximately

Free Galilean algebra

Add new generators iteratively for $1/c$ corrections to Galilei.

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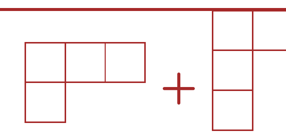
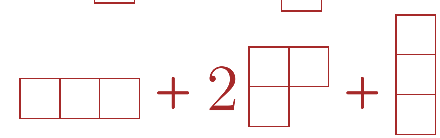
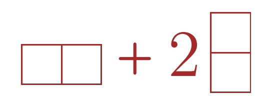
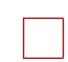
Allowing the most general extension at every step leads to a free Lie algebra generated by $\tilde{G}_i$ and $\tilde{H}$ [Gomis, AK Palmkvist]

	$\ell=0$	$\ell=1$	$\ell=2$	$\ell=3$	$\ell=4$	$\cdots$
$m=0$	$\tilde{J}_{ij}$	$\tilde{G}_i$	$\tilde{S}_{ij}$	$\tilde{Y}_i\,,\tilde{Y}_{ij,k}$		$\cdots$
$m=1$		$\tilde{H}$	$\tilde{P}_i$	$\tilde{N}\,,\tilde{A}_{ij}\,,\tilde{B}_{i,j}$		$\cdots$
$m=2$				$\tilde{Z}_i$		$\cdots$
$m=3$						$\cdots$

Contains much more than corrections towards Poincaré

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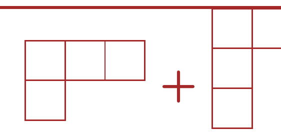
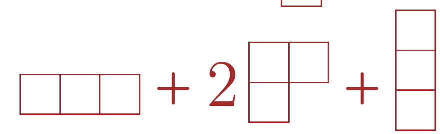
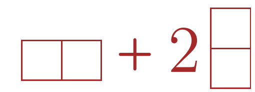
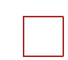
$m=0$	$\tilde{J}_{ij}$	$\tilde{G}_i$	$\tilde{S}_{ij}$	$\tilde{Y}_i$, ~~$\tilde{Y}_{ij,k}$~~		$\cdots$
$m=1$		$\tilde{H}$	$\tilde{P}_i$	$\tilde{N}$, ~~$\tilde{A}_{ij}$~~ , ~~$\tilde{B}_{i,j}$~~		$\cdots$
$m=2$				~~$\tilde{Z}_i$~~		$\cdots$
$m=3$						$\cdots$

Contains much more than corrections towards Poincaré

Take quotient to obtain only $1/c$ corrections predicted by Poincaré. Reproduces algebras of [Hansen, Hartong Obers] [Ozdemir, Ozkan Tunca, Zorba]

Galilei$_\infty$ algebra

Denote the quotient to all orders in $1/c$ by $\mathfrak{G}_\infty$ ($m, n \geq 0$)

$$\left[J_{ij}^{(m)}, J_{kl}^{(n)}\right] = 4\delta_{k][j} J_{i][l}^{(m+n)}, \qquad \left[G_i^{(m)}, G_j^{(n)}\right] = J_{ij}^{(m+n+1)}$$

$$\left[J_{ij}^{(m)}, G_k^{(n)}\right] = 2\delta_{k[j} G_{i]}^{(m+n)}, \qquad \left[J_{ij}^{(m)}, P_k^{(n)}\right] = 2\delta_{k[j} P_{i]}^{(m+n)}$$

$$\left[H^{(m)}, G_i^{(n)}\right] = P_i^{(m+n)}, \qquad \left[G_i^{(m)}, P_j^{(n)}\right] = -\delta_{ij} H^{(m+n+1)}$$

Here $J_{ij}^{(0)} = \tilde{J}_{ij}$, $J_{ij}^{(1)} = \tilde{S}_{ij}$, $H^{(0)} = \tilde{H}$, $H^{(1)} = \tilde{N}$ etc.

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Similar construction in [Khasanov Kuperstein]. Algebra can also be viewed as (parabolic part of twisted) affine algebra.

Lie algebra expansion

$$J_{ij}^{(m)} = c^{-2m} J_{ij}, \qquad G_i^{(m)} = c^{-2m-1} J_{0i}$$

$$P_i^{(m)} = c^{-2m} P_i, \qquad H^{(m)} = c^{-2m+1} P_0$$

[Peñafiel Ravera] [Hatsuda Sakaguchi] [Izaurieta, Salgado Rodriguez] [de Azcarraga, Picon Izquierdo, Varela] [Bergshoeff, Izquierdo Ortín, Romano]

Generalised Minkowski space

What does the infinite symmetry algebra act on?

Take non-linear realisation of $\mathfrak{G}_\infty$ w.r.t. $\mathfrak{L}_\infty = \langle J_{ij}^{(m)}, G_i^{(m)} \rangle$

Coset element $$g = \exp\left[\sum_{m\geq 0}\left(t_{(m)}H^{(m)} + x^i_{(m)}P_i^{(m)}\right)\right]$$

Infinite-dim'l space: generalised Minkowski space $\mathcal{M}_\infty$

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Take non-linear realisation of $\mathfrak{G}_\infty$ w.r.t. $\mathfrak{L}_\infty = \langle J_{ij}^{(m)}, G_i^{(m)} \rangle$

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Infinite-dim'l space: generalised Minkowski space $\mathcal{M}_\infty$

For infin. $\sum_{m\geq 0}\left(\frac{1}{2}\alpha^{ij}_{(m)} J_{ij}^{(m)} + v^i_{(m)} G_i^{(m)} + \epsilon_{(m)} H^{(m)} + \epsilon^i_{(m)} P_i^{(m)}\right) \in \mathfrak{G}_\infty$

$$\delta x^i_{(m)} = \epsilon^i_{(m)} + \sum_{n=0}^{m}\left(v^i_{(n)} t_{(m-n)} - \delta_{jk}\alpha^{ij}_{(n)} x^k_{(m-n)}\right)$$

$$\delta t_{(m)} = \epsilon_{(m)} + \sum_{n=0}^{m-1} \delta_{ij} v^i_{(n)} x^j_{(m-n-1)}$$

Relation to standard Minkowski space

Standard Minkowski space $\mathbb{R}^{1,D-1}$ is a quotient of $\mathcal{M}_\infty$

$$X^i = \sum_{m\geq 0} c^{-2m} x^i_{(m)}\,, \qquad X^0 = \sum_{m\geq 0} c^{-2m+1} t_{(m)}$$

Fixed X^i and X^0 describe hypersurfaces of co-dimension D

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$$\theta^i = \sum_{m\geq 0} c^{-2m-1} v^i_{(m)}$$

on the quotient by the hypersurfaces is

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$\mathfrak{G}_\infty$ transformations move between different hypersurfaces

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  Ideal $\mathfrak{I}$ generated by relations

  $$J^{(m)}_{ij} = J^{(n)}_{ij}\,, \quad G^{(m)}_i = G^{(n)}_i\,, \quad P^{(m)}_i = P^{(n)}_i\,, \quad H^{(m)} = H^{(n)}$$

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- More quotients of $\mathcal{M}_\infty$ possible $\Rightarrow$ other symmetries?

Particle dynamics

Generalised Minkowski space $\mathcal{M}_\infty$ has invariant metric

$$ds^2 = dX^a dX_a = \sum_{m,n\geq 0} c^{-2(m+n)} \left(-c^2 dt_{(m)} dt_{(n)} + d\vec{x}_{(m)} \cdot d\vec{x}_{(n)}\right)$$

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Probe geometry of $\mathcal{M}_\infty$ by considering geodesics

$$S = -mc \int d\tau \sqrt{-\dot{X}^a \dot{X}_a} = S_{(0)} + S_{(1)} + S_{(2)} + \dots$$

i.e.

$$S_{(0)} = -m\, c^2 \int d\tau\, \dot{t}_{(0)}$$

$$S_{(1)} = m \int d\tau \left[-\dot{t}_{(1)} + \frac{\dot{\vec{x}}^{\,2}_{(0)}}{2\,\dot{t}_{(0)}} \right]$$

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← Bargmann inv.

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Particle dynamics (II)

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$$S = -mc \int d\tau \sqrt{-\dot{X}^a \dot{X}_a} = S_{(0)} + S_{(1)} + S_{(2)} + \dots$$

has full $\mathfrak{G}_\infty$ symmetry (even every individual $S_{(m)}$)

Dynamical system with ∞ variables. Truncate at $m \leq N$

To analyse fix gauge & reparam. invariance $c^{-2m} t_{(m)} = t = \tau$

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Project $c^{-2m} x^i_{(m)} = x^i$. Breaks $\mathfrak{G}_\infty$ symmetry

$$\Rightarrow \quad S_{(2)} = \int dt \left[-mc^2 + \frac{m}{2} \dot{\vec{x}}^{\,2} + \frac{m}{8c^2} \dot{\vec{x}}^{\,4} \right]$$

Associated conserved energy and momentum

$$E = mc^2 + \frac{m}{2} \dot{\vec{x}}^{\,2} + \frac{3m}{8c^2} \dot{\vec{x}}^{\,4} , \qquad \vec{P} = m\dot{\vec{x}} + \frac{m}{2c^2} \dot{\vec{x}}^{\,2} \dot{\vec{x}} \qquad ✓$$

Non-relativistic gravity

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Same philosophy also applies to (A)dS and its contractions Newton–Hooke and Carroll. (A)dS starting point

$$[J_{ab}, J_{cd}] = 4\eta_{c][b}J_{a][d}\,, \quad [J_{ab}, P_c] = 2\eta_{c[b}P_{a]}\,, \quad [P_a, P_b] = \sigma J_{ab}$$

($\sigma = +1$: AdS, $\sigma = -1$: dS.) Newton–Hooke $c \to \infty$ contraction

$$[\tilde{G}_i, \tilde{H}] = \tilde{P}_i\,, \qquad [\tilde{J}_{ij}, \tilde{P}_k] = 2\delta_{k[j}\tilde{P}_{i]}\,, \quad [\tilde{J}_{ij}, \tilde{G}_k] = 2\delta_{k[j}\tilde{G}_{i]}$$

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effect of curvature

Choose AdS for definiteness.

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Choose AdS for definiteness.

To go from this non-relativistic algebra to a relativistic one, again expand generators [Gomis, AK, Palmkvist Salgado-Rebolledo]

$$J_{ij}^{(m)} = c^{-2m}\tilde{J}_{ij}\,, \ H^{(m)} = c^{-2m}\tilde{H}\,, \ G_i^{(m)} = c^{-2m-1}\tilde{G}_i\,, \ P_i^{(m)} = c^{-2m-1}\tilde{P}_i$$

Non-relativistic gravity (II)

Truncate at a finite order in $1/c$. E.g. c^{-3}, new generators

$$J_{ij}^{(1)}, \quad H^{(1)}, \quad G_i^{(1)}, \quad P_i^{(1)}$$

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Can be done in any dimension D and to any order in $1/c$.

Post-Newtonian gravity

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Can be done in any dimension D and to any order in $1/c$.
Post-Newtonian gravity

Obtain again an infinite-dimensional algebra $\mathfrak{nh}_\infty$

- quotient to Newton–Hooke
- quotient to AdS
- relation to affine Kac–Moody algebras

Can repeat steps for particle model with cosmological constant

Non-relativistic Chern–Simons

For gravity theory consider Chern–Simons in $D = 3$

$$S_{\mathrm{CS}}[A] = \int d^3x \left\langle AdA + \frac{2}{3} A \wedge A \wedge A \right\rangle = S_{(0)} + S_{(1)} + S_{(2)} + \ldots$$

with connection one-form A in $\mathfrak{nh}_\infty$

$$A = \sum_{m \geq 0} \left(e^i_{(m)} P_i^{(m)} + \omega^i_{(m)} G_i^{(m)} + \tau_{(m)} H^{(m)} + \frac{1}{2} \omega^{ij}_{(m)} J_{ij}^{(m)} \right)$$

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Expanded action

- $S_{(0)}$ Galilei gravity
- $S_{(1)}$ extended Newton–Hooke gravity $\supset$ extended Bargmann [Papageorgiou Schroers] [Bergshoeff Rosseel] [Hartong Lei, Obers]
- $S_{(2)}$ generalises [Ozdemir, Ozkan Tunca, Zorba] to $\Lambda \neq 0$

Also other choices of invariant form since $\mathfrak{so}(2,2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

Small curvatures

Apply the same idea to $1/R$ expansion? [Gomis, AK, Roest Salgado-Rebolledo]

Go from $[P_a, P_b] = 0$ to $[P_a, P_b] \neq 0$...

Introduce generators $J_{ab}^{(m)} = R^{-2m} J_{ab}$ and $P_a^{(m)} = R^{-2m-1}$ (expansion of AdS algebra)

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- construct infinite coset space, with quotients to Minkowski and AdS space
- geodesic particle action on this space, truncated to finite order gives expansion of AdS geodesics
- same for dS
- embedded in larger formalism of free Lie algebras (Maxwell$_\infty$) that also occurs in electro-magnetism [Schrader] [Bacry Levy-Leblond] [Bonanos Gomis] [Gomis AK]

Summary and outlook

- systematic kinematics for small correction $1/c$ or $1/R$
- simplest quotients recover directly corrections for particles and gravity
- Other quotients $\longrightarrow$ direct constructions of new NR theories?
- kinematics also applicable to extended objects (brane algebras)
- Applications to post-Newtonian GR and gravitational waves?

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Thank you for your attention!