Lectures on "non-lorentzian" G-structures (July 2020)

What are the possible geometries of space & time? An answer to this question was given (subject to some assumptions) by Bacry & Lévy-Leblond, who initiated the classification of kinematical symmetries. Taking their ideas to their logical conclusion, it is possible to give a complete classification of (spatially isotropic, homogeneous) spacetimes in any dimension but up to covering.

One finds that these spacetimes are of one of several types: lorentzian, galilean, carrollian and aristotelian. (The classification also gives some riemannian spaces and in 2d some spacetimes without any discernible structure.)

Being homogeneous, they serve as models for more realistic geometries, in the same way that Minkowski spacetime serves as a model for the lorentzian spacetimes in GR. Technically the realistic spacetimes are Cartan geometries modelled on the kinematical Klein geometries which derive from the pioneering work of Bacry & Lévy-Leblond, Bacry & Nuyts, ending with my paper with Stefan Prohazka.

A natural next step in this direction is to further refine the classification into lorentzian, galilean, carrollian and aristotelian types, by re-interpreting each type in terms of G-structures & then to classify the possible intrinsic torsion of the (adapted) connections.

This may seem strange coming from the direction of GR because in lorentzian geometry (without any additional structure) the intrinsic torsion of a metric connection vanishes, but we will see that for galilean, carrollian, aristotelian (and even Bargmann) G-structures, the intrinsic torsion gives us some information.

For example, the classification of galilean (a.k.a. NC) structures by intrinsic torsion coincides with the classification into torsion-free, twistless torsional and torsional NC geometries.

We will see that there are 4 types of carrollian structures, 16 types of aristotelian structures and 13 types of Bargmann structures which relate in a precise way to galilean structures (via null reduction) and to carrollian structures on a distinguished foliation by null hypersurfaces. Moreover, each type can be characterised geometrically without reference to any connection.

The plan of the lectures is the following:

- Basic notions about G-structures
  - frame bundle
  - soldering form
  - G-structures & associated bundles
- The intrinsic torsion of a G-structure
  - Ehresmann connections
  - Adapted affine connections
  - Intrinsic torsion
- Non-lorentzian G-structures & their intrinsic torsion:
  - The three types of NC geometries
  - The four types of carrollian geometries
  - The sixteen types of aristotelian geometries
  - The thirteen types of Bargmann geometries

1 Basic notions about G-structures

Let M be an n-dimensional smooth manifold and let $p \in M$. By a frame at p we mean a vector space isomorphism $u: \mathbb{R}^n \to T_pM$. Since $\mathbb{R}^n$ has a distinguished basis (the elementary vectors e_i), then its image under u is a basis $(X_1 = u(e_1), X_2, \dots, X_n)$ for T_pM. If u, u' are two frames at p, then $g := u^{-1} \circ u' \in GL(n, \mathbb{R})$. We can write this as $u' = u \circ g$ and in this way we define a right action of $GL(n,\mathbb{R})$ on the collection $F_p(M)$ of frames at p. This action is transitive (any two frames are so related) and free (if $u = u \circ g$ then $g = id$). This makes $F_p(M)$ into a "torsor" or "principal homogeneous space" of $GL(n,\mathbb{R})$. In other words, it is like $GL(n,\mathbb{R})$ forgetting the identity. (cf. affine space vs. vector space)

The collection $F(M) = \bigsqcup_{p\in M} F_p(M)$ can be made into a principal $GL(n,\mathbb{R})$-bundle called the frame bundle of M. In particular, we have a smooth right action of $GL(n,\mathbb{R})$: a diffeomorphism $R_g : F(M) \to F(M)$ associated with every $g \in GL(n,\mathbb{R})$, where $R_g u = u \circ g$ for every frame $u \in F(M)$. Let $\pi : F(M) \to M$ be the smooth map sending $u \in F_p(M)$ to $p \in M$. Then $\pi \circ R_g = \pi$ $\forall g \in GL(n,\mathbb{R})$, since $GL(n,\mathbb{R})$ acts on the frames at p. A local section $s : U \to F(M)$, $U \subset M$, is nothing but a moving frame $s = (X_1, \dots, X_n)$ in U.

Moving frames always exist: if $(U, x^1, \dots, x^n)$ is a local coordinate chart, then $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ is a moving frame in U. If $(V, y^1, \dots, y^n)$ is an overlapping coordinate chart, then in the overlap $U \cap V$ the moving frames $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$ and $\left(\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right)$ are related by a local $GL(n,\mathbb{R})$ transformation $U \cap V \to GL(n,\mathbb{R})$ which is the jacobian matrix of the change of coordinates:

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \qquad \left[\frac{\partial y^j}{\partial x^i}\right] : U \cap V \longrightarrow GL(n,\mathbb{R})$$

It may happen that we can choose "distinguished" moving frames which are related on overlaps by local G transformations, for some subgroup $G \leq GL(n,\mathbb{R})$. For example, we may cover a riemannian manifold (M, g) by open sets on each of which we have an orthonormal moving frame. Orthonormality says that on overlaps they transform by a local $O(n) < GL(n,\mathbb{R})$ transformation.

Continuing with this example...
Let $p \in M$ and define $P_p \subset F_p(M)$ to be the set of ON frames at p. Then $\gamma \in O(n)$ acts on P_p by sending an ON frame $u \in P_p$ to $u' = u \circ \gamma$, which is also an ON frame.
The collection $P = \bigsqcup_{p \in M} P_p$ of ON frames defines a ppal $O(n)$-subbundle of $F(M)$. We call $P \subset F(M)$ an $O(n)$-structure on M.

If M is paracompact (ie: it admits smooth partitions of unity) then M always admits a riemannian metric and hence there is no obstruction to the existence of an $O(n)$-structure. This is in sharp contrast with the case of lorentzian structures (ie: $O(n-1,1)$-structures) which may be topologically obstructed.

Definition 1

> Let $G < GL(n,\mathbb{R})$. A **G-structure** on M^n is a principal G-subbundle $P \subset F(M)$.

As in the example of riemannian geometry, a G-structure on M can be defined in terms of tensor fields on M, but this requires introducing a couple of additional concepts.

Let $P \subset F(M)$ be a G-structure on M. Then $P \to M$ is a principal G-bundle & hence we have the usual notions of Ehresmann connections (see tomorrow's lecture) and associated vector bundles. Let $\rho : G \to GL(\mathbb{V})$ be a representation of ρ, with $\mathbb{V}$ some vector space. Then we can define a (right) action of G on $P \times \mathbb{V}$ by

$$(u, v) \cdot g := (u \circ g, \rho(g^{-1}) \cdot v)$$

This action is free (since G acts freely on P) and the quotient $P \times_G \mathbb{V}$ is the total space of a vector bundle over M with typical fibre $\mathbb{V}$: the associated vector bundle to $(P, \mathbb{V})$. Sections of $P \times_G \mathbb{V}$ may be identified with functions $P \xrightarrow{\sigma} \mathbb{V}$ which are G-equivariant:

$$\Gamma(P \times_G \mathbb{V}) \cong C^\infty_G(P; \mathbb{V}) = \{ \sigma : P \to \mathbb{V} \mid \sigma(u \cdot g) = \rho(g^{-1}) \cdot \sigma(u) \}$$

If $\mathbb{W}$ is another representation, a G-equivariant linear map $\phi : \mathbb{V} \to \mathbb{W}$ defines a bundle map $P \times_G \mathbb{V} \xrightarrow{\Phi} P \times_G \mathbb{W}$. The corresponding map on sections sends $\sigma \in C^\infty_G(P; \mathbb{V})$ to $\phi \circ \sigma \in C^\infty_G(P; \mathbb{W})$.

In our case $P \subset F(M)$ and this gives us an additional structure: an $\mathbb{R}^n$-valued 1-form θ on P. Suppose that $X_u \in T_u P$ for $u \in P_p$. Then $\theta_u(X_u) := u^{-1}(\pi_* X_u)$, where $\pi : P \to M$ with $\pi(u) = p$. In words, $\theta_u(X_u)$ is the coordinate vector of $\pi_* X_u \in T_p M$ relative to the frame $u : \mathbb{R}^n \to T_p M$.
Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then write $\theta = \theta^i e_i$ (Einstein summation convention!). If $s = (X_1, \dots, X_n) : U \to P$ is a local moving frame, $s^* \theta^i \in \Omega^1(U)$ and $(s^*\theta^1, \dots, s^*\theta^n)$ is the dual coframe: $(s^*\theta^i)(X_j) = \delta^i_j$. We call $\theta \in \Omega^1(P; \mathbb{R}^n)$ the soldering form of the G-structure.

The soldering form defines an isomorphism $TM \xrightarrow{\cong} P \times_G \mathbb{R}^n$ and in general allows us to identify tensor bundles over M with the corresponding associated vector bundles to P. We will use this often (and often tacitly) in these lectures.

Let $\rho : G \to GL(\mathbb{V})$ be a representation and let $0 \neq v \in \mathbb{V}$ be G-invariant : $\rho(g) \cdot v = v \quad \forall g \in G$.
Then the constant function $f : P \to \mathbb{V}$, $u \mapsto v$
obeys $f(u \cdot g) = \rho(g^{-1}) \cdot v$ and therefore gives a section of the associated vector bundle $P \times_G \mathbb{V}$ and, via the soldering form, a tensor field on M. For example, let $\mathbb{V} = \odot^2(\mathbb{R}^n)^*$ and $\delta \in \mathbb{V}$ be such that $\delta(e_i, e_j) = \delta_{ij}$. δ is $O(n)$-invariant & indeed $O(n)$ is precisely the subgroup of $GL(n,\mathbb{R})$ leaving δ invariant. If P is an $O(n)$-structure, the constant function $P \to \odot^2(\mathbb{R}^n)^*$, $u \mapsto \delta$ defines a section of $P \times_{O(n)} \odot^2(\mathbb{R}^n)^*$, and via the soldering form a section of $\odot^2 T^*M$, which relative to a local ON frame $s = (X_1, \dots, X_n) : U \to P$ takes the form $\delta_{ij}\, s^*\theta^i\, s^*\theta^j$ <u>ie</u>: the riemannian metric on M which defines the $O(n)$-structure.

In these lectures we shall be interested in several different G-structures besides lorentzian :

- **galilean** $\quad G \cong O(n-1) \ltimes \mathbb{R}^{n-1} < GL(n,\mathbb{R})$ the subgroup fixing $e_n^* \in (\mathbb{R}^n)^*$ and $\sum_{i=1}^{n-1} e_i e_i \in \odot^2 \mathbb{R}^n$.

$$\begin{pmatrix} 0^T & 1 \end{pmatrix} \begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix} = \begin{pmatrix} 0^T & 1 \end{pmatrix} \qquad \begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad G = \left\{ \begin{pmatrix} A & v \\ 0^T & 1 \end{pmatrix} \in GL(n,\mathbb{R}) \;\middle|\; \begin{array}{l} A \in O(n-1) \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$$

- **carrollian** $\quad G \cong O(n-1) \ltimes \mathbb{R}^{n-1} < GL(n,\mathbb{R})$ the subgroup fixing $e_n \in \mathbb{R}^n$ and $\sum_{i=1}^{n-1} e_i^* e_i^* \in \odot^2(\mathbb{R}^n)^*$.

$$\begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & v \\ w^T & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad G = \left\{ \begin{pmatrix} A & 0 \\ w^T & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} A \in O(n-1) \\ w \in \mathbb{R}^{n-1} \end{array} \right\}$$

Notice that $G_{gal} \cong G_{car}$ but they are not conjugate subgroups of $GL(n,\mathbb{R})$. They are related by transposition

- **aristotelian** $G = G_{gal} \cap G_{car}$ since aristotelian spacetimes admit simultaneously a galilean and a carrollian structure

$$\Rightarrow \quad G = \left\{ \begin{pmatrix} A & 0 \\ 0^T & 1 \end{pmatrix} \,\middle|\, A \in O(n-1) \right\} \cong O(n-1) < GL(n,\mathbb{R})$$

- **Bargmann** $G \cong O(n-1) \ltimes \mathbb{R}^{n-1} < O(n,1) < GL(n+1,\mathbb{R})$
  the subgroup of $O(n,1)$ which fixes a null vector.

$$\Rightarrow$$

$$G = \left\{ \begin{pmatrix} 1+\frac{1}{2}x^Tx & x^TA & -\frac{1}{2}x^Tx \\ x & A & -x \\ \frac{1}{2}x^Tx & x^TA & 1-\frac{1}{2}x^Tx \end{pmatrix} \in GL(n+1,\mathbb{R}) \,\middle|\, \begin{array}{l} A \in O(n-1) \\ x \in \mathbb{R}^{n-1} \end{array} \right\} \cong O(n-1) \ltimes \mathbb{R}^{n-1} < O(n,1) < GL(n+1,\mathbb{R})$$

Notice that, abstractly, $G^{(n+1)}_{Barg} \cong G^{(n)}_{gal} \cong G^{(n)}_{car}$ but the geometries are different.

In the next lecture we will introduce connections adapted to a G-structure and discuss its torsion and, in particular, the component of the torsion which doesn't depend on the choice of connection: the "intrinsic" torsion.

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