Geometry and Dynamics of Spin Matrix Strings

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Based on **several ongoing projects** with Leo Bidussi, Troels Harmark, Jelle Hartong and Niels Obers

Main message: ***target space becomes phase space!***

$$\{X(\sigma), P(\sigma')\} \sim \delta(\sigma - \sigma') \qquad \Longrightarrow \qquad \{X(\sigma), X(\sigma')\} \sim \delta(\sigma - \sigma')$$

Outline

- Motivation: Spin Matrix Theory limit of $\mathcal{N} = 4$ SYM
  - zoom in on BPS bound $E \geq Q$
  - keep non-relativistic dynamics

- Bulk dual: Spin Matrix strings
  - from non-relativistic worldsheet limit
  - for each BPS bound get U(1)-Galilean geometry
  - large Q limit commutes with Penrose limit

- Dynamics: U(1)-Galilean parametrizes phase space!

Spin Matrix Theory limits of $\mathcal{N} = 4$

Spin Matrix Theory: [Kruczenski] [Harmark, Kristjansson, Orselli]

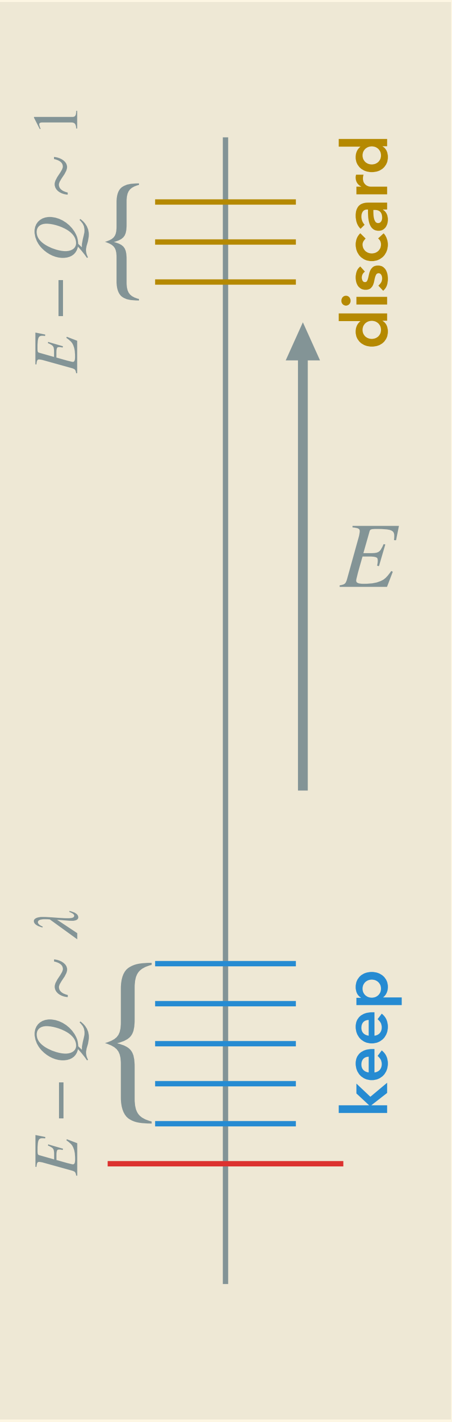
Starting from of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times S^3$, zoom in on BPS bound

$E \geq Q = \sum a^i S_i + b^i J_i$ (S^3 isometries S_i and R-charges J_i)

by taking the limit

$$\lambda \to 0, \qquad N = \text{fixed}, \qquad \frac{E - Q}{\lambda} = \text{fixed}$$

Here, focus on $N \to \infty$ and large Q $\implies$ sigma models



Spin Matrix Theory limits of $\mathcal{N} = 4$

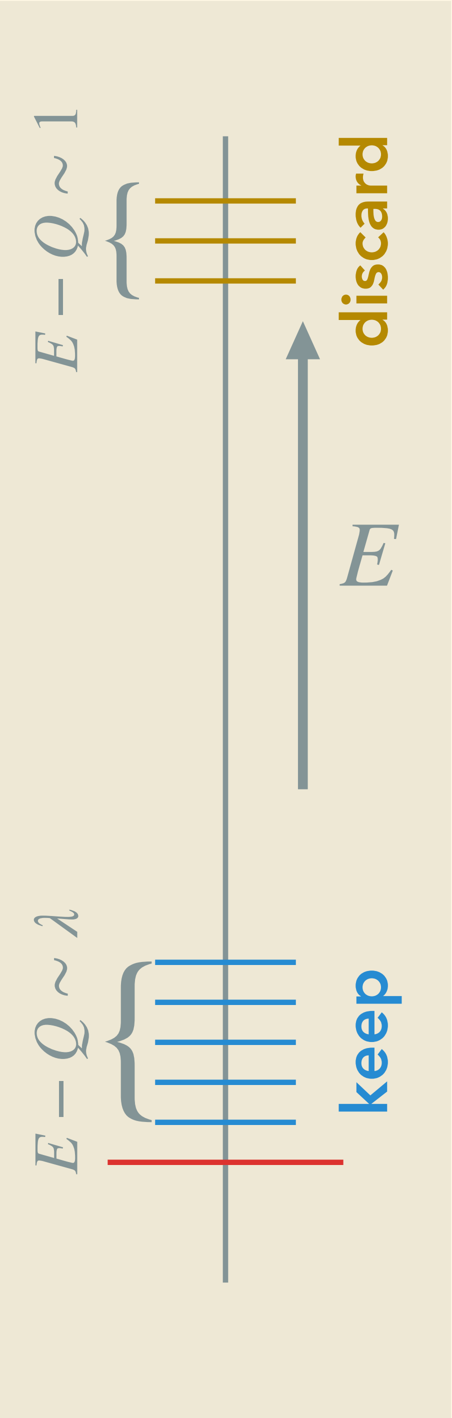
Example: $SU(2)$ Landau-Lifshitz model from $Q = J_1 + J_2$
[Kruczenski] [Harmark, Kristjansson, Orselli]

$$S = \frac{Q}{4\pi} \int d^2\sigma \left[\dot{\varphi} \cos\theta - \frac{1}{4} \left[(\theta')^2 + \sin^2\theta \left(\varphi'\right)^2 \right] \right]$$

Goal: understand this from *dynamics of non-relativistic string!*

- Where are these directions in $AdS_5 \times S^5$?
- How does non-relativistic behavior arise?
- How to quantize?

Build on [Harmark, Hartong, Obers, Menculini, Yan]



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Spin Matrix strings

Bulk dual of Spin Matrix limit with $E \geq Q = \sum a^i S_i + b^i J_i$

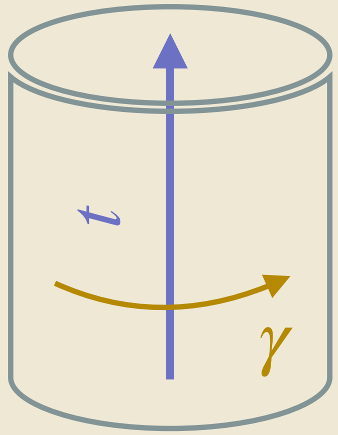
$$g_s \to 0, \qquad N = \text{fixed}, \qquad \frac{E - Q}{g_s} = \text{fixed}$$

Procedure: [Harmark-Hartong-Obers]

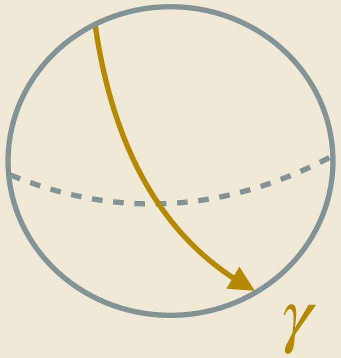
- find a combination of angles γ such that $Q = -i\partial_\gamma$
- define $x^0 = (t + \gamma)/2$ and $u = \gamma - t$ and rescale $x^0 \sim \tilde{x}^0 / g_s$

$$i\partial_{\tilde{x}^0} = \frac{E - Q}{g_s} \text{ and } -i\partial_u = (E + Q)/2$$

- keeps only dynamics on submanifold where $\left(\partial_u\right)^2 = 0$



AdS$_5$



S^5

Spin Matrix strings

Example: $SU(2)$ Spin Matrix string from $Q = J_1 + J_2$

Parametrize the S^5 using $|z_1|^2 + |z_2|^2 + |z_3|^2 = R^2$

$\Longrightarrow$ charges J_i correspond to phases of z_i

To get $Q = -i\partial_\gamma$ use Hopf coordinates

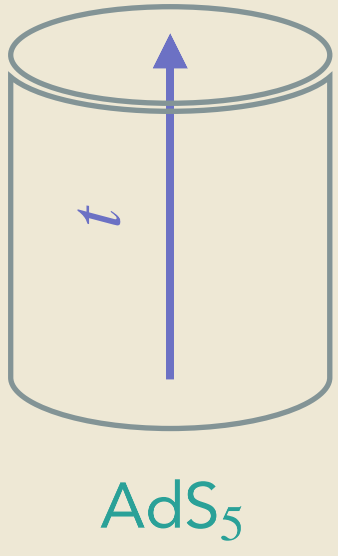
$z_1 = R\sin(\beta/2)\sin(\theta/2)e^{i(\gamma+\varphi/2)}$

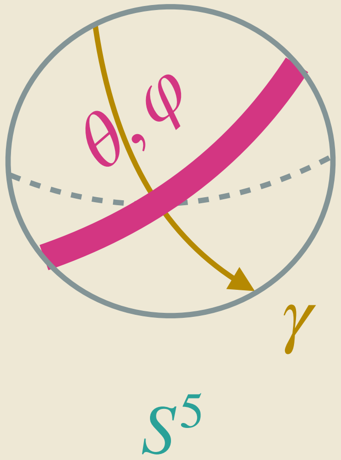
$z_2 = R\sin(\beta/2)\cos(\theta/2)e^{i(\gamma-\varphi/2)}$

$z_3 = R\cos(\beta/2)e^{i\alpha}$

Then restrict to $\rho = 0$ and $\beta = \pi$

$$ds^2\Big|_M = -R^2dt^2 + R^2\left[\frac{1}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + \left(d\gamma + \sin^2(\theta/2)d\varphi\right)^2\right]$$





Spin Matrix strings

Example: $SU(2)$ Spin Matrix string from $Q = J_1 + J_2$

Using $u = \gamma - t$ and $x^0 = (t+\gamma)/2$, this gives

$$ds^2\Big|_M = 2\tau\left(R^2 du - m\right) + h,$$

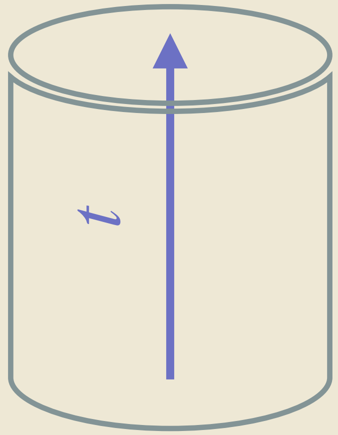
$$\tilde{\tau} = d\tilde{x}^0, \quad m = -\frac{R^2}{2}\cos\theta d\varphi, \quad h = \frac{R^2}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

corresponding to Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow \mathbf{CP}^1 \simeq S^2$

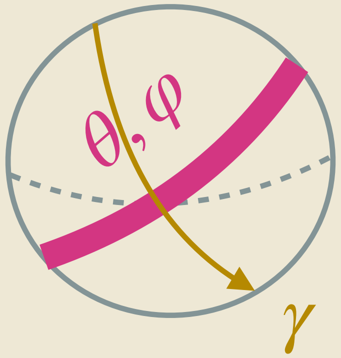
Likewise, $Q = J_1 + J_2 + J_3$ leads to $S^1 \hookrightarrow S^5 \rightarrow \mathbf{CP}^2$ for $SU(2|3)$

$\Longrightarrow$ can restrict to $S^3 \subset S^5$ and $\mathbf{CP}^1 \subset \mathbf{CP}^2$ above

Obtain *all SMT backgrounds* from $Q = S_1 + S_2 + J_1 + J_2 + J_3$ case!



AdS$_5$



S^5

Spin Matrix strings

Split $x^\mu = (x^0, x^a)$, then clock form

$$\tau_\mu dx^\mu \sim \tau_0 d\tilde{x}^0 / g_s + \mathcal{O}(g_s^0)$$

Then (rescaled) Galilean boosts $\tilde{\lambda}^a$ and U(1) local transformations σ are

$$\delta\tau_\mu = 0\,,$$
$$\delta m_\mu = \partial_\mu \sigma$$
$$\delta h_{\mu\nu} = 2\tilde{\tau}_{(\mu} E^a_{\nu)} \tilde{\lambda}_a$$

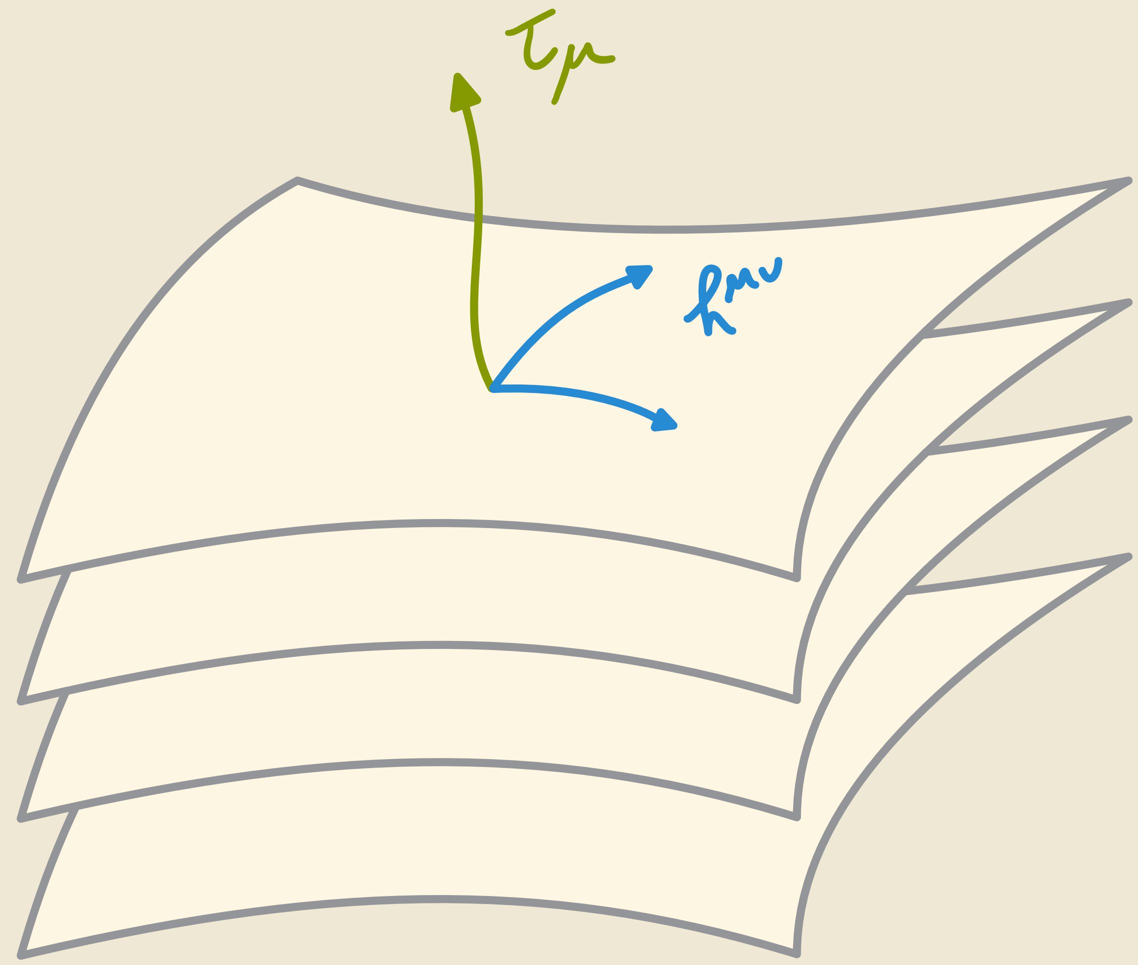
$\Longrightarrow$ 'mass' one-form decouples

$\Longrightarrow$ no boost-invariant $\bar{h}_{\mu\nu}$

Hence 'U(1)-Galilean geometry'

[Harmark-Hartong-Obers]

$\Longrightarrow$ *role in string sigma model?*



$$\tau \wedge d\tau = 0$$

Spin Matrix strings

Spin Matrix string 'light-cone gauge' action [Harmark-Hartong-Obers]

$$S = -\frac{Q}{2\pi}\int d^2\sigma \left[m_\mu \dot{X}^\mu + \frac{1}{2} h_{\mu\nu} X'^\mu X'^\nu \right]$$

Unlike TNC *cannot have* $m_\mu = 0$, otherwise no dynamics in X^μ,

$\Longrightarrow$ role of U(1)-Galilean geometry?

Example: for $SU(2)$ background

$$\tau = d\tilde{x}^0, \quad m = -\frac{1}{2}\cos\theta d\varphi, \quad h = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

this reproduces the $SU(2)$ Landau-Lifshitz action

$$S = \frac{Q}{4\pi}\int d^2\sigma \left[\dot{\varphi}\cos\theta - \frac{1}{4}\left[(\theta')^2 + \sin^2\theta \left(\varphi'\right)^2 \right] \right]$$

Can quantize [Klose-Zarembo] but difficult, simplify?

Spin Matrix strings

Example: $SU(2)$ Spin Matrix string from $Q = J_1 + J_2$

$$\tau = d\tilde{x}^0, \quad m = -\frac{1}{2}\cos\theta d\varphi, \quad h = \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Simplification: take $Q \to \infty$ with $\tilde{x}^0$ fixed and

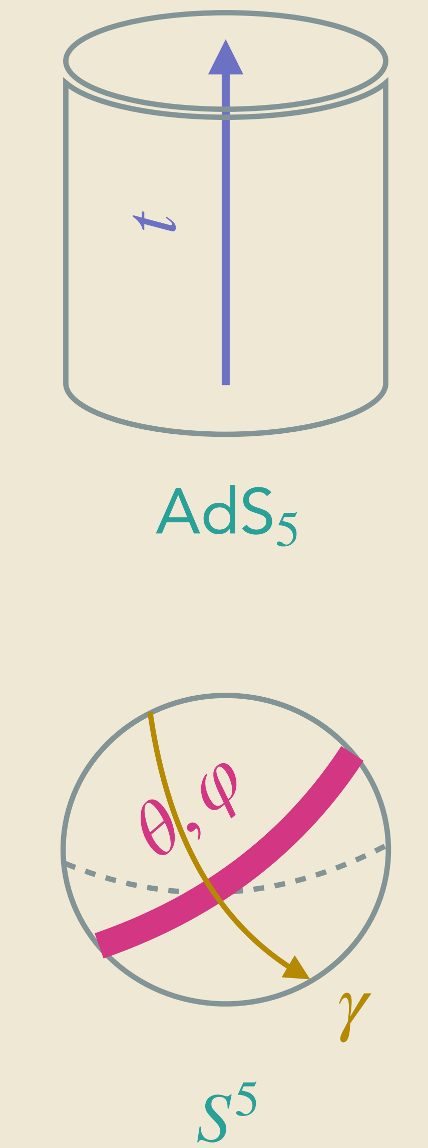
$$u = \frac{\tilde{u}}{Q} \quad \theta = \frac{\pi}{2} + \frac{x}{\sqrt{Q}}, \quad \varphi = \frac{y}{\sqrt{Q}}$$

This leads to the 'flat' background

$$\tau = d\tilde{x}^0, \quad m = \frac{1}{2}xdy, \quad h = \frac{1}{4}\left(dx^2 + dy^2\right)$$

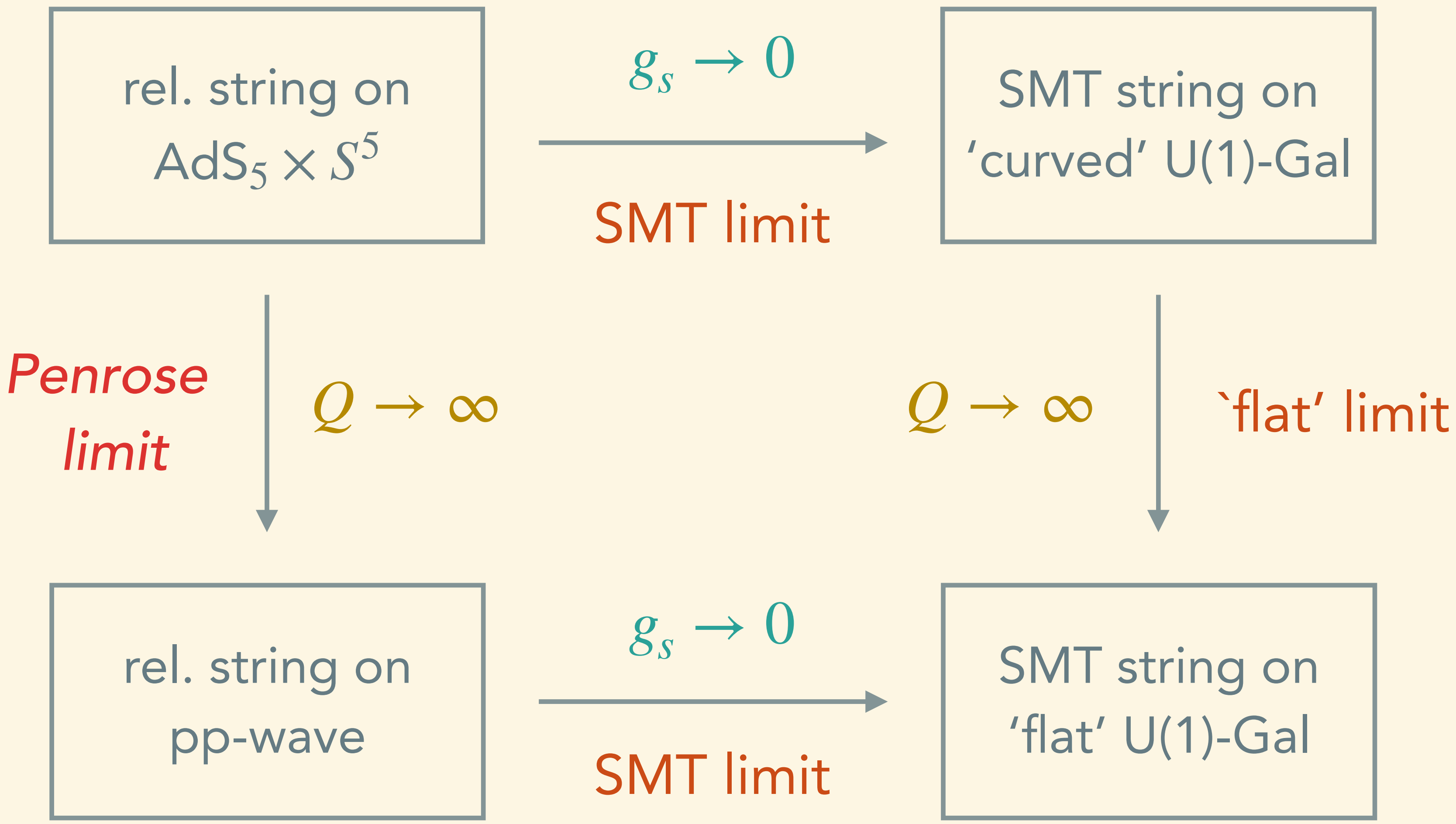
and the 'light-cone' string action

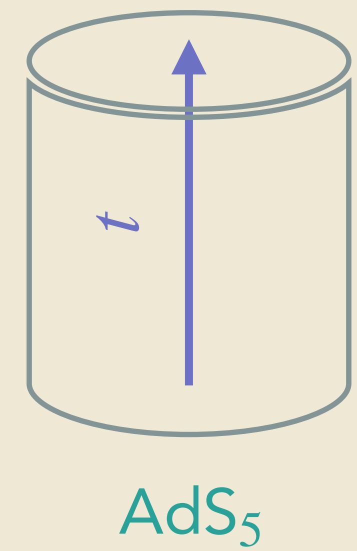
$$S = \frac{1}{4\pi}\int d^2\sigma \left(x\dot{y} - \frac{1}{4}\left[(x')^2 + \left(y'\right)^2\right]\right)$$

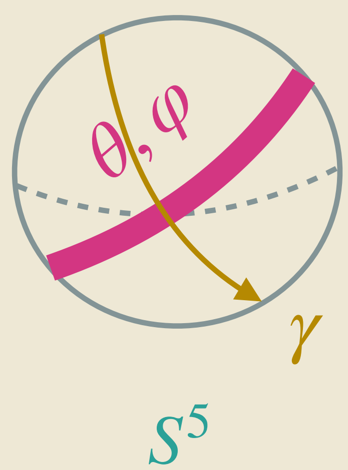


Spin Matrix strings

Penrose and SMT limit *commute!*







Spin Matrix strings

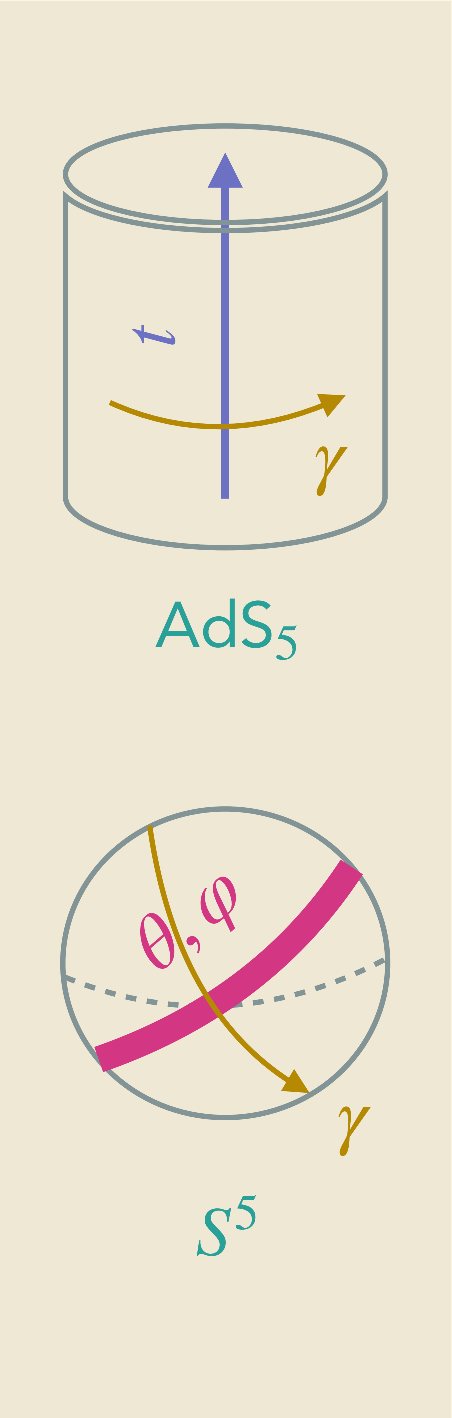
Penrose limit $Q \to \infty$ of $\mathrm{AdS}_5 \times S^5$ gives pp-wave geometry

$$ds^2 = 2dx^0 du - 2m_\alpha dx^\alpha du + d\mathbf{x}^2 - \delta_{ij} x^i x^j \left(dx^0\right)^2$$

Split coordinates (u, x^0, x^α, x^i) where [Bertolini ea., Grignani ea.]

- x^i feel quadratic potential $\Longrightarrow$ decouple in SMT limit
- x^α are 'flat' $\Longrightarrow$ parametrize SMT dynamics

Agrees with 'flat limit' $Q \to \infty$ of 'curved' U(1)-Galilean!



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- Dynamics: U(1)-Galilean parametrizes phase space!

Dynamics and U(1)-Galilean geometry

In Hamiltonian dynamics, frequently use phase space action

$$S = \int dt \left[\dot{q}^I p_I - H(q,p)\right], \qquad \{q^I, p_J\} = \delta^I_J$$

No problem if not diagonal: with presymplectic form $\Theta_I(q)$

$$S = \int dt \left[\dot{q}^I \Theta_I(q) - H(q)\right], \qquad \omega_{IJ} = \partial_I \Theta_J - \partial_J \Theta_I\,, \qquad \{q^I, q^J\} = \omega^{IJ}$$

Compare: Spin Matrix string 'light-cone gauge' action (with $\tau = d\tilde{x}^0$)

$$S = -\frac{Q}{2\pi}\int d^2\sigma \left[m_\mu \dot{X}^\mu + \frac{1}{2} h_{\mu\nu} X'^\mu X'^\nu\right]$$

- symplectic form $\omega \sim dm$,
- Poisson bracket $\{X^\mu, X^\nu\} = \omega^{\mu\nu}$,
- Hamiltonian $H \sim \oint d\sigma^1 h_{\mu\nu} X'^\mu X'^\nu$

$\Longrightarrow$ U(1)-Galilean geometry defines *phase space* of SMT action!

Dynamics and U(1)-Galilean geometry

Example: SU(2) Landau-Lifshitz model, $Q = J_1 + J_2$

On 'curved' $\mathbf{CP}^1 \simeq S^2$ background,

$$S = \frac{Q}{4\pi} \int d^2\sigma \left[\cos\theta\,\dot{\varphi} - \frac{1}{4} \left[(\theta')^2 + \sin^2\theta\,(\varphi')^2 \right] \right]$$

- symplectic form $\omega \sim \sin\theta d\theta \wedge d\varphi$
- Poisson bracket $\{\theta(\sigma), \varphi(\sigma')\} \sim \dfrac{\delta(\sigma - \sigma')}{\sin\theta} \implies$ quantization is hard

Simplification: 'flat' background from $Q \to \infty$,

$$S = \frac{1}{4\pi} \int d^2\sigma \left(x\dot{y} - \frac{1}{4} \left[(x')^2 + (y')^2 \right] \right)$$

- symplectic form $\omega \sim dx \wedge dy$
- Poisson bracket $\{x(\sigma), y(\sigma')\} \sim \delta(\sigma - \sigma') \implies$ quantization is simpler!

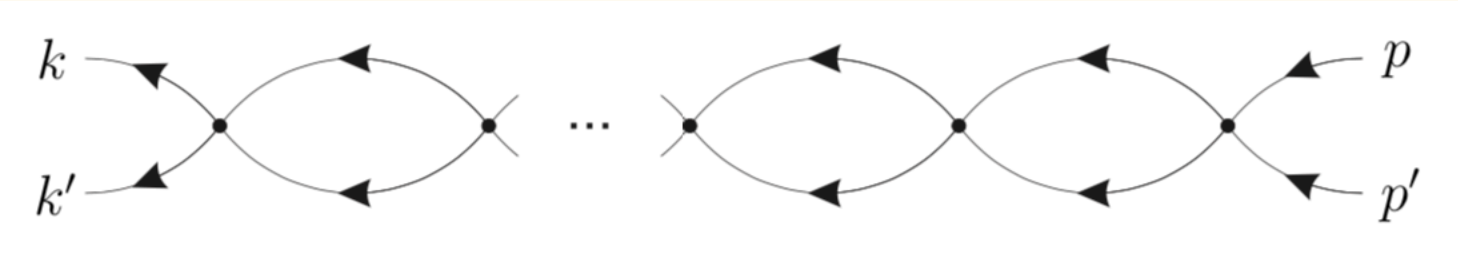
Dynamics and U(1)-Galilean geometry

$$S = \frac{Q}{4\pi}\int d^2\sigma \left[\cos\theta\,\dot{\varphi} - \frac{1}{4}\left[(\theta')^2 + \sin^2\theta\,(\varphi')^2\right]\right]$$

- parametrize $p = \cos\theta$ and $q = \varphi$, *use Darboux coordinates!*
- then $\omega \sim \sin\theta d\theta \wedge d\varphi \sim dp \wedge dq$!
- leads to Schrödinger-type propagator for $\phi = q + ip$

$$\mathscr{L}_{\text{eff}} = \frac{i}{2}\left(\bar{\phi}\dot{\phi} - \phi\dot{\bar{\phi}}\right) - \left|\phi'\right|^2 - \frac{1}{2}\left[\left(\bar{\phi}\phi'\right)^2 + \left(\phi\bar{\phi}'\right)^2\right]$$

- only particular type of diagram contributes to four-point function [Klose-Zarembo]



Can be used to compute exact S-matrix! Generalize to other SMT models?

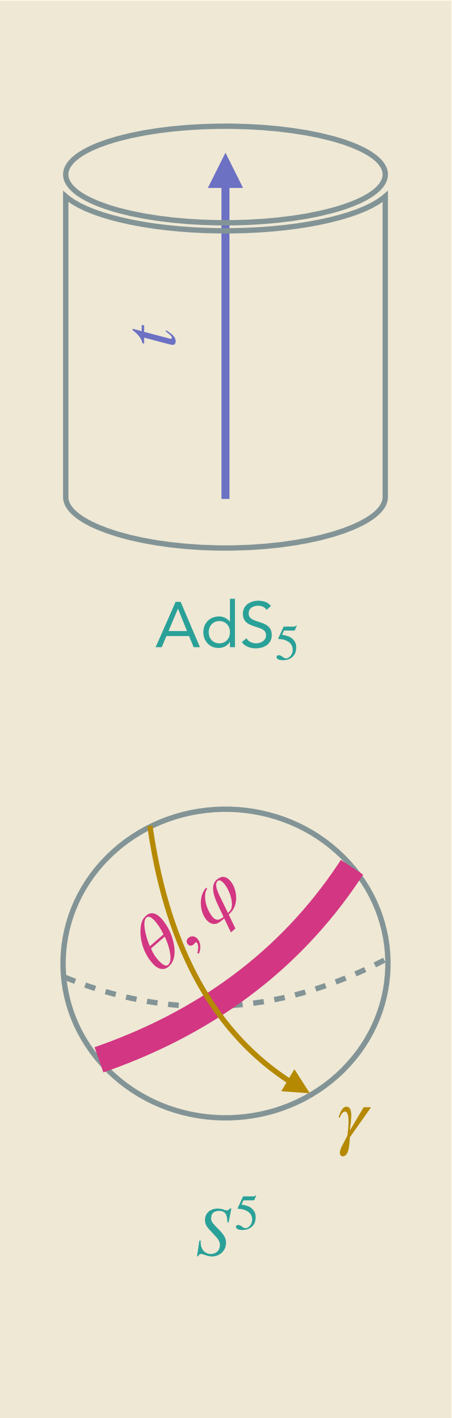
Conclusions and outlook

Spin Matrix strings

- arise from non-relativistic limit of strings on $\mathrm{AdS}_5 \times S^5$
- couple to U(1)-Galilean geometry
- which is a phase space for physical excitations

$$S = -\frac{Q}{2\pi}\int d^2\sigma \left[m_\mu \dot{X}^\mu + \frac{1}{2} h_{\mu\nu} X'^\mu X'^\nu \right]$$

symplectic form $\omega \sim dm$ and Hamiltonian $H \sim \oint d\sigma^1 h_{\mu\nu} X'^\mu X'^\nu$



Upcoming work:

- light-cone quantization
  - symplectic structure from limit of light-cone action
  - anomalies in global symmetry algebra?
- covariant quantization
  - GCA generators from Polyakov action, anomalies?
  - Hamiltonian reduction to $\omega \sim dm$