Non-relativistic corners of $\mathcal{N} = 4$ super Yang-Mills: (1+1)-dimensional theories

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Outline

AdS/CFT correspondence

$\mathcal{N}=4$ SYM with gauge group $SU(N) \leftrightarrow$ type IIB string theory on $AdS_5 \times S^5$

Believed to be true for all couplings [Maldacena, 1997][Gubser, Klebanov, Polyakov , 1998][Witten, 1998]. Great successes:

- Planar limit $N=\infty$ and the power of integrability [Minahan, Zarembo, 2002][Beisert, Kristjansen, Staudacher, 2003]
- Planar limit and the Hagedorn behaviour [Harmark, Wilhelm, 2017-18]
- Supersymmetric localization [Pestun, 2007]

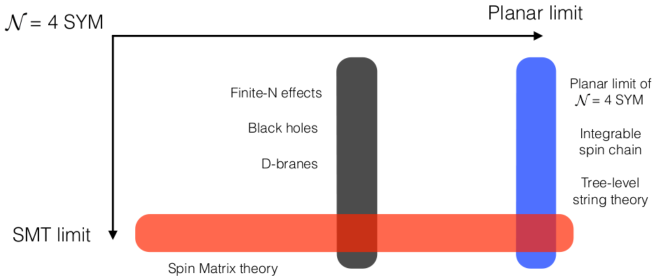
Problem:

- Planar limit: gravity enters as $1/N$ perturbative corrections $\Rightarrow$ No access to black holes and D-branes

Spin Matrix Theory

Controlled finite N effects: Spin Matrix Theory limits [Harmark, Orselli, 2014].

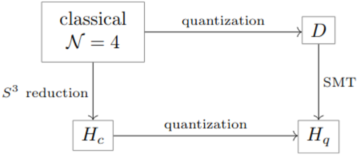
- Decoupling limits of $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3$ $\Rightarrow$ the theory reduces to a subsector with only one-loop contributions of the dilatation operator [Harmark, Orselli, 2006][Harmark, Kristjansson, Orselli, 2006-07]
- Approach zero-temperature critical points in the grand-canonical ensemble
- Approach unitarity (BPS) bounds in the microcanonical ensemble



Spin Matrix Theory is non-relativistic

- Emergent $U(1)$ global symmetry corresponding to mass conservation
- The bulk duals are non-relativistic string theories with non-Lorentzian geometries [Harmark, Hartong, Obers, 2017][Harmark, Hartong, Menculini, Obers, Yan, 2018][Harmark, Hartong, Menculini, Obers, Oling, 2019]
- Insights on holography and quantum gravity from this corner of physics
- New ways to obtain non-relativistic field theories

Scheme of the procedure [Harmark, Wintergerst, 2019]:



Sectors with effective (1+1)-dimensional theories

Focus on BPS bounds

$$H \geq S_1 + \sum_{i=1}^{3} \omega_i Q_i \tag{1}$$

- S_1, S_2 Cartan generators for rotations on S^3
- Q_i Cartan generators of $SU(4)$ R-symmetry group
- ω_i chemical potentials characterizing the bound

Spin Matrix Theory limit

$$\lambda \to 0\,, \qquad \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{\lambda} \text{ finite }, \qquad N \text{ fixed} \tag{2}$$

Sectors	Combination of $SU(4)$ Cartan charges $\sum_{i=1}^{3} \omega_i Q_i$
SU$(1,1)$ bosonic	Q_1
SU$(1,1)$ fermionic	$\frac{2}{3}(Q_1 + Q_2 + Q_3)$
SU$(1,1\|1)$	$Q_1 + \frac{1}{2}(Q_2 + Q_3)$
SU$(1,1\|2)$	$Q_1 + Q_2$

Hamiltonian analysis

Focus on gauge kinetic term + minimal coupling

$$S = \int \sqrt{-g}\, \mathrm{tr}\left(-\frac{1}{4} F_{\mu\nu}^2 + A^\mu j_\mu \right) \tag{3}$$

Canonical momenta:

$$\Pi_0 = \frac{\delta S}{\delta \dot{A}_0} = 0 \,, \qquad \Pi_i = \frac{\delta S}{\delta \dot{A}_i} = F_{0i} \tag{4}$$

We work in Coulomb gauge, imposed via a Lagrange multiplier η. Hamiltonian:

$$H = \int \sqrt{-g}\, \mathrm{tr}\left(\frac{1}{2}\Pi_i^2 + \frac{1}{4} F_{ij}^2 - A_0 (\nabla_i \Pi^i - j_0) \right) - A^i j_i + \eta \nabla_i A^i \,, \tag{5}$$

with constraints

$$\nabla_i \Pi^i - j_0 = 0 \,, \quad \nabla_i A^i = 0 \tag{6}$$

- We treat A_0 as a Lagrange multiplier enforcing Gauss' law: it is no longer a dynamical variable.
- They are two second-class constraints, corresponding to the two unphysical degrees of freedom.

Decomposing into spherical harmonics on S^3 [Ishiki, Takayama, Tsuchiya, 2006] the constraints become

$$2i\sqrt{J(J+1)}\Pi_{(0)}^{Jm\tilde{m}} + j_0^{\dagger\, Jm\tilde{m}} = 0\,, \quad A_{(0)}^{Jm\tilde{m}} = 0\,. \tag{7}$$

We can directly solve the constraints for $A_{(0)}^{Jm\tilde{m}}$ and its simplectic partner $\Pi_{(0)}^{Jm\tilde{m}}$ and plug the solution into the Hamiltonian without changing the Poisson brackets:

$$\begin{aligned} H = \mathrm{tr} \sum_{J,m,\tilde{m}} \Bigg\{ \sum_{\rho=\pm 1} \left(\frac{1}{2} |\Pi_{(\rho)}^{Jm\tilde{m}}|^2 + \frac{1}{2}(2J+2)^2 |A_{(\rho)}^{Jm\tilde{m}}|^2 + A_{(\rho)}^{Jm\tilde{m}} j_{(\rho)}^{\dagger\, Jm\tilde{m}} \right) \\ + \frac{1}{8J(J+1)} |j_0^{Jm\tilde{m}}|^2 \Bigg\}\,. \end{aligned} \tag{8}$$

General procedure:

- Isolate the propagating modes in a given near-BPS limit from the quadratic classical Hamiltonian
- Derive the form of the current which couple to the gauge field from the $\mathcal{N} = 4$ SYM action
- Integrate out additional non-dynamical modes giving rise to effective interactions in a given near-BPS limit
- Derive the interacting Hamiltonian from

$$H_{\text{int}} = \lim_{g \to 0} \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{g^2 N} \tag{9}$$

The $SU(1,1|1)$ sector: field content and constraints

Imposing at the quadratic level $H_0 - S_1 - Q_1 - \frac{1}{2}(Q_2 + Q_3) = 0$ gives a set of conditions.
Relevant constraints

$$A^{Jm\tilde{m}}_{(\rho)} = -\frac{1}{(2J+2)^2 - (m-\tilde{m})^2} j^{Jm\tilde{m}}_{(\rho)} \tag{10}$$

$$\Pi_1^{J,-J,J} + i(2J+1)\Phi_1^{\dagger\, J,-J,J} = 0 \tag{11}$$

$$\psi^{A=1}_{J,-J-\frac{1}{2},J;\kappa=1} \quad \text{unconstrained} \tag{12}$$

- They are non-relativistic constraints (anti-particles decouple)
- Eq. (11) imposes the Dirac brackets $\{\Phi_1^{J,-J,J}, (\Phi_1^{J',-J',J'})^\dagger\} = \frac{i}{2(2J+1)}\delta_{JJ'}$

Conventions (standard normalization of Dirac brackets):

$$\Phi_s \equiv \sqrt{2(s+1)}\,\Phi_1^{\frac{s}{2},-\frac{s}{2},\frac{s}{2}}\,, \qquad \psi_s \equiv \psi^{A=1}_{\frac{s}{2},-\frac{s}{2}-\frac{1}{2},\frac{s}{2};\kappa=1} \tag{13}$$

$SU(1,1|1)$ sector: interacting Hamiltonian

$$H_{\text{int}} = \frac{1}{2N}\sum_{l=1}^{\infty}\frac{1}{l}\,\text{tr}\left((q_l^{\text{tot}})^\dagger q_l^{\text{tot}}\right) + \frac{1}{N}\sum_{s_1,s_2,l=0}^{\infty}\frac{1}{\sqrt{(s_1+l+1)(s_2+l+1)}}\,\text{tr}\left([\Phi_{s_2},\psi^\dagger_{s_2+l}][\psi_{s_1+l},\Phi^\dagger_{s_1}]\right) + \frac{1}{8N}\,\text{tr}\left((q_0^{\text{tot}})^\dagger q_0^{\text{tot}}\right) - \frac{1}{4N}\sum_{s_1=0}^{\infty}\frac{1}{s_1+1}\,\text{tr}\left([\Phi^\dagger_{s_1},\Phi_{s_1}]\tilde{q}_0^{\text{tot}}\right) .$$

where the charge densities are

$$q_l \equiv \sum_{s\geq 0}[\Phi^\dagger_s,\Phi_{s+l}] , \qquad \tilde{q}_l \equiv \sum_{s\geq 0}\frac{\sqrt{s+1}}{\sqrt{s+l+1}}\{\psi^\dagger_s,\psi_{s+l}\} , \qquad q_l^{\text{tot}} \equiv q_l + \tilde{q}_l \quad (14)$$

- Global $U(1)$ symmetry: conservation of particle number
- Commutes with the generators $L_0, L_\pm$ of the $SU(1,1)$ subgroup
- Supersymmetric: can be shown explicitly using a superfield formulation

Quantization of the $SU(1,1|1)$ Hamiltonian

Define quantum operators $a_s \equiv \Phi_s$, $a_s^\dagger \equiv \Phi_s^\dagger, b_s \equiv \psi_s$, $b_s^\dagger \equiv \psi_s^\dagger$ such that

$$[(a_r)^i{}_j, (a_s^\dagger)^k{}_l] = \delta^i{}_l \delta^k{}_j \delta_{rs} \,, \qquad \{(b_r)^i{}_j, (b_s^\dagger)^k{}_l\} = \delta^i{}_l \delta^k{}_j \delta_{rs} \tag{15}$$

Directly promote the classical result to a quantum-mechanical Hamiltonian

$$\begin{aligned} H_{\text{qm}} = \operatorname{tr} & \left(\sum_{s=0}^{\infty} \left(s + \frac{1}{2} \right) a_s^\dagger a_s + \sum_{s=0}^{\infty} (s+1) b_s^\dagger b_s + \frac{g_0}{2N} \sum_{s=1}^{\infty} \frac{1}{s} (q_s^{\text{tot}})^\dagger q_s^{\text{tot}} \right) \\ & + \frac{g_0}{N} \sum_{s,s_1,s_2=0}^{\infty} \frac{1}{\sqrt{(s_1+s+1)(s_2+s+1)}} \operatorname{tr} \left([a_{s_2}, b^\dagger_{s_2+s}][b_{s_1+s}, a^\dagger_{s_1}] \right) \\ & + \frac{g_0}{8N} \operatorname{tr} \left((q_0^{\text{tot}})^\dagger q_0^{\text{tot}} \right) - \frac{g_0}{4N} \sum_{s_1=0}^{\infty} \frac{1}{s_1+1} \operatorname{tr} \left([a^\dagger_{s_1}, a_{s_1}] \tilde{q}_0^{\text{tot}} \right) \,. \end{aligned} \tag{16}$$

Normal ordering gives rise to self-energy corrections

$$\sum_{s=1}^{\infty} \frac{1}{s} \operatorname{tr}\left((q_s^{\text{tot}})^\dagger q_s^{\text{tot}} \right) = \sum_{s=1}^{\infty} \frac{1}{s} \operatorname{tr}\left(: (q_s^{\text{tot}})^\dagger q_s^{\text{tot}} : \right) + 2N \sum_{s=0}^{\infty} h(s) \operatorname{tr}\left(a_s^\dagger a_s \right)$$
$$- 2 \sum_{s=0}^{\infty} h(s) \operatorname{tr}\left(a_s^\dagger \right) \operatorname{tr}(a_s) + 2N \sum_{s=0}^{\infty} h(s+1) \operatorname{tr}\left(b_s^\dagger b_s \right) + 2 \sum_{s=0}^{\infty} h(s+1) \operatorname{tr}\left(b_s^\dagger \right) \operatorname{tr}(b_s)$$

Similar manipulations also work for the terms proportional to the singlet constraint.

Comparison with one-loop dilatation operator

Interacting Hamiltonian:

$$
\begin{aligned}
H_{\text{int}} &= \frac{1}{4N} \sum_{s=0}^{\infty} \sum_{s_1,s_2=0}^{s} \text{tr}\left(: [a^\dagger_{s_1}, a_{s_2}][a^\dagger_{s_1-s}, a_{s_2-s}] : \right) \left(\delta_{s_1,s_2} \left(h(s_1) + h(s_2 - s) \right) - \frac{1-\delta_{s_1,s_2}}{|s_1 - s_2|} \right) \\
&+ \frac{1}{4N} \sum_{s=0}^{\infty} \sum_{s_1,s_2=0}^{s} \sqrt{\frac{(s_1+1)(s_2-s+1)}{(s_2+1)(s_1-s+1)}}\, \text{tr}\left(: \{b^\dagger_{s_1}, b_{s_2}\}\{b^\dagger_{s_1-s}, b_{s_2-s}\} : \right) \\
&\times \left(\delta_{s_1,s_2} \left(h(s_1+1) + h(s_2 - s + 1) \right) - \frac{1-\delta_{s_1,s_2}}{|s_1 - s_2|} \right) \\
&+ \frac{1}{4N} \sum_{s=0}^{\infty} \sum_{s_1,s_2=0}^{s} \sqrt{\frac{s_1 - s + 1}{s_2 - s + 1}}\, \text{tr}\left(: [a^\dagger_{s_1}, a_{s_2}]\{b^\dagger_{s_1-s}, b_{s_2-s}\} : \right) \left(\delta_{s_1,s_2} h(s_1) - \frac{1-\delta_{s_1,s_2}}{|s_1 - s_2|} \right) \\
&+ \frac{1}{4N} \sum_{s=0}^{\infty} \sum_{s_1,s_2=0}^{s} \sqrt{\frac{s_2+1}{s_1+1}}\, \text{tr}\left(: \{b^\dagger_{s_1}, b_{s_2}\}[a^\dagger_{s_1-s}, a_{s_2-s}] : \right) \left(\delta_{s_1,s_2} h(s_1+1) - \frac{1-\delta_{s_1,s_2}}{|s_1 - s_2|} \right) \\
&+ \frac{1}{N} \sum_{s,s_1,s_2=0}^{\infty} \frac{1}{\sqrt{(s_1+s+1)(s_2+s+1)}}\, \text{tr}\left(: [a_{s_2}, b^\dagger_{s_2+s}][b_{s_1+s}, a^\dagger_{s_1}] : \right)
\end{aligned}
$$

The red terms match with the one-loop dilatation operator [Beisert, 2004][Bellucci, Casteill, Morales, 2005]

Semi-local formulation of the $SU(1,1|1)$ sector

Surviving states of the $SU(1,1|1)$ near-BPS limit

$$|d_1^n Z\rangle \quad \text{(spin 1/2 representation of } su(1,1))$$
$$|d_1^n \chi_1\rangle \quad \text{(spin 1 representation of } su(1,1))$$

- One quantum number n $\Rightarrow$ There is a one-dimensional spatial direction
- n is quantized $\Rightarrow$ We put the theory on a circle
- The action of the $su(1,1)$ generators $L_0, L_\pm$ on physical states $|d_1^n Z\rangle, |d_1^n \chi_1\rangle$ induces the representation

$$\Phi(t,x) = \sum_{n=0}^{\infty} \Phi_n(t) e^{i(n+\frac{1}{2})x}, \qquad \psi(t,x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} \psi_n(t) e^{i(n+1)x} \quad (17)$$

Local fields + positivity contraint on the modes.

Non-standard equal-time (anti)commutators

$$[\Phi(t,x),\Phi(t,x')] = 0 \ , \quad [\Phi(t,x), -i\Phi^\dagger(t,x')] = iS_{\frac{1}{2}}(x-x') \tag{18}$$

$$\{\psi(t,x),\psi(t,x')\} = 0 \ , \quad \{\psi(t,x), \partial_{x'}\psi^\dagger(t,x')\} = iS_1(x-x') \tag{19}$$

where

$$S_j(x) = \sum_{n=0}^{\infty} e^{i(n+j)x} \tag{20}$$

Superfield formulation of the $SU(1,1|1)$ sector

Semi-local formulation of the Hamiltonian in the $SU(1,1|1)$ near-BPS limit with a superfield formalism $\Rightarrow$ manifest supersymmetric invariance

- Anti-commutator $\{Q, Q^\dagger\} = -i\partial_x = H_0 \Rightarrow$ superspace contains a single complex Grassmannian variable $(\theta, \theta^\dagger)$
- Representation of supercharges

$$Q = \frac{\partial}{\partial\theta} - \frac{i}{2}\theta^\dagger\partial_x\,, \qquad Q^\dagger = \frac{\partial}{\partial\theta^\dagger} - \frac{i}{2}\theta\partial_x$$

- Representation of supersymmetric covariant derivatives

$$D = i\tfrac{\partial}{\partial\theta} - \tfrac{1}{2}\theta^\dagger\partial_x \qquad D^\dagger = -i\tfrac{\partial}{\partial\theta^\dagger} + \tfrac{1}{2}\theta\partial_x\,,$$
$$\{D, Q\} = \{D^\dagger, Q^\dagger\} = \{D, Q^\dagger\} = \{D^\dagger, Q\} = 0\,, \qquad \{D, D^\dagger\} = -H_0$$

- (Anti)chiral fermionic superfield

$$\Psi(t,x,\theta,\theta^\dagger) = \psi + \theta\Phi + \frac{i}{2}\theta\theta^\dagger\partial_x\psi\,, \quad \Psi^\dagger(t,x,\theta,\theta^\dagger) = \psi^\dagger + \theta^\dagger\Phi^\dagger - \frac{i}{2}\theta\theta^\dagger\partial_x\psi^\dagger$$

satisfying

$$D^\dagger\Psi = 0\,, \qquad D\Psi^\dagger = 0$$

(Anti)chiral bosonic superfield:

$$\mathcal{A}(t,x,\theta,\theta^\dagger) = A(t,x) + \theta\lambda(t,x) + \tfrac{i}{2}\theta\theta^\dagger\partial_x A(t,x)\,,$$
$$\mathcal{A}^\dagger(t,x,\theta,\theta^\dagger) = A^\dagger(t,x) - \theta^\dagger\lambda^\dagger(t,x) - \tfrac{i}{2}\theta\theta^\dagger\partial_x A^\dagger$$

A, λ are residual gauge field and gaugino after quantization in Coulomb gauge.
Action of the $SU(1,1|1)$ sector:

$$S = \int dtdx \int d\theta^\dagger d\theta \ \mathrm{tr}\left(i\Psi^\dagger(\mathcal{D}_0 - \mathcal{D}_x)\Psi + \mathcal{A}^\dagger\mathcal{A}\right) \tag{21}$$

with $\mathcal{D}_0 \equiv \partial_0\,, \mathcal{D}_x \equiv \partial_x - ig_0\mathcal{A} - ig_0\mathcal{A}^\dagger$.
Formulation in components after integrating out the gaugino λ

$$S = \int dtdx \ \mathrm{tr}\left\{\partial_x\psi^\dagger(\partial_0 - \partial_x)\psi - i\Phi^\dagger(\partial_0 - \partial_x)\Phi + iA^\dagger\partial_x A \right.$$
$$\left. + g_0 A j + g_0 A^\dagger j^\dagger + g_0^2[\Phi,\psi^\dagger][\psi,\Phi^\dagger]\right\} \tag{22}$$

where

$$j(t,x) = i\{\partial_x\psi^\dagger,\psi\} + [\Phi^\dagger,\Phi] \tag{23}$$

In momentum space A can also be integrated out, giving the correct SMT Hamiltonian.

Conclusions

- Derivation of novel interacting non-relativistic theories from near-BPS limits of $\mathcal{N} = 4$ SYM
- Equivalence between two procedures: sphere reduction + quantization $\leftrightarrow$ quantum corrections of dilatation operator + SMT limit
- Local interpretation of the effective (1+1)-dimensional field theory with non-standard Dirac brackets
- Superfield formulation for the $SU(1,1|1)$ sector

Future developments

- Investigation of SMT limits in sectors with (2+1)-dimensional effective behaviour (work in progress with T. Harmark, N. Wintergerst)
- Exploration of properties of these new non-relativistic field theories
- Holographic investigations: TNC and SNC strings [Bergshoeff, Gomis, Gürsoy, Harmark, Hartong, Oling, Rosseel, Simsek, Yan, Zinnato, . . .]
- Relation to black holes for the $SU(1,2|3)$ sector [Gutowski, Reall, 2004]

Thank you!