Type I Torsional Newton-Cartan from String Theory

Natale Zinnato

Based on 1906.01607 and ongoing work with
U. Gürsoy, D. Gallegos and S. Verma

Utrecht University

Outline

- The string action and its symmetries
- Background field method
- Geodesic equation: twistless torsion
- Beta functions, equations of motion
- Results from Double Field Theory
- Solutions

The worldsheet theory

Usual Bosonic string

- Riemannian metric:

$$ds^2 = g_{MN} dx^M dx^M$$

- Geometry and KR matter:

$$\mathcal{L} = -\frac{\sqrt{-\gamma}}{4\pi l_s^2}\gamma^{\alpha\beta} g_{\alpha\beta} - \frac{1}{4\pi l_s^2}\epsilon^{\alpha\beta}\mathcal{B}_{\alpha\beta}$$

$$g_{\alpha\beta} \equiv g_{\mu\nu}\partial_\alpha X^M \partial^\alpha X^M$$

- Dilaton: $\mathcal{L}_\phi = \frac{1}{16\pi}\sqrt{-\gamma}\mathcal{R}\phi$

Non-relativistic string

- Reduction along null direction:

$$ds^2 = 2\tau(du - m) + h_{mn} dx^m dx^n$$

- Geometry and KR matter:

$$\mathcal{L} = -\frac{1}{4\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\bar{h}_{\alpha\beta} + \epsilon^{\alpha\beta}\bar{B}_{\alpha\beta}\right) - \frac{1}{4\pi l_s^2}\left(\sqrt{-\gamma}\gamma^{\alpha\beta}\tau_\alpha - \epsilon^{\alpha\beta}\aleph_\alpha\right)\partial_\beta X^u$$

- Dilaton: $\mathcal{L}_\phi = \frac{1}{16\pi}\sqrt{-\gamma}\mathcal{R}\phi$

The string action

- Introduce Lagrange multipliers to impose momentum conservation along X^u:

$$\mathcal{L} = \frac{1}{4\pi l_s^2} e \left[e_+^\alpha e_-^\beta \left(\bar{h}_{\alpha\beta} + \bar{B}_{\alpha\beta} \right) + \lambda_+ e_-^\beta \left(\partial_\beta \eta + \aleph_\beta + \tau_\beta \right) + \lambda_- e_+^\beta \left(\partial_\beta \eta + \aleph_\beta - \tau_\beta \right) \right]$$

- The Dilaton Lagrangian stays the same

Symmetries

- Spacetime symmetries:

$$\delta\tau_s = \pounds_\xi \tau_s$$
$$\delta e^i_s = \pounds_\xi e^i_s + \lambda^i \tau_s + \lambda^i{}_j e^j_s$$
$$\delta v^s = \lambda^i e^s_i$$
$$\delta e^s_i = \pounds_\xi e^s_i$$

$$\delta m_s = \pounds_\xi m_s + \lambda_i e^i_s + \partial_s \sigma$$
$$\delta \bar{B}_{mn} = \pounds_\xi \bar{B}_{mn} + 2\aleph_{[m}\partial_{n]}\sigma$$
$$\delta \aleph_m = \pounds_\xi \aleph_m \, ,$$
$$\delta\phi = \pounds_\xi \phi$$

$$\delta\lambda_\pm = \mp e^\alpha_\pm \partial_\alpha \sigma$$

- $\pounds_\xi$: Diffeomorphisms
- λ^i : Galilean boosts
- $\lambda^i{}_j$: Rotations
- $\partial_s \sigma$: mass U(1)

Symmetries

- KR U(1) symmetries:

$$\delta_\Lambda \mathcal{B}_{MN} = \partial_M \Lambda_N - \partial_N \Lambda_M \longrightarrow \begin{aligned} \delta_\Lambda \bar{B}_{mn} &= \partial_m \Lambda_n - \partial_n \Lambda_m \\ \delta_\Lambda \aleph_m &= \partial_m \Lambda_u \\ \delta_\Lambda \eta &= -\Lambda_u \end{aligned}$$

- Worldsheet diffeomorphisms: $\gamma^{ab} = e^{-2\rho}\eta^{ab}$ $\quad \sqrt{-\gamma}\mathcal{R} = -2\partial^2\rho$

Weyl invariance?

The background field method

- Expansion in powers of l_s:

$$X^m = X_0^m + l_s \bar{Y}^m$$
$$\lambda_\pm = \lambda_\pm^0 + l_s \bar{\Lambda}_\pm$$
$$\eta = \eta_0 + l_s \bar{H}$$

- One-loop effective action:

$$e^{i\bar{\Gamma}[\Psi_0](0)} = \int D\Psi \; e^{i\bar{S}[\Psi_0,\Psi](0)}$$

- Require Weyl invariance:

$$\delta_\psi \bar{\Gamma}[\Psi_0](0) = \delta_\psi \left\langle S^{[1]} + S^{[2]} \right\rangle_0 + \frac{i}{2}\delta_\psi \left\langle S^{[1]} S^{[1]} \right\rangle_0 - i\delta_\psi \left\langle S^{[1]} \right\rangle_0 \left\langle S^{[1]} \right\rangle_0 - i\delta_\psi \log(Z_0 Z_{FP}) + \mathcal{O}\left(D^3\right)$$

- Usual computation to get critical dimension

The TNC geodesic equation

$N \equiv \tau_p \dot{x}^p \qquad F_{mn} = 2\partial_{[m}\tau_{n]}$

$$\mathcal{S}_{\text{part}} = \int d\lambda \frac{m}{2} \frac{\bar{h}_{mn}\dot{x}^m \dot{x}^n}{\tau_s \dot{x}^s} \Longrightarrow \ddot{x}^t + \Gamma^t_{mn}\dot{x}^m\dot{x}^n = \frac{\dot{N}}{N}\dot{x}^t - \frac{\bar{h}_{mn}F_{sr}h^{st}}{2N}\dot{x}^m\dot{x}^n\dot{x}^r$$

- Ansatz: $x^m = X_0^m + \lambda l_s Y^m + \frac{\lambda^2}{2} l_s^2 Y_2^m + \mathcal{O}(l_s^3) = X_0^m + l_s \bar{Y}^m$
- Solution: $\bar{Y}^m = Y^m - \frac{l_s}{2}\left(\Gamma^m_{rs} + \frac{1}{2}\bar{h}_{rs}a_n h^{mn}\right) Y^r Y^s + \mathcal{O}\left(l_s^2\right)$
- Solution exists as long as $h^{mr}h^{ns}F_{mn}F_{rs} = 0$

The TNC geodesic equation

- The twistless condition on torsion appears as a requirement to have a covariant expansion, rather than an equation of motion

- It is natural to define a new, symmetric, U(1) invariant connection:

$$\mathring{\Gamma}^m_{rs} = \Gamma^m_{rs} + \hat{\upsilon}^m \partial_{[r}\tau_{s]} + \frac{1}{2} a_n h^{mn} \bar{h}_{rs}$$

Final steps

- Flat indices decomposition to compute correlators (just a field redefinition):

$$Y^m = -\hat{\upsilon}^m\left(\tau_s Y^s\right) + e_i^m\left(\delta^{ij} e_j^r \bar{h}_{rs} Y^s\right) \equiv -\hat{\upsilon}^m \frac{Y^0}{\sqrt{2\Phi}} + e_i^m Y^i \equiv e_I^m Y^I$$

$$\hat{H} = \frac{H}{\sqrt{2\Phi}}, \qquad \bar{\Lambda}_\pm = \sqrt{2\Phi}\Lambda_\pm$$

- Variation of the effective action:

$$\delta_\psi \bar{\Gamma}[\Psi_0](0) = -\int d^2\sigma \frac{\psi}{4\pi}\Big[\beta_{rs}\eta^{\alpha\beta}\partial_\alpha X^r \partial_\beta X^s + \bar{\beta}_{rs}\epsilon^{\alpha\beta}\partial_\alpha X^r \partial_\beta X^s$$
$$+\beta_m \Delta\lambda^\beta \partial_\beta X_0^m + \bar{\beta}_m \Sigma\lambda^\beta \partial_\beta X_0^m + \beta\lambda_+^0\lambda_-^0\Big]$$

Setting the Beta functions to zero

- Twistless condition:

$$\frac{1}{4}\left(b_{mr}b_{ns} - F_{mr}F_{ns}\right)h^{mn}h^{rs} = 0$$

$\Rightarrow$

$$F_{mn} = a_m\tau_n - a_n\tau_m$$

$$\begin{aligned} b_{mn} &= 2\partial_{[m}\aleph_{n]} \\ &= \mathfrak{e}_m\tau_n - \mathfrak{e}_n\tau_m \end{aligned}$$

- Two scalar equations:

$$D\cdot a + a^2 = 2\mathfrak{e}^2 + 2\left(a\cdot D\phi\right)$$

$$D\cdot\mathfrak{e} = 2\left(\mathfrak{e}\cdot D\phi\right)$$

Setting the Beta functions to zero

- Symmetric tensor equation:

$$R_{(mn)} - \frac{H_{rs(m}H_{n)tw}h^{rt}h^{sw}}{4} + 2D_{(m}D_{n)}\phi = \frac{\mathfrak{e}^2\left(2\Phi\tau_m\tau_n - \bar{h}_{mn}\right) - \mathfrak{e}_m\mathfrak{e}_n}{2} - \mathfrak{e}_r h^{rs}\left(\Delta_T\right)^t_{(m}H_{n)ts} + \left(\Delta_S\right)^t_{(m}D_{n)}a_t + \frac{a_m a_n}{2} - a^2\,\Phi\tau_m\tau_n$$

- Antisymmetric tensor equation:

$$\frac{1}{2}h^{rs}D_r H_{smn} - h^{rp}H_{pmn}D_r\phi = \left(\Delta_S\right)^t_{[m}D_{n]}\mathfrak{e}_t - \left(\Delta_T\right)^r_{[m}D_{n]}\mathfrak{e}_r - a_{[m}\mathfrak{e}_{n]} - \frac{1}{2}a_r h^{rs}H_{smn} + \left(\hat{\upsilon}^t D_t\phi - \frac{D_t\hat{\upsilon}^t}{2}\right)b_{mn}$$

Simpler case: no torsion

- Vanishing torsion implies

$$a_m = \mathfrak{e}_m = 0$$

- More familiar equations:

$$R_{(mn)} - \frac{H_{rs(m}H_{n)tw}h^{rt}h^{sw}}{4} + 2D_{(m}D_{n)}\phi = 0$$

$$\frac{1}{2}h^{rs}D_r H_{smn} - h^{rp}H_{pmn}D_r\phi = 0$$

Newton's law

- Time-time projection of symmetric equation:

$$D^2\Phi + 3\,(a \cdot D\Phi) + m_\Phi^2\,\Phi = \rho_\kappa + \rho_{\rm m}$$

- Mass and mass densities are:

$$\begin{aligned} m_\Phi^2 &\equiv a^2 + 2\,\mathfrak{e}^2 + 4\,a \cdot D\phi \\ \rho_{\mathcal{K}} &\equiv \mathcal{K}_{rs}\mathcal{K}_{tw}h^{rt}h^{sw} - \hat{\upsilon}^n D_n\left(\mathcal{K}_{rs}h^{rs}\right) \\ \rho_{\rm m} &\equiv \frac{1}{4}\hat{\upsilon}^m\hat{\upsilon}^n h^{rt}h^{sw}H_{rsm}H_{twn} - 2\hat{\upsilon}^m\hat{\upsilon}^n D_m D_n\phi \end{aligned}$$

The role of U(1) mass symmetry

- It changes the O(D) of the action, i.e. not compatible with our expansion

$$\delta \int \frac{d^2\sigma\, e}{4\pi} D_r D_s \bar{h}_{mn} Y^r Y^s \partial_\alpha X_0^m \partial^\alpha X_0^n = -2 \int \frac{d^2\sigma\, e}{4\pi} \tau_m D_r D_s D_n \sigma Y^r Y^s \partial_\alpha X_0^m \partial^\alpha X_0^n$$

- Solution: *impose* it at the quantum level

The role of U(1) mass symmetry

- Rearrange the O(D) terms such that the beta functions are invariant:

$$\int d^2\sigma d^2\sigma' \left(V^{\alpha}_{(IJ)}V^{\beta}_{(KL)}\right)\langle Y^I(\sigma)\partial_\alpha Y^J(\sigma)Y^K(\sigma')\partial_\beta Y^L(\sigma')\rangle = \mathcal{O}\left(D^3\right)$$

- Beta functions are then *uniquely* determined

- Considering the action at all orders in derivatives should give the correct beta functions without the need for this trick

Action and equations of motion from DFT

- Formulation of string theory in terms of a generalized metric and generalized dilaton
- Can derive action and EOM’s for many different target spacetimes, TNC included
- E.g. Type I TNC with a dilaton:

$$\mathcal{H} = \begin{pmatrix} \bar{h}_{mn} & 0 & 0 & \tau_r \\ 0 & 2\Phi & -\hat{v}^s & 0 \\ 0 & -\hat{v}^s & h^{mn} & 0 \\ \tau_r & 0 & 0 & 0 \end{pmatrix}$$

$$e^{-2d} = \sqrt{\frac{\det \bar{h}}{2\Phi}} e^{-2\phi}$$

Action and equations of motion from DFT

- Type I TNC action including KR matter:

$$S = \int d^d x \, e \, h^{\mu\nu} \Big[\mathcal{R}_{\mu\nu} + \frac{1}{2} a_\mu a_\nu + \frac{1}{2} \mathfrak{e}_\mu \mathfrak{e}_\nu - 4 a_\mu D_\nu \phi + 4 D_\mu \phi D_\nu \phi - \frac{1}{12} H_{\mu\rho\lambda} H_{\nu\sigma\kappa} h^{\rho\sigma} h^{\lambda\kappa}$$
$$- \frac{1}{2} \hat{\upsilon}^\lambda h^{\rho\sigma} H_{\mu\rho\lambda} b_{\nu\sigma} - \frac{1}{2} h^{\rho\sigma} \left(F_{\mu\rho} F_{\nu\sigma} + b_{\mu\rho} b_{\nu\sigma} \right) \Phi \Big]$$

- EOM's seem to match with the ones obtained from the requirement of Weyl invariance:

$$D \cdot a + a^2 = 2 \mathfrak{e}^2 + 2 \left(a \cdot D\phi \right)$$

- Newton's law cannot be obtained from this action, but can be obtained from the DFT action directly (known problem with these kinds of geometries) Park, Cho, 1909.10711

Solutions

- No matter, non-zero torsion, spherically symmetric static solution in $d+2$ dimensions

- Ansatz:

$$\tau = f(r)\,dt \qquad \bar{h}_{mn} = \begin{pmatrix} -2f(r)^2\Phi(r) & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & h(r)\gamma_{ij} \end{pmatrix}$$

- Solution:

$$\Phi(r) = \frac{\Phi_1}{f} + \Phi_2 \frac{\log f}{f} \qquad g(r) = g_1 h(r)^d f'(r)^2$$

$$h(r) = h_2 f^{\frac{2\sqrt{d}-\sqrt{2d+2}}{\sqrt{d}(1-d)}} \left(h_1 + f^{\sqrt{\frac{2+2d}{d}}}\right)^{-2/(d-1)} \qquad h_2 = \left(-\frac{h_1(2d+2)}{g_1\mathfrak{R}(d-1)}\right)^{1/(d-1)}$$

- f is a free parameter

Solutions

- In four dimensions we have

$$\bar{h}_{mn}dx^m dx^n = -2R^2\,\Phi(R)dt^2 + g_1 h(R)^2 dR^2 + h(R)d\Omega_{S^2}$$

$$h(R) = h_2 R^{-(2-\sqrt{3})}\left(h_1 + R^{\sqrt{3}}\right)^{-2} \qquad \Phi(R) = \frac{\Phi_1}{R} + \Phi_2\frac{\log R}{R}$$

- Quite different from the Type II TNC analogue (*strong* gravity expansion of Schwarzschild metric) Van den Bleeken, 1703.03459

$$\bar{h}_{mn}dx^m dx^n = \left(1 - \frac{2G_N\,M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{S^2}$$

$$\tau = \sqrt{1 - \frac{2G_N\,M}{r}}\,dt$$

Solutions

- If we assume vanishing torsion the solution is much simpler ($d > 1$):

$$\bar{h}_{mn} dx^m dx^n = -2\Phi(r) dt^2 + dr^2 + r^2 d\Omega_{S^d} \qquad \Phi(r) = \tfrac{\Phi_1}{r^{d-1}}$$

- Very similar from the Type II TNC analogue in four dimensions (*weak* gravity expansion of Schwarzschild metric) Hansen, Hartong, Obers, 2001.10277

$$\bar{h}_{mn} dx^m dx^n = -2\Phi(r) dt^2 + dr^2 + r^2 d\Omega_{S^2}$$

$$\Phi(r) = \tfrac{-G_N m}{r} \qquad \Phi_{mn} dx^m dx^n = -2\Phi(r) dr^2$$

Future directions

- Full action and EOM’s from DFT
- Study solutions in detail: conserved quantities, geodesics, etc.
- Couple to QFT
- String spectrum, vertex operators, interactions, etc.
- D-Branes

Thanks!