② Intrinsic torsion of a G-structure

Notation alert : From now on, we will let $V = \mathbb{R}^n$. ie: V is just the name for $\mathbb{R}^n$, not an abstract vector space. Then $GL(V)$ will denote $GL(n,\mathbb{R})$ and $\mathfrak{gl}(V)$ its Lie algebra: the space of $n\times n$ real matrices.

Let $G < GL(V)$ and let $\mathfrak{g} < \mathfrak{gl}(V)$ denote its Lie algebra. Let $P \xrightarrow{\pi} M^n$ be a G-structure and let $\theta \in \Omega^1(P;V)$ be the soldering form.

If $u \in P$ is a frame at $p = \pi(u)$, then $(\pi_*)_u : T_uP \to T_pM$ is a surjective linear map, whose kernel $\mathcal{V}_u := \ker(\pi_*)_u$ is called the vertical subspace of T_uP. The rank theorem says that $\dim \mathcal{V}_u = \dim \mathfrak{g}$. It defines a distribution $\mathcal{V} \subset TP$ which is G-invariant : $\pi \circ R_g = \pi \Rightarrow (R_g)_*$ preserves the kernel of π_*. It is also involutive : the corresponding foliation is the foliation of P by its fibres $\pi^{-1}(p)$.

By an Ehresmann connection on P we mean a G-invariant distribution $\mathcal{H} \subset TP$ complementary to $\mathcal{V}$. At every frame $u \in P_t$, $T_uP = \mathcal{V}_u \oplus \mathcal{H}_u$ and $(\pi_*)_u : \mathcal{H}_u \xrightarrow{\cong} T_pM$. We will denote by $h : T_uP \to \mathcal{H}_u$ the horizontal projector.

One can equivalently define an Ehresmann connection via a connection one-form $\omega \in \Omega^1(P;\mathfrak{g})$ defined uniquely by the properties :

$$\ker \omega_u = \mathcal{H}_u \qquad \text{and} \qquad \omega(\xi_X) = X \quad \forall X \in \mathfrak{g}.$$

where ξ_X is the fundamental vector field on P corresponding to $X \in \mathfrak{g}$. $\left(\xi_X|_u = \frac{d}{dt}\left(u \circ e^{tX}\right)\Big|_{t=0} \right)$

One can show that $R_g^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation.

Recall that if $\rho: G \to GL(\mathbb{V})$ is a representation of G, then $P \times_G \mathbb{V}$ is a vector bundle on M associated to $(P, \mathbb{V})$ whose sections can be identified with G-equivariant functions $\sigma: P \to \mathbb{V}$: $\sigma(u \circ g) = \rho(g^{-1}) \cdot \sigma(u)$.
More generally, we define the basic forms ($\mathbb{V}$-valued)

$$\Omega^p_G(P;\mathbb{V}) := \Big\{ \varphi \in \Omega^p(P;\mathbb{V}) \;\Big|\; \underbrace{R_g^* \varphi = \rho(g^{-1}) \cdot \varphi}_{\text{“invariant”}} \;\&\; \underbrace{h^*\varphi = \varphi}_{\text{“horizontal”}} \Big\}$$

where $(h^*\varphi)(Y_1, \dots, Y_p) := \varphi(hY_1, \dots, hY_p)$, with h the horizontal projector.

The reason why $\varphi \in \Omega^p_G(P;\mathbb{V})$ is called basic, is that it defines a p-form on M with values in the associated bundle $P \times_G \mathbb{V}$. Indeed, as $C^\infty(M)$-modules

$$\Omega^p_G(P;\mathbb{V}) \cong \Omega^p(M; P\times_G \mathbb{V}).$$

An Ehresmann connection on P defines a covariant derivative on sections of any associated vector bundle. The expression is particularly transparent in terms of the functions in $C^\infty_G(P;\mathbb{V})$. If $\sigma: P \to \mathbb{V}$ is equivariant we define $\nabla\sigma := h^* d\sigma$. It follows that

$$\nabla : \Omega^0_G(P;\mathbb{V}) \longrightarrow \Omega^1_G(P;\mathbb{V})$$

or, equivalently, down at the base,

$$\nabla : \Gamma(P\times_G\mathbb{V}) \longrightarrow \Omega^1(M; P\times_G\mathbb{V})$$

A standard exercise in this subject is to show that

$$\nabla\sigma = d\sigma + \rho_*(\omega)\cdot\sigma$$

where $\rho_*: \mathfrak{g} \to \mathfrak{gl}(\mathbb{V})$ is the representation of $\mathfrak{g}$ induced by ρ.

The soldering form $\theta \in \Omega^1(P;V)$ is actually basic:

(horizontal) $$(h^*\theta_u)(X_u) = \theta_u(hX_u) = u^{-1}\pi_*(hX_u)$$
$$= u^{-1}\pi_*(X_u - (X_u)^{vert})$$
$$= u^{-1}\pi_* X_u$$
$$= \theta_u(X_u)$$

(invariant) $$(R_g^*\theta)_u(X_u) = \theta_{u\cdot g}((R_g)_* X_u)$$
$$= (u\cdot g)^{-1}\pi_*(R_g)_* X_u$$
$$= g^{-1}\cdot u^{-1}\pi_* X_u$$
$$= g^{-1}\cdot \theta_u(X_u)$$

Therefore $\theta \in \Omega^1_G(P;V)$ and hence it defines on M a one-form with values in $P\times_G V$; that is, a section of $(P\times_G V)\otimes T^*M = \mathrm{Hom}(TM, P\times_G V)$. In other words, none other than the isomorphism $TM \xrightarrow{\cong} P\times_G V$.

The Koszul connection on $P\times_G V$ induces an affine connection on TM, which is said to be adapted to the G-structure P.

$$\begin{tikzcd} \Gamma(P\times_G V) \arrow[r, "\nabla"] \arrow[d, "\cong"'] & \Omega^1(M; P\times_G V) \arrow[d, "\cong"] \\ \mathfrak{X}(M) \arrow[r, "\nabla"] & \Omega^1(M;TM) \end{tikzcd}$$

Its torsion $T^\nabla \in \Omega^2(M;TM)$, defined by

$$T^\nabla(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \qquad \forall X,Y\in\mathfrak{X}(M),$$

is represented by $\Theta \in \Omega^2_G(P;V)$, defined by $\Theta = h^* d\theta$, or equivalently by the first structure equation

$$\Theta = d\theta + \omega\wedge\theta$$

where the second term in RHS involves also the action of $\omega \in \Omega^1(P;\mathfrak{g})$ on $\theta\in\Omega^1(P;V)$ via the embedding $\mathfrak{g} < \mathfrak{gl}(V)$.
If $X,Y\in\mathfrak{X}(P)$,

$$\Theta(X,Y) = d\theta(X,Y) + \omega(X)\theta(Y) - \omega(Y)\theta(X)$$

We remark that the tensor fields defining the G-structure are covariantly constant with respect to any adapted connection. Indeed, they are represented in P as constant G-equivariant functions $P \xrightarrow{\sigma} \mathbb{V}$, with $\mathbb{V}$ the relevant space of tensors, and hence $d\sigma = 0$ so that in particular $h^*d\sigma = 0$.
This is a way to recognise the adapted connections.

Now let us see how the torsion changes when we change the connection. Let $\mathcal{H}' \subset TP$ be a second Ehresmann connection with connection one-form $\omega' \in \Omega^1(P;\mathfrak{g})$.

Let $\kappa := \omega' - \omega \in \Omega^1(P;\mathfrak{g})$. Since ω, ω' are invariant, so is κ. But now κ is now also horizontal and hence it descends to a one-form with values in $\operatorname{ad} P := P \times_G \mathfrak{g}$.

In general, the difference $\nabla' - \nabla$ between two affine connections belongs to $\Omega^1(M; \operatorname{End} TM)$ but if they are adapted, then $\nabla' - \nabla$ is a 1-form with values in the sub-bundle of $\operatorname{End} TM$ which corresponds to $\operatorname{ad} P$ via the soldering form.

Let Θ' be the torsion 2-form of $\mathcal{H}'$. From the structure equation

$$\Theta' - \Theta = \kappa \wedge \theta \quad \Rightarrow \quad (\Theta' - \Theta)(X,Y) = \kappa(X)\theta(Y) - \kappa(Y)\theta(X)$$

Of course κ, Θ, Θ' are horizontal so this is an equation for sections of the corresponding AVBs:

$$\begin{array}{ccc} \partial : \Omega^1(M; P \times_G \mathfrak{g}) & \longrightarrow & \Omega^2(M; P \times_G V) \\ \| & & \| \\ \Gamma(P \times_G (\mathfrak{g} \otimes V^*)) & & \Gamma(P \times_G (V \otimes \Lambda^2 V^*)) \end{array}$$

induced by the linear map (also denoted ∂):

$$\begin{array}{ccc} \mathfrak{g} \otimes V^* & \xrightarrow{\partial} & V \otimes \Lambda^2 V^* \\ \| & & \| \\ \operatorname{Hom}(V, \mathfrak{g}) & & \operatorname{Hom}(\Lambda^2 V, V) \end{array} \qquad \text{(cf. Spencer)}$$

$\forall v, w \in V \quad (\partial \kappa)(v,w) = \kappa_v w - \kappa_w v \quad$ where $\kappa : V \to \mathfrak{g}$, $v \mapsto \kappa_v$

Proposition 2

Let $P \xrightarrow{\pi} M$ be a G-structure and $\omega \in \Omega^1(P;\mathfrak{g})$ the one-form of an Ehresmann connection with torsion $\Theta \in \Omega^2(P;V)$. If $\omega' = \omega + \kappa$ is another Ehresmann connection, then its torsion $\Theta' = \Theta + \partial\kappa$, where $\partial : \Omega^1_G(P;\mathfrak{g}) \to \Omega^2_G(P;V)$ is induced from the linear map (also called ∂)

$$\partial : \mathrm{Hom}(V,\mathfrak{g}) \longrightarrow \mathrm{Hom}(\Lambda^2 V, V)$$

defined by $(\partial\kappa)(v,w) = \kappa_v(w) - \kappa_w(v) \quad \forall\, v,w \in V$.

Under $\mathrm{Hom}(V,\mathfrak{g}) \cong \mathfrak{g}\otimes V^*$ and $\mathrm{Hom}(\Lambda^2V,V) \cong V\otimes\Lambda^2V^*$, the map ∂ is the composition

$$\mathfrak{g}\otimes V^* \xrightarrow{i\otimes \mathrm{id}_{V^*}} V\otimes V^*\otimes V^* \xrightarrow{\mathrm{id}_V\otimes\wedge} V\otimes\Lambda^2V^*$$

where $i : \mathfrak{g} \to V\otimes V^*$ is the embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V) \cong V\otimes V^*$
and $\wedge : V^*\otimes V^* \to \Lambda^2 V^*$ is skew-symmetrisation.

To any linear map (such as $\partial : \mathfrak{g}\otimes V^* \to V\otimes\Lambda^2V^*$) there is associated an exact sequence:

$$0 \to \ker\partial \to \mathfrak{g}\otimes V^* \xrightarrow{\partial} V\otimes\Lambda^2V^* \to \mathrm{coker}\,\partial \to 0$$

where $\mathrm{coker}\,\partial = V\otimes\Lambda^2V^* / \mathrm{im}\,\partial$, which induces an exact sequence of associated VBs:

$$0 \to \underbrace{P\times_G \ker\partial}_{\text{"contorsions which do not alter the torsion"}} \to \underbrace{P\times_G(\mathfrak{g}\otimes V^*)}_{\text{"contorsion"}} \to \underbrace{P\times_G(V\otimes\Lambda^2V^*)}_{\text{"torsion"}} \to \underbrace{P\times_G \mathrm{coker}\,\partial}_{\text{"intrinsic torsion"}} \to 0$$

We see that $\Theta' - \Theta = \partial(\kappa)$ or suggestively in terms of affine connections: $T^{\nabla'} - T^{\nabla} = \partial(\nabla' - \nabla)$ and hence the image $[T^\nabla] \in \Gamma(P\times_G \mathrm{coker}\,\partial)$ does not depend on the connection: it is called the intrinsic torsion of the G-structure.

The game is then to study coker ∂ as a representation of G and determine all subrepresentations. Each such subrepresentation determines a different type of G-structure which we can characterise in terms of the tensors which determine the G-structure.

As an example, consider lorentzian structures, so $\mathfrak{g} = \underline{so}(V)$, and η the lorentzian inner product on V.

Lemma 3

> The map $\partial : \underline{so}(V) \otimes V^* \longrightarrow V \otimes \Lambda^2 V^*$ is an isomorphism.

Proof Notice that

$$\dim (\mathfrak{g} \otimes V^*) = \dim (V \otimes \Lambda^2 V^*) = \frac{n^2(n-1)}{2},$$

so that the result will follow if we show that $\ker \partial = 0$.

Let $K \in \underline{so}(V) \otimes V^* \cong \mathrm{Hom}(V, \underline{so}(V))$. Then $\partial K \in V \otimes \Lambda^2 V^* \cong \mathrm{Hom}(\Lambda^2 V, V)$ is such that

$$(\partial K)(v,w) = K_v w - K_w v$$

Since $K_v \in \underline{so}(V)$, $\langle K_v w, z \rangle = - \langle w, K_v z \rangle = - \langle K_v z, w \rangle$

If $\partial K = 0$, then also $\langle K_v w, z \rangle = + \langle K_w v, z \rangle$

So letting

$$T(v,w,z) := \langle K_v w, z \rangle$$

$\partial K = 0$ iff

$$T(v,w,z) = - T(v,z,w) = T(w,v,z)$$

But this means $T = 0$ and since $\langle , \rangle$ is non-degenerate, that $K = 0$. ■

It follows that coker $\partial = 0$ and hence any adapted connection (here any metric connection) can be modified to be torsion free. And since $\ker \partial = 0$, there is a unique torsion-free connection. Indeed that's the fundamental theorem of (pseudo) riemannian geometry.

This has the following consequence: if $G < O(n-1,1) < GL(n,\mathbb{R})$ then $\ker \partial = 0$, since ∂ is the restriction to $\mathfrak{g} < \mathfrak{so}(n-1,1)$ of the map in the lemma, which has zero kernel. Therefore the exact sequence becomes short exact:

$$0 \longrightarrow \mathfrak{g}\otimes V^* \xrightarrow{\partial} V\otimes\Lambda^2V^* \longrightarrow \operatorname{coker}\partial \longrightarrow 0$$

and

$$\dim \operatorname{coker}\partial = n\left(\binom{n}{2} - \dim\mathfrak{g}\right).$$

Another example (not relevant to non-Lorentzian geometry) is that of almost symplectic geometry. An almost symplectic manifold (M,ω) consists of a nondegenerate $\omega\in\Omega^2(M)$. This requires $n=\dim M$ to be even. So let $n=2m$. By the symplectic version of Gram-Schmidt, can choose local Darboux frames which are related on overlaps by local $Sp(2m,\mathbb{R}) < GL(2m,\mathbb{R})$ transformations. Again let $V=\mathbb{R}^{2m}$. Then the Lie algebra $\mathfrak{sp}(V)\cong\odot^2V$ and the Spencer sequence is now

$$0\to\odot^3V\longrightarrow\odot^2V\otimes V^*\longrightarrow V\otimes\Lambda^2V^*\longrightarrow\Lambda^3V^*\longrightarrow 0$$

where we have used ω to identify V and V^*. The intrinsic torsion is therefore a 3-form on M: none other than $d\omega\in\Omega^3(M)$. (Exercise!)

As $G=Sp(V)$ representation, $\Lambda^3V^*\cong\Lambda^3_0V\oplus V^*$, where Λ^3_0V are the ω-traceless 3-forms & V^* the ω-trace. This gives 4 types of almost symplectic manifolds:

- symplectic : $d\omega=0$
- locally conformal symplectic : $d\omega=\omega\wedge\varphi$ $\exists\varphi\in\Omega^1(M)$
  ($\Rightarrow d\varphi=0$ so locally $\varphi=df$ & $d(e^{-f}\omega)=0$)
- balanced $d\omega^{m-1}=0$
  (thanks to Robert Bryant via MathOverflow!)
- generic

In Friday's lecture we will calculate coker ∂ for the non-lorentzian G-structures and will therefore divide each class of non-lorentzian manifold into subclasses depending on their intrinsic torsion.

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