Non-Relativistic Sector of General Relativity

Non-Lorentzian Geometry Zoom Meeting, 24/06/2020

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Motivation

Why non-relativity is interesting

- Non-relativistic (NR) physics is still relevant to study!

Why non-relativity is interesting

- Non-relativistic (NR) physics is still relevant to study!
- Many things are effectively non-relativistic: Condensed matter systems, throwing a ball, fluid dynamics, ...
- Crucial: Simpler causal structure - instantaneous interactions.
- New corners of existing theories to be studied:
  - String theory [Jaume Gomis et al. 2001], [Danielsson et al. 2001], [Jaume Gomis et al. 2005], [Andringa et al. 2012], [Batlle et al. 2017], [Harmark et al. 2017], [E. Bergshoeff et al. 2018], [Harmark et al. 2019], [Gallegos et al. 2019], [E. A. Bergshoeff et al. 2020], [Yan et al. 2020], [Roychowdhury 2020].
  - AdS/CFT correspondence and generalized holography [Son 2008], [Taylor 2008], [Balasubramanian et al. 2008], [Janiszewski et al. 2013], [Griffin et al. 2013], [Son 2013], [Christensen et al. 2014], [Hartong et al. 2015].
- Philosophically: What effects are *really* relativistic?
  - Capacitors in electrodynamics are [Le Bellac et al. 1973].
  - Einstein's three classical tests of general relativity (GR) are not.

Newton–Cartan Geometry and Gravity

Newton–Cartan geometry I

- Newton–Cartan (NC) geometry [Cartan 1923], [Cartan 1924]:

$$\begin{aligned} \tau_\mu &: \text{ Clock form - proper time interval } \textstyle\int_{\mathcal{C}} \tau . \\ h^{\mu\nu} &: \text{ Inverse spatial metric, signature } (0,+,\dots,+). \\ v^\mu, h_{\mu\nu} &: \text{ Projective inverses - under local Galilean boosts:} \end{aligned}$$

$$\delta_B \tau_\mu = \delta_B h^{\mu\nu} = 0\,, \quad \delta_B v^\mu = h^{\mu\nu}\lambda_\nu\,, \quad \delta_B h_{\mu\nu} = \tau_\mu \lambda_\nu + \tau_\nu \lambda_\mu\,.$$

- Completeness relations:

$$\tau_\mu h^{\mu\nu} = v^\mu h_{\mu\nu} = 0\,, \quad v^\mu \tau_\mu = -1\,, \quad \delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\lambda} h_{\lambda\nu}\,, \quad \text{so}$$

$$h^{\mu\lambda} h_{\lambda\nu} X^\nu = (\delta^\mu_\nu + v^\mu \tau_\nu) X^\nu \neq X^\mu\,.$$

- Newton–Cartan connections $\Gamma^\rho_{\mu\nu}$:
  - Not possible to simultaneously have metric-compatibility ($\nabla_\mu \tau_\nu = \nabla_\mu h^{\nu\lambda} = 0$) and torsionlessness ($2\Gamma^\rho_{[\mu\nu]} = 0$).
  - No unique metric compatible connection.
  - Any m.c. connection must satisfy $2\tau_\rho \Gamma^\rho_{[\mu\nu]} = 2\partial_{[\mu}\tau_{\nu]}$.

Newton–Cartan geometry II

- Three classes of NC geometry [Christensen et al. 2014], [Bekaert et al. 2014]:

$$\begin{aligned} \mathrm{d}\boldsymbol{\tau} = 0 &: \text{ No torsion, absolute time } (\oint_{\mathcal{C}} \boldsymbol{\tau} = 0). \\ \boldsymbol{\tau} \wedge \mathrm{d}\boldsymbol{\tau} = 0 &: \text{ Twistless torsion, hypersurfaces of simultaneity.} \\ \boldsymbol{\tau} \text{ unconstrained} &: \text{ Acausal spacetime [Geracie et al. 2015].} \end{aligned}$$

- A natural, torsionful, connection:

$$\begin{aligned} \check{\Gamma}^{\rho}_{\mu\nu} &= -v^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right), \\ 2\check{\Gamma}^{\rho}_{[\mu\nu]} &= -2v^{\rho}\partial_{[\mu}\tau_{\nu]}\,. \end{aligned}$$

- Riemann tensor:

$$\left[\check{\nabla}_{\mu}, \check{\nabla}_{\nu}\right] X_{\sigma} = \check{R}_{\mu\nu\sigma}{}^{\rho} X_{\rho} - 2\check{\Gamma}^{\rho}_{[\mu\nu]}\check{\nabla}_{\rho} X_{\sigma}\,.$$

- Ricci tensor has an antisymmetric part:

$$\check{R}_{\mu\nu} = \check{R}_{\mu\rho\nu}{}^{\rho}\,, \quad \check{R}_{[\mu\nu]} \neq 0\,.$$

$1/c^2$ expansion of Lorentzian geometry I

- **Goal:** Systematically NR expand Lorentzian geometry.

$1/c^2$ expansion of Lorentzian geometry I

- **Goal:** Systematically NR expand Lorentzian geometry.
- Expand in $1/c^2 = \sigma/\hat{c}^2 = \sigma$ [Dautcourt 1990], [Van den Bleeken 2017].
  - See [Van den Bleeken 2019], [Ergen et al. 2020] for $1/c$ expansions.
- 'Pre-NR' parameterisation of GR is useful [Hansen et al. 2019]:

$$g_{\mu\nu} = -\sigma^{-1} T_\mu T_\nu + \Pi_{\mu\nu}\,, \quad g^{\mu\nu} = -\sigma T^\mu T^\nu + \Pi^{\mu\nu}\,,$$
$$T_\mu \Pi^{\mu\nu} = T^\mu \Pi_{\mu\nu} = 0\,, \quad T^\mu T_\mu = -1\,, \quad \delta^\mu_\nu = -T^\mu T_\nu + \Pi^{\mu\lambda}\Pi_{\lambda\nu}\,.$$

- Einstein-Hilbert action can be written equivalently as:

$$\mathcal{L}_{\mathrm{EH}} = \sqrt{-\det\left(-T_\alpha T_\beta + \Pi_{\alpha\beta}\right)}\Big[\frac{1}{4}\Pi^{\mu\nu}\Pi^{\rho\sigma} T_{\mu\rho} T_{\nu\sigma} + \sigma \Pi^{\mu\nu} \overset{(C)}{R}_{\mu\nu} - \sigma^2 T^\mu T^\nu \overset{(C)}{R}_{\mu\nu}\Big]\,, \quad T_{\mu\nu} = 2\partial_{[\mu} T_{\nu]}\,,$$

with connection from expansion of the Levi–Civita connection:

$$C^\rho_{\mu\nu} = -T^\rho \partial_\mu T_\nu + \Pi^{\rho\sigma}\left(\partial_{(\mu}\Pi_{\nu)\sigma} - \frac{1}{2}\partial_\sigma \Pi_{\mu\nu}\right)\,, \qquad \overset{(C)}{\nabla}_\mu T_\nu = \overset{(C)}{\nabla}_\mu \Pi^{\nu\rho} = 0\,.$$

$1/c^2$ expansion of Lorentzian geometry II

- Expand all background fields in σ (assuming analyticity):

$$
\begin{aligned}
T_\mu &= \tau_\mu + \sigma m_\mu + \sigma^2 B_\mu + \mathcal{O}(\sigma^3)\,,\\
T^\mu &= v^\mu + \sigma v^\mu v^\rho m_\rho + \mathcal{O}(\sigma^2)\,,\\
\Pi_{\mu\nu} &= h_{\mu\nu} + \sigma \Phi_{\mu\nu} + \sigma^2 \psi_{\mu\nu} + \mathcal{O}(\sigma^3)\,,\\
\Pi^{\mu\nu} &= h^{\mu\nu} + \sigma\left(2h^{\rho(\mu}v^{\nu)}m_\rho - h^{\mu\rho}h^{\nu\sigma}\Phi_{\rho\sigma}\right) + \mathcal{O}(\sigma^2)\,.
\end{aligned}
$$

- LO are Newton–Cartan fields and NLO are gauge fields.
- For the metric we get in terms of boost-invariant fields:

$$
g_{\mu\nu} = -\sigma^{-1}\tau_\mu\tau_\nu + \Big[\underbrace{h_{\mu\nu} - 2\tau_{(\mu}m_{\nu)}}_{\equiv \bar{h}_{\mu\nu}}\Big] + \sigma\Big[\underbrace{\Phi_{\mu\nu} - m_\mu m_\nu - 2B_{(\mu}\tau_{\nu)}}_{\equiv \bar{\Phi}_{\mu\nu}}\Big] + \mathcal{O}(\sigma^2)\,,
$$

$$
g^{\mu\nu} = h^{\mu\nu} - \sigma\Big[\underbrace{(v^\mu - h^{\mu\rho}m_\rho)}_{\equiv \hat{v}^\mu}(v^\mu - h^{\mu\sigma}m_\sigma) + h^{\mu\rho}h^{\nu\sigma}\bar{\Phi}_{\rho\sigma}\Big] + \mathcal{O}(\sigma^2)\,.
$$

Expansion of EH Lagrangian I

- The expansion can be done in terms of $1/c^2$. Starts at order c^6 as can be found by dimensional analysis.
- The expansion of the EH Lagrangian takes the form

$$\mathcal{L}_{\text{EH}} = c^6 \left(\overset{(-6)}{\mathcal{L}}_{\text{LO}} + \sigma \overset{(-4)}{\mathcal{L}}_{\text{NLO}} + \sigma^2 \overset{(-2)}{\mathcal{L}}_{\text{NNLO}} + \mathcal{O}(\sigma^3) \right) ,$$

where

$$\overset{(-6)}{\mathcal{L}}_{\mathrm{LO}} = \frac{e}{16\pi G_N} \frac{1}{4} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho} \tau_{\nu\sigma} ,$$

$$\overset{(-4)}{\mathcal{L}}_{\mathrm{NLO}} = \frac{e}{16\pi G_N} \left[h^{\mu\nu} \check{R}_{\mu\nu} - 2 \overset{(-6)}{G}{}^{\mu}_{\tau} m_\mu - \overset{(-6)}{G}{}^{\mu\nu}_{h} \Phi_{\mu\nu} \right] ,$$

and $e = \sqrt{-\det(-\tau_\mu \tau_\nu + h_{\mu\nu})}$ and $\tau_{\mu\nu} \equiv 2\partial_{[\mu} \tau_{\nu]}$.

- $\overset{(-6)}{G}{}^{\mu}_{\tau}$, $\overset{(-6)}{G}{}^{\mu\nu}_{h} \propto \tau \wedge \mathrm{d}\tau$ are LO EOMs we will see.

Expansion of EH Lagrangian II

- At NNLO order we find

$$\overset{(-2)}{\mathcal{L}}_{\text{NNLO}} = \frac{e}{16\pi G_N}\Bigg[-v^\mu v^\nu \check{R}_{\mu\nu} - 2m_\nu \check{\nabla}_\mu \left(h^{\mu\rho} h^{\nu\sigma} K_{\rho\sigma} - h^{\mu\nu} h^{\rho\sigma} K_{\rho\sigma}\right)$$
$$+ \Phi h^{\mu\nu} \check{R}_{\mu\nu} + \frac{1}{4} h^{\mu\rho} h^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} - \Phi_{\rho\sigma} h^{\mu\rho} h^{\nu\sigma} \Big(\check{R}_{\mu\nu} - \check{\nabla}_\mu a_\nu - a_\mu a_\nu$$
$$- \frac{1}{2} h_{\mu\nu} h^{\kappa\lambda} \check{R}_{\kappa\lambda} + h_{\mu\nu} e^{-1} \partial_\kappa \left(e h^{\kappa\lambda} a_\lambda\right)\Big)\Bigg] + \text{terms} \propto \text{LO EOMs},$$

$$F_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu - a_\mu m_\nu + a_\nu m_\mu\,,$$
$$\Phi = -v^\mu m_\mu\,, \quad a_\mu = \mathcal{L}_v \tau_\mu\,, \quad K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_v h_{\mu\nu}\,.$$

- LO and NLO EOMs are reproduced at this order by fields of subleading order.

Equations of motion I

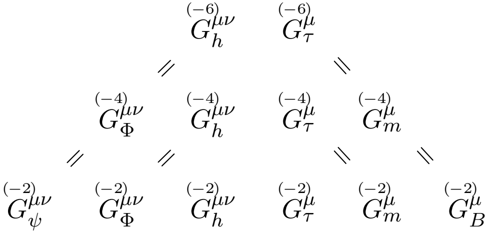

Figure 1: Structure of vacuum EOMs.

- Define

$$\frac{1}{16\pi G_N}\overset{(2n-6)}{G}{}_h^{\alpha\beta} \equiv -e^{-1}\frac{\delta \overset{(2n-6)}{\mathcal{L}}{}_{\mathrm{N}^n\mathrm{LO}}}{\delta h_{\alpha\beta}},$$

$$\frac{1}{8\pi G_N}\overset{(2n-6)}{G}{}_\tau^{\alpha} \equiv -e^{-1}\frac{\delta \overset{(2n-6)}{\mathcal{L}}{}_{\mathrm{N}^n\mathrm{LO}}}{\delta \tau_{\alpha}},$$

  and likewise for more subleading fields.
- EOMs are nested because of variational identities.

Equations of motion II

- At leading order:

$$
\begin{aligned}
\overset{(-6)}{G}{}_h^{\alpha\beta} &= -\frac{1}{8} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho}\tau_{\nu\sigma} h^{\alpha\beta} + \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} \tau_{\mu\rho}\tau_{\nu\sigma}\,, \\
\overset{(-6)}{G}{}_\tau^{\alpha} &= \frac{1}{8} h^{\mu\nu} h^{\rho\sigma} \tau_{\mu\rho}\tau_{\nu\sigma} v^\alpha + \frac{1}{2} a_\mu h^{\mu\nu} h^{\rho\alpha} \tau_{\nu\rho} \\
&\quad + \frac{1}{2} e^{-1} \partial_\mu \left(e h^{\mu\nu} h^{\rho\alpha} \tau_{\nu\rho} \right) .
\end{aligned}
$$

- Crucial: $\tau \wedge \mathrm{d}\tau = 0$ can be imposed off shell, yielding simplifications.
- $\mathrm{d}\tau = 0$ cannot be imposed off shell: This must be determined dynamically by the matter sources as we shall see.
- This reveals something deep about Newtonian gravity.

Coupling to Matter and Newtonian Gravity

Coupling to matter I

- Assume a matter Lagrangian $\mathcal{L}_{\mathrm{mat}} = \mathcal{L}_{\mathsf{mat}}(1/c^2, \phi, \partial_\mu \phi)$ with the expansion

$$\mathcal{L}_{\mathsf{mat}} = c^N \overset{(-N)}{\mathcal{L}}_{\mathsf{mat},\mathrm{LO}} + c^{N-2} \overset{(2-N)}{\mathcal{L}}_{\mathsf{mat},\mathrm{NLO}} + \mathcal{O}(c^{N-4})\,.$$

- Might begin at a different order than the EH action.
- General structure of matter coupled EOMs:

$$\overset{(2n-6)}{G}{}^{\mu\nu}_{\phi} = 8\pi G_N \overset{(2n-6)}{T}{}^{\mu\nu}_{\phi}\,, \qquad \overset{(2n-6)}{G}{}^{\mu}_{\phi} = 8\pi G_N \overset{(2n-6)}{T}{}^{\mu}_{\phi}\,.$$

- The currents are zero if the geometric field does not appear at the corresponding order.

Coupling to matter II

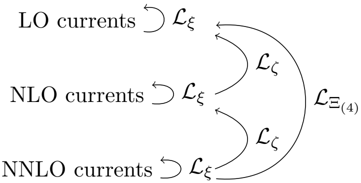

Figure 2: Structure of diffeomorphic WIs.

- On-shell Ward identities (i.e. up to matter EOMs) are useful.
- Energy and momentum conservation at each order from diffeomorphisms $\Xi^\mu = \xi^\mu + \sigma\zeta^\mu + \sigma^2 \Xi^\mu_{(4)} + \mathcal{O}(\sigma^3)$.
- There are also boost and subleading boost currents.
- Imposing $\tau \wedge \mathrm{d}\tau = 0$ off shell we write NNLO currents as:

$$\mathcal{G}^{\mu\nu}_\phi \equiv \overset{(-2)}{G}{}^{\mu\nu}_{\phi} \big|_{\tau\wedge\mathrm{d}\tau=0} \,, \qquad \mathcal{T}^{\mu\nu}_\phi \equiv \overset{(-2)}{T}{}^{\mu\nu}_{\phi} \big|_{\tau\wedge\mathrm{d}\tau=0} \,.$$

The Road to Newtonian (or $\mathrm{d}\tau = 0$) gravity I

- Assuming that $\tau \wedge \mathrm{d}\tau = 0$, we have WLOG

$$\tau = N \mathrm{d} T \,,$$

  where N is the lapse function describing local time dilation and T is the time function.

- Up to boundary conditions and topology of the manifold, an equation for N follows from the NLO EOMs:

$$8\pi G_N \left(-(d-2)\tau_\mu \mathcal{T}_m^\mu + h_{\mu\nu} \mathcal{T}_\Phi^{\mu\nu} \right) = (d-1) e^{-1} \partial_\mu \left(e h^{\mu\nu} a_\nu \right) \,,$$

  where $h^{\mu\nu} a_\nu = h^{\mu\nu} N^{-1} \partial_\nu N$.

- In conclusion: A necessary condition for Newtonian gravity is

$$(d-2)\tau_\mu \mathcal{T}_m^\mu = h_{\mu\nu} \mathcal{T}_\Phi^{\mu\nu} \,.$$

The Road to Newtonian (or $\mathrm{d}\tau = 0$) gravity II

- EOMs sourced by a static mass density ρ gives $\mathrm{d}\tau = 0$ and

$$\bar{R}_{\mu\nu} = 8\pi G_N \frac{d-2}{d-1} \times \rho\, \tau_\mu \tau_\nu\,, \quad \bar{\Gamma}^\rho_{\mu\nu} \equiv \check{\Gamma}^\rho_{\mu\nu} - h^{\rho\sigma} \tau_{(\mu} F_{\nu)\sigma}\,.$$

- This is exactly the covariant Newtonian Poisson equation [Trautman 1963] - In flat gauge:

$$\partial_i \partial_i m_0 \propto \rho\,.$$

- This equation is not compatible with Bargmann symmetry.
- Local symmetry group is larger than Bargmann: Can be studied systematically using Lie algebra expansions [Khasanov et al. 2011], [Hansen et al. 2019], [E. Bergshoeff et al. 2019], [Hansen et al. 2020].
- Thus the NRG theory described by $\mathcal{L}_{\mathrm{NRG}} \equiv \overset{(-2)}{\mathcal{L}}_{\mathrm{NNLO}}$ (with $\tau \wedge \mathrm{d}\tau = 0$ off shell) is much richer than Newtonian gravity.

Solutions

Schwarzschild-like solution I

- Schwarzschild solution is canonical in GR:

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2G_N m}{c^2 r}\right)\mathrm{d}t^2 + \left(1-\frac{2G_N m}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{S^2}\,.$$

- Two types of expansions: (Covariant) Post-Newtonian expansion and Strong field expansion.
- Post-Newtonian expansion - Weak fields (m const.):

$$\begin{aligned}
\tau_\mu \mathrm{d}x^\mu &= \mathrm{d}t\,,\\
m_\mu \mathrm{d}x^\mu &= -\frac{G_N m}{r}\mathrm{d}t = \Phi\mathrm{d}t\,,\\
h_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= \mathrm{d}r^2 + r^2\mathrm{d}\Omega_{S^2}\,,\\
\Phi_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= \frac{2G_N m}{r}\mathrm{d}r^2 = -2\Phi\mathrm{d}r^2\,,\\
\Phi^{(2n)}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= (-2\Phi)^n\mathrm{d}r^2\,.
\end{aligned}$$

Schwarzschild-like solution II

- The other possibility is a strong field expansion (m/c^2 const.)
- The expansion terminates immediately [Van den Bleeken 2017]:

$$\begin{aligned}
\tau_\mu \mathrm{d}x^\mu &= \sqrt{1-\frac{2G_N M}{r}}\mathrm{d}t\,, \\
m_\mu \mathrm{d}x^\mu &= 0\,, \\
h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \left(1-\frac{2G_N M}{r}\right)^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_{S^2}\,, \\
\Phi_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= 0\,.
\end{aligned}$$

- We can reproduce GR geodesics for orbits (but they are not so well understood) [Hansen et al. 2019]:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \frac{r^4}{l^2} - \left(1-\frac{2G_N M}{r}\right)\left(\frac{C^2}{l^2}r^4 + r^2\right)\,, \qquad C = \mathrm{const}\,.$$

Summary

Conclusions and Outlook

- Conclusions:
  - Non-relativistic gravity is much richer than Newtonian gravity.
  - $1/c^2$ expansion of GR is naturally formulated as a torsionful Newton–Cartan geometry.
  - Newtonian gravity requires special matter sources.
  - Many canonical solutions of GR are also solutions of NRG.

Conclusions and Outlook

- Conclusions:
  - Non-relativistic gravity is much richer than Newtonian gravity.
  - $1/c^2$ expansion of GR is naturally formulated as a torsionful Newton–Cartan geometry.
  - Newtonian gravity requires special matter sources.
  - Many canonical solutions of GR are also solutions of NRG.

- Outlook:
  - Touch base with Post-Newtonian expansion [Tichy et al. 2011].
  - Apply this framework to non-relativistic holography.
  - NRG in first-order formalism (work in progress).
  - Backreaction in Schrödinger–Newton theory.
  - Study when gravitational waves show up in the expansion.
  - Ultra-relativistic expansion of GR - Carrollian geometry and beyond (work in progress).

Thank you

Transformations of fields

$$
\begin{aligned}
\delta\tau_\mu &= \mathcal{L}_\xi \tau_\mu \,, \\
\delta h_{\mu\nu} &= \mathcal{L}_\xi h_{\mu\nu} + \tau_\mu \lambda_\nu + \tau_\nu \lambda_\mu \,, \\
\delta h^{\mu\nu} &= \mathcal{L}_\xi h^{\mu\nu} \,, \\
\delta \bar{h}_{\mu\nu} &= \mathcal{L}_\xi \bar{h}_{\mu\nu} - \tau_\mu \mathcal{L}_\zeta \tau_\nu - \tau_\nu \mathcal{L}_\zeta \tau_\mu \,, \\
\delta \hat{v}^\mu &= \mathcal{L}_\xi \hat{v}^\mu - h^{\mu\rho} \mathcal{L}_\zeta \tau_\rho \,, \\
\delta \bar{\Phi}_{\mu\nu} &= \mathcal{L}_\xi \bar{\Phi}_{\mu\nu} + \mathcal{L}_\zeta \bar{h}_{\mu\nu} \,,
\end{aligned}
$$

where $\lambda_\mu \equiv e_\mu^a \lambda_a$ is the Galiliean boost parameter which obeys $v^\mu \lambda_\mu = 0$. ξ^μ generates diffeomorphisms and ζ^μ generates gauge transformations.

Local non-relativistic algebra

$$[H\,,G_a] = P_a\,, \qquad [P_a\,,G_b] = N\delta_{ab}\,,$$
$$[N\,,G_a] = T_a\,, \qquad [H\,,B_a] = T_a\,, \qquad [S_{ab}\,,P_c] = \delta_{ac}T_b - \delta_{bc}T_a\,,$$
$$[S_{ab}\,,G_c] = \delta_{ac}B_b - \delta_{bc}B_a\,, \qquad [G_a\,,G_b] = -S_{ab}\,,$$
$$[J_{ab}\,,X_c] = \delta_{ac}X_b - \delta_{bc}X_a\,,$$
$$[J_{ab}\,,J_{cd}] = \delta_{ac}J_{bd} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc} + \delta_{bd}J_{ac}\,,$$
$$[J_{ab}\,,S_{cd}] = \delta_{ac}S_{bd} - \delta_{bc}S_{ad} - \delta_{ad}S_{bc} + \delta_{bd}S_{ac}\,,$$

where X_a denotes P_a, T_a, G_a, B_a.

Equations of motion (boost inv. formulation) I

$$
\begin{aligned}
\mathcal{G}_{\hat{\Phi}} &= \frac{1}{2} h^{\mu\nu} \bar{R}_{\mu\nu} \\
\mathcal{G}^{\mu\nu}_{\bar{\Phi}} &= h^{\mu\rho} h^{\nu\sigma} \left(\bar{R}_{\rho\sigma} - a_\rho a_\sigma - \bar{\nabla}_\rho a_\sigma \right) - \frac{1}{2} h^{\mu\nu} \left(h^{\rho\sigma} \bar{R}_{\rho\sigma} - 2 e^{-1} \partial_\rho \left(e h^{\rho\sigma} a_\sigma \right) \right) , \\
h^{\rho\mu} \mathcal{G}^{\hat{v}}_{\mu} &= h^{\rho\mu} \hat{v}^\nu \bar{R}_{\mu\nu} , \\
2 \hat{v}^\mu \mathcal{G}^{\hat{v}}_{\mu} &= 2 \hat{\Phi} E^{\mathrm{g}}_{\hat{\Phi}} - \bar{\Phi}_{\mu\nu} E^{\mu\nu}_{\mathrm{g}} + h^{\mu\nu} \bar{\Phi}_{\mu\nu} e^{-1} \partial_\rho \left(e h^{\rho\sigma} a_\sigma \right) - h^{\mu\rho} h^{\nu\sigma} \bar{\Phi}_{\mu\nu} \left(\bar{\nabla}_\rho a_\sigma + a_\rho a_\sigma \right) \\
&\quad + \left(h^{\rho\sigma} \bar{K}_{\rho\sigma} \right)^2 - h^{\rho\sigma} h^{\kappa\lambda} \bar{K}_{\rho\kappa} \bar{K}_{\sigma\lambda} - \bar{\nabla}_\mu \left[h^{\mu\rho} h^{\nu\sigma} \left(\bar{\nabla}_\rho \bar{\Phi}_{\nu\sigma} - \bar{\nabla}_\nu \bar{\Phi}_{\rho\sigma} \right) \right] ,
\end{aligned}
$$

Equations of motion (boost inv. formulation) II

$$
\begin{aligned}
\mathcal{G}_h^{\alpha\beta} = &+ \left(h^{\mu\alpha} h^{\nu\beta} \bar{\Phi}_{\mu\nu} - \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} \bar{\Phi}_{\mu\nu}\right) \left(e^{-1} \partial_\rho \left(e h^{\rho\sigma} a_\sigma\right) - E^{\mathrm{g}}_{\hat{\Phi}}\right) \\
&+ \frac{1}{2} h^{\alpha\beta} \bar{\Phi}_{\mu\nu} E_{\mathrm{g}}^{\mu\nu} - h^{\mu\alpha} \bar{\Phi}_{\mu\rho} E_{\mathrm{g}}^{\rho\beta} - h^{\mu\beta} \bar{\Phi}_{\mu\rho} E_{\mathrm{g}}^{\rho\alpha} + \frac{1}{2} h^{\rho\sigma} \bar{\Phi}_{\rho\sigma} E_{\mathrm{g}}^{\alpha\beta} + \hat{\Phi} E_{\mathrm{g}}^{\alpha\beta} \\
&+ \frac{1}{2} h^{\alpha\beta} \left[\left(h^{\mu\nu} \bar{K}_{\mu\nu}\right)^2 - h^{\mu\rho} h^{\nu\sigma} \bar{K}_{\mu\nu} \bar{K}_{\rho\sigma}\right] - \bar{\nabla}_\rho \left[\hat{v}^\rho h^{\mu\alpha} h^{\nu\beta} \bar{K}_{\mu\nu} - \hat{v}^\rho h^{\alpha\beta} h^{\mu\nu} \bar{K}_{\mu\nu}\right] \\
&- h^{\mu\alpha} h^{\nu\beta} \bar{\nabla}_\mu \partial_\nu \hat{\Phi} - h^{\mu\alpha} h^{\nu\beta} \left(a_\mu \partial_\nu \hat{\Phi} + a_\nu \partial_\mu \hat{\Phi}\right) + h^{\alpha\beta} h^{\mu\nu} \bar{\nabla}_\mu \partial_\nu \hat{\Phi} + 2 h^{\alpha\beta} h^{\mu\nu} a_\mu \partial_\nu \hat{\Phi} \\
&- \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} h^{\rho\sigma} \left(\bar{\nabla}_\mu + a_\mu\right) \left(\bar{\nabla}_\rho + a_\rho\right) \bar{\Phi}_{\nu\sigma} \\
&+ h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} \left(\bar{\nabla}_\rho + a_\rho\right) \left(\bar{\nabla}_{(\mu} \bar{\Phi}_{\nu)\sigma} - \frac{1}{2} \bar{\nabla}_\sigma \bar{\Phi}_{\mu\nu}\right) \\
&+ \frac{1}{2} h^{\alpha\beta} h^{\mu\nu} h^{\rho\sigma} \left(\bar{\nabla}_\mu + a_\mu\right) \bar{\nabla}_\nu \bar{\Phi}_{\rho\sigma} - \frac{1}{2} h^{\mu\alpha} h^{\nu\beta} h^{\rho\sigma} \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi}_{\rho\sigma} \,.
\end{aligned}
$$

Expansion generalities

- Expand fields

$$\phi^I(x;\sigma) = \phi^I_{(0)}(x) + \sigma\phi^I_{(2)}(x) + \sigma^2\phi^I_{(4)}(x) + \mathcal{O}(\sigma^3)\,.$$

- Expansion of Lagrangians:

$$\tilde{\mathcal{L}}(\sigma) = \tilde{\mathcal{L}}(0) + \sigma\tilde{\mathcal{L}}'(0) + \frac{1}{2}\sigma^2\tilde{\mathcal{L}}''(0) + \mathcal{O}(\sigma^3)\,.$$

- So:

$$\begin{aligned}\tilde{\mathcal{L}}(\sigma) &= \tilde{\mathcal{L}}(0) + \sigma\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\sigma} + \frac{\partial\phi}{\partial\sigma}\frac{\partial\tilde{\mathcal{L}}}{\partial\phi} + \frac{\partial\partial_\mu\phi}{\partial\sigma}\frac{\partial\tilde{\mathcal{L}}}{\partial\partial_\mu\phi}\right)\Bigg|_{\sigma=0} + \cdots \\ &= \tilde{\mathcal{L}}(0) + \sigma\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\sigma}\Big|_{\sigma=0} + \phi_{(2)}\frac{\partial\tilde{\mathcal{L}}(0)}{\partial\phi_{(0)}} + \partial_\mu\phi_{(2)}\frac{\partial\tilde{\mathcal{L}}(0)}{\partial\partial_\mu\phi_{(0)}}\right) + \cdots\,.\end{aligned}$$

- We can also derive the TOV equation in this framework.
  - Consider a static, spherical symmetric Ansatz:

$$
\begin{aligned}
\tau_\mu &= N(r)\delta^t_\mu = e^{\alpha(r)}\delta^t_\mu\,, \\
v^\mu &= -e^{-\alpha(r)}\delta^\mu_t\,, \\
h^{\mu\nu} &= \mathrm{diag}\left(0, e^{-2\beta(r)}, 1/r^2, 1/(r^2\sin^2\theta)\right)\,, \\
h_{\mu\nu} &= \mathrm{diag}\left(0, e^{+2\beta(r)}, r^2, r^2\sin^2\theta\right)\,.
\end{aligned}
$$

  - We source by a perfect fluid beginning at order $\mathcal{O}(c^4)$ of pressure P and energy E.
  - With total mass $M(r)$ we then obtain the TOV equation:

$$
P' = -\frac{G_N}{r^2}(P+E)\left(M(r) + 4\pi r^3 c^{-4} P\right)\left(1 - \frac{2M(r)G_N}{r}\right)^{-1}\,.
$$

FLRW Cosmology

- Consider a d-dimensional maximally symmetric space with metric σ_{ij} and scale factor $a(t)$. Then our Ansatz is:

$$\tau_\mu = \delta^t_\mu \,, \qquad h_{\mu\nu} = a(t)^2 \sigma_{ij} \delta^i_\mu \delta^j_\nu \,, \quad \text{yielding} \quad a_\mu = 0 \,, \quad K_{\mu\nu} = \frac{\dot{a}}{a} h_{\mu\nu} \,.$$

- We source by a perfect fluid beginning with non-zero energies and pressures $E_{(-4)}$, $P_{(-4)}$, $E_{(-2)}$ and $P_{(-2)}$.
- After resumming pressures and energies it is then easy to see that one obtains the Friedmann equations:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_N}{3c^2} E - \frac{c^2 k}{a(t)^2} \,, \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 &= -\frac{4\pi G_N}{3c^2} (E + 3P) \,, \qquad k = -1, 0, 1 \,. \end{aligned}$$

Perfect fluid

- Expansion of fluid velocity:

$$U^\mu = u^\mu + \frac{1}{c^2} u^\mu_{(2)} + \mathcal{O}(c^{-4}), \qquad \tau_\mu u^\mu = 1, \quad \tau_\mu u^\mu_{(2)} = \frac{1}{2} \bar{h}_{\mu\nu} u^\mu u^\nu \,.$$

- EM tensor:

$$T^{\mu\nu} = \frac{E+P}{c^2} U^\mu U^\nu + P g^{\mu\nu} \,.$$

- Expansion of energy and pressure:

$$\begin{aligned} E &= c^4 E_{(-4)} + c^2 E_{(-2)} + E_{(0)} + \mathcal{O}(c^{-2}) \,, \\ P &= c^4 P_{(-4)} + c^2 P_{(-2)} + P_{(0)} + \mathcal{O}(c^{-2}) \,. \end{aligned}$$

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