(3) Non-lorentzian G-structures & their intrinsic torsion

In contrast to the case of lorentzian structures, the other kinematical G-structures do admit intrinsic torsion and we can use the G-module structure of coker ∂ to distinguish different types and then try to characterise each type in terms of the tensor fields defining the G-structure.

We start with galilean (a.k.a., Newton–Cartan) structures. The reformulation of a galilean structure in terms of a G-structure goes back at least to a 1972 paper of Künzle. He does not consider the intrinsic torsion, but that was a new subject then. Kobayashi's book was published in 1972 and Sternberg's 1964 book does not mention it. They do appear implicitly (but not by name) in Guillemin's 1964 paper on the integrability of G-structures.

Intrinsic torsion of a galilean G-structure

A galilean structure on a manifold M^n can be defined in at least two equivalent ways:

- by specifying $\tau \in \Omega^1(M)$ & $\gamma \in \Gamma(\odot^2 TM)$ with τ nowhere-vanishing, $\ker \gamma^\# = \langle \tau \rangle$ and γ positive-semi-definite.

- by a G-structure $P \xrightarrow{\pi} M$ where $G < GL(V)$ is conjugate to the subgroup $\left\{ \left(\begin{array}{c|c} A & v \\ \hline 0^T & 1 \end{array} \right) \middle| \begin{array}{l} A \in O(n-1) \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$

The Lie algebra $\mathfrak{g}$ of G is conjugate to $\left\{ \left(\begin{array}{c|c} X & v \\ \hline 0^T & 0 \end{array} \right) \middle| \begin{array}{l} X^T = -X \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$

We choose basis P_a, H for V with canonical dual basis π^a, η for V^*. We choose basis $J_{ab} = -J_{ba}$, B_a for $\mathfrak{g}$, where $a, b = 1, \dots, n-1$. The brackets are

$$[J_{ab}, J_{cd}] = \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc}$$
$$[J_{ab}, B_c] = \delta_{bc} B_a - \delta_{ac} B_b$$

The actions of $\mathfrak{g}$ on V and V^* are given by

$$J_{ab} \cdot P_c = \delta_{bc} P_a - \delta_{ac} P_b \qquad J_{ab} \cdot \pi^c = \left(-\delta_b^c \delta_{ad} + \delta_a^c \delta_{bd}\right) \pi^d$$
$$J_{ab} \cdot H = 0 \qquad J_{ab} \cdot \eta = 0$$
$$B_a \cdot P_b = 0 \qquad B_a \cdot \pi^b = -\delta_a^b \eta$$
$$B_a \cdot H = P_a \qquad B_a \cdot \eta = 0$$

We see that neither V nor V^* are irreducible:

$$\langle P_a \rangle \subset V \quad \text{and} \quad \langle \eta \rangle \subset V^* \quad \text{are submodules;}$$

but they are indecomposable, since there are no complementary submodules.

It is straight-forward to determine the map $\partial : \mathfrak{g} \otimes V^* \to V \otimes \Lambda^2 V^*$ and we see that

$$\partial(J_{ab} \otimes \pi^c) = (\delta_{bd} P_a - \delta_{ad} P_b) \otimes \pi^d \wedge \pi^c$$
$$\partial(J_{ab} \otimes \eta) = (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \wedge \eta$$
$$\partial(B_a \otimes \pi^b) = P_a \otimes \eta \wedge \pi^b$$
$$\partial(B_a \otimes \eta) = 0$$

$$\therefore \ \ker \partial = \langle B_a \otimes \eta,\ J_{ab} \otimes \eta + (\delta_{bc} B_a - \delta_{ac} B_b) \otimes \pi^c \rangle$$

$$J_{ab} \cdot (B_c \otimes \eta) = (\delta_{bc} B_a - \delta_{ac} B_b) \otimes \eta$$
$$B_a \cdot (B_b \otimes \eta) = 0$$
$$\begin{aligned} J_{ab} \cdot (J_{cd} \otimes \eta + (\delta_{de} B_c - \delta_{ce} B_d) \otimes \pi^e) &= (\delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc}) \otimes \eta \\ &\quad + \delta_{de}(\delta_{bc} B_a - \delta_{ac} B_b) - \delta_{ce}(\delta_{bd} B_a - \delta_{ad} B_b) \otimes \pi^e \\ &\quad + (\delta_{de} B_c - \delta_{ce} B_d) \otimes (-\delta_a^e \delta_{bf} + \delta_b^e \delta_{af}) \pi^f \end{aligned}$$

$$\therefore \quad J_{ab}\cdot\left(J_{cd}\otimes\eta + (\delta_{de}B_c - \delta_{ce}B_d)\otimes\pi^e\right) = \delta_{bc}\left(J_{ad}\otimes\eta + (\delta_{de}B_a - \delta_{ae}B_b)\otimes\pi^e\right)$$
$$- \delta_{ac}\left(J_{bd}\otimes\eta + (\delta_{de}B_b - \delta_{be}B_d)\otimes\pi^e\right)$$
$$- \delta_{bd}\left(J_{ac}\otimes\eta + (\delta_{ce}B_a - \delta_{ae}B_c)\otimes\pi^e\right)$$
$$+ \delta_{ad}\left(J_{bc}\otimes\eta + (\delta_{ce}B_b - \delta_{be}B_c)\otimes\pi^e\right)$$

$$B_a\cdot\left(J_{cd}\otimes\eta + (\delta_{de}B_c - \delta_{ce}B_d)\otimes\pi^e\right) = (\delta_{ac}B_d - \delta_{ad}B_c)\otimes\eta$$
$$+ (\delta_{de}B_c - \delta_{ce}B_d)\otimes(-\delta_a^e\eta)$$
$$= 2(\delta_{ac}B_d - \delta_{ad}B_c)\otimes\eta$$

Lemma 4

> As $\mathfrak{g}$-modules $\ker\partial \cong \Lambda^2 V^*$.

Proof The $\mathfrak{g}$ action on Λ^2V^* is given by the obvious action of $\underline{so}(n-1)$ and then

$$B_c\cdot\pi^a\wedge\pi^b = -\delta_c^a\,\eta\wedge\pi^b - \delta_c^b\,\pi^a\wedge\eta$$

$$B_c\cdot\pi^a\wedge\eta = 0$$

So define the following linear map $\ker\partial \xrightarrow{\varphi} \Lambda^2V^*$:

$$B_a\otimes\eta \overset{\varphi}{\longmapsto} \delta_{ab}\,\pi^b\wedge\eta$$

$$J_{ab}\otimes\eta + (\delta_{bc}B_a - \delta_{ac}B_b)\otimes\pi^c \overset{\varphi}{\longmapsto} 2\,\delta_{ac}\delta_{bd}\,\pi^c\wedge\pi^d$$

The map φ is clearly $O(n-1)$-equivariant. To check it is $\mathfrak{g}$-equivariant, we show it commutes with the action of B:

$$B_a\cdot\left(J_{cd}\otimes\eta + (\delta_{de}B_c - \delta_{ce}B_d)\otimes\pi^e\right) = 2\,(\delta_{ac}B_d - \delta_{ab}B_c)\otimes\eta$$

$$\begin{array}{ccc} \Big\downarrow \varphi & & \Big\downarrow \varphi \\ & & 2\,(\delta_{ac}\delta_{de} - \delta_{ab}\delta_{ce})\,\pi^e\wedge\eta \end{array}$$

$$B_a\cdot\left(2\,\delta_{ce}\delta_{df}\,\pi^e\wedge\pi^f\right) = 2\,\delta_{ce}\delta_{df}\left(-\delta_a^e\,\eta\wedge\pi^f - \pi^e\wedge\delta_a^f\,\eta\right)$$

$$= 2\left(\delta_{ac}\delta_{de} - \delta_{ad}\delta_{ce}\right)\pi^e\wedge\eta \qquad \blacksquare$$

The cokernel of ∂ is spanned by the image in coker ∂ of

$$\langle H \otimes \pi^a \wedge \pi^b, H \otimes \eta \wedge \pi^a \rangle$$

Lemma 5

> As $\mathfrak{g}$-modules coker $\partial \cong \Lambda^2 V^*$.

Proof Define φ : coker $\partial \longrightarrow \Lambda^2 V^*$ by

$$[H \otimes \pi^a \wedge \pi^b] \longmapsto \pi^a \wedge \pi^b$$
$$[H \otimes \eta \wedge \pi^a] \longmapsto \eta \wedge \pi^a$$

(ie: contracting with η)

It is manifestly $\mathfrak{so}(n-1)$-equivariant. In addition,

$$\begin{aligned} B_c \cdot [H \otimes \pi^a \wedge \pi^b] &= [B_c \cdot H \otimes \pi^a \wedge \pi^b] \\ &= [P_c \otimes \pi^a \wedge \pi^b + H \otimes (-\delta^a_c \eta \wedge \pi^b - \delta^b_c \pi^a \wedge \eta)] \\ &= (-\delta^a_c \delta^b_d + \delta^b_c \delta^a_d)[H \otimes \eta \wedge \pi^d] \\ &\overset{\varphi}{\longmapsto} (-\delta^a_c \delta^b_d + \delta^b_c \delta^a_d)\, \eta \wedge \pi^d \end{aligned}$$

$$[H \otimes \pi^a \wedge \pi^b] \overset{\varphi}{\longmapsto} \pi^a \wedge \pi^b \overset{B_c}{\longmapsto} (-\delta^a_c \delta^b_d + \delta^b_c \delta^a_d)\, \eta \wedge \pi^d . \qquad \blacksquare$$

As G-module, $\Lambda^2 V^*$ is indecomposable, but not irreducible:

$$0 \subset \langle \eta \wedge \pi^a \rangle \subset \Lambda^2 V^* \qquad \text{(filtered module)}$$

$\therefore$ There are three types of galilean structures depending on whether the intrinsic torsion vanishes, lands in the submodule of type $\langle \eta \wedge \pi^a \rangle$ or is generic.

Notice that the sequence

$$0 \longrightarrow \operatorname{im} \partial \longrightarrow V \otimes \Lambda^2 V^* \longrightarrow \operatorname{coker} \partial \longrightarrow 0$$

does not split (as G-modules); although it does split as vector spaces. This means that whereas it is possible to find a vector subspace of $V \otimes \Lambda^2 V^*$ complementary to im ∂, it is not possible to demand in addition that it should be stable under G.

In this example, we have chosen the span of $\langle H\otimes\pi^a\wedge\pi^b, H\otimes\pi^a\wedge\eta\rangle$ as the complement of $\operatorname{im}\partial$ in $V\otimes\Lambda^2V^*$. But this is not preserved under G on the nose, but only modulo $\operatorname{im}\partial$. This has the following geometrical consequence. Having intrinsic torsion in the submodule $\mathcal{E}:=\langle H\otimes\pi^a\wedge\eta\rangle$ does **NOT** mean that there exists an adapted connection ∇ whose torsion T^∇ is a section of $P\times_G\mathcal{E}$. What it **does** mean is that relative to some local moving frame (in P), the torsion 2-form will be represented by a function

$$T^\nabla : U \longrightarrow \mathcal{E}$$

If we change the frame, then since $\mathcal{E}$ is not stable under G this will stop being the case, but one can then modify the connection s.t. relative to the new adapted connection, the torsion 2-form is represented again by a function $T^{\nabla'}: U \longrightarrow \mathcal{E}$. This is why it is important to derive consequences of the fact that the intrinsic torsion lands in $\mathcal{E}$ which are independent of the choice of adapted connection.

This is something one seldom sees in riemannian G-structures, since if $G<O(n)$ then G is reductive and sequences split and modules are fully reducible into irreducibles. This is why the results of this kind are typically simpler to state.

Since $\operatorname{coker}\partial \cong \Lambda^2V^*$, $P\times_G \operatorname{coker}\partial \cong \Lambda^2T^*M$ and therefore the intrinsic torsion is captured by a 2-form. It should not come as a surprise that it is $d\tau$, where $\tau\in\Omega^1(M)$ is the clock one-form.

Let ∇ be an adapted affine connection. Then $\nabla\tau=0$. This says that $\forall X,Y\in\mathfrak{X}(M)$, $(\nabla_X\tau)(Y)=0$, which expands to

$$X\cdot\tau(Y) = \tau(\nabla_X Y).$$

Therefore

$$\begin{aligned} X\cdot\tau(Y) - Y\cdot\tau(X) &= \tau(\nabla_X Y - \nabla_Y X) \\ &= \tau([X,Y] + T^\nabla(X,Y)) \end{aligned}$$

$\therefore\ d\tau(X,Y) = X\cdot\tau(Y) - Y\cdot\tau(X) - \tau([X,Y]) = \tau(T^\nabla(X,Y))$

or $\quad d\tau = \tau\circ T^\nabla \quad$ but τ is represented by the constant function $P \longrightarrow V^*$ sending $u \mapsto \eta$ and the isomorphism coker $\partial \longrightarrow \Lambda^2 V^*$ is precisely contracting with η.

If the intrinsic torsion vanishes, $d\tau = 0$.

If the intrinsic torsion lands in $\langle \eta\wedge\pi^a\rangle$, then $d\tau$ is represented by a G-equivariant function $P \xrightarrow{\sigma} \langle\eta\wedge\pi^a\rangle$ so $\sigma(u) = \sigma_a(u)\,\eta\wedge\pi^a = \eta\wedge\sigma_a(u)\pi^a$ which when viewed as a 2-form on M has the form $\tau\wedge\alpha$ $\exists\,\alpha\in\Omega^1(M)$.

Therefore $\quad d\tau = \tau\wedge\alpha \iff \tau\wedge d\tau = 0$

Finally, the generic case is such that $\tau\wedge d\tau \neq 0$.

Theorem 6

There are three types of galilean structures according their intrinsic torsion:

- $d\tau = 0$ — torsion-free Newton-Cartan geometry
- $\tau\wedge d\tau = 0$ — twistless torsional NC geometries
- $\tau\wedge d\tau \neq 0$ — torsional NC geometries.

Examples: All spatially isotropic homogeneous galilean spacetimes have $d\tau = 0$, but there are homogeneous examples of all three kinds. Null reductions of "supersymmetric spacetimes" $\mathbb{R}\times AdS_3$, $\mathbb{R}\times S^3$ and NW_4 along the Dirac current of a Killing spinor give examples of TTNC, TNC & TNC, respectively.

Intrinsic torsion of a carrollian G-structure

A carrollian structure on a manifold M^n can be defined in at least two equivalent ways:

- by specifying $\xi \in \mathcal{X}(M)$ & $h \in \Gamma(\odot^2 T^*M)$ with ξ nowhere-vanishing, $\ker h^\flat = \langle \xi \rangle$ and h positive-semi-definite.

- by a G-structure $P \xrightarrow{\pi} M$ where $G < GL(V)$ is conjugate to the subgroup $\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline v^T & 1 \end{array} \right) \middle| \begin{array}{l} A \in O(n-1) \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$.

The Lie algebra $\mathfrak{g}$ of G is conjugate to $\left\{ \left(\begin{array}{c|c} X & 0 \\ \hline v^T & 0 \end{array} \right) \middle| \begin{array}{l} X^T = -X \\ v \in \mathbb{R}^{n-1} \end{array} \right\}$

With the same notation for the bases of $\mathfrak{g}$, V and V^*, the only differences with the galilean example is that now

$$B_a \cdot P_b = \delta_{ab} H \qquad B_a \cdot \pi^b = 0$$
$$B_a \cdot H = 0 \qquad B_a \cdot \eta = -\delta_{ab} \pi^b$$

Again, V & V^* are indecomposable but not irreducible

$$0 \subset \langle H \rangle \subset V \qquad 0 \subset \langle \pi^a \rangle \subset V^*$$

Let us introduce the notation W for the $\mathfrak{g}$-module $\langle \pi^a \rangle$. $\quad \langle \pi^a \rangle = \langle H \rangle^\circ$ (annihilator)

The map $\partial : \mathfrak{g} \otimes V^* \longrightarrow V \otimes \Lambda^2 V^*$ is such that

$$\partial (J_{ab} \otimes \eta) = (\delta_{bc} P_a - \delta_{ac} P_b) \otimes \pi^c \wedge \eta$$
$$\partial (J_{ab} \otimes \pi^c) = (\delta_{bd} P_a - \delta_{ad} P_b) \otimes \pi^d \wedge \pi^c$$
$$\partial (B_a \otimes \eta) = \delta_{ab} H \otimes \pi^b \wedge \eta$$
$$\partial (B_a \otimes \pi^b) = \delta_{ac} H \otimes \pi^c \wedge \pi^b$$

Therefore $\ker \partial = \langle (\delta_{bc} B_a + \delta_{ac} B_b) \otimes \pi^c \rangle \cong \odot^2 W$

$$(\delta_{bc} B_a + \delta_{ac} B_b) \otimes \pi^c \longmapsto \delta_{ac} \delta_{bd} \pi^c \pi^d$$

Similarly, $\operatorname{coker} \partial$ is spanned by the image in $\operatorname{coker} \partial$ of

$$\langle (\delta_{bc} P_a + \delta_{ac} P_b) \otimes \eta \wedge \pi^c \rangle$$

Therefore $\operatorname{coker} \partial \cong \odot^2 W$ as well.

The module $\odot^2 W$ breaks up into two submodules:

$\underbrace{\odot^2_0 W}_{\text{symmetric traceless}} \oplus \underbrace{\mathbb{R}}_{\text{trace}}$ and hence so does $\operatorname{coker} \partial$:

$$\operatorname{coker} \partial = \underbrace{\left\langle \left[\left(\delta_{bc} P_a + \delta_{ac} P_b - \frac{2}{n-1} \delta_{ab} P_c \right) \otimes \eta \wedge \pi^c \right] \right\rangle}_{\mathcal{C}_1} \oplus \underbrace{\langle [P_a \otimes \eta \wedge \pi^a] \rangle}_{\mathcal{C}_2}$$

In summary, there are four types of carrollian geometries depending on which submodule of $\operatorname{coker} \partial$ the intrinsic torsion lands: 0, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_1 \oplus \mathcal{C}_2$.

To characterise them geometrically, observe that the intrinsic torsion is captured by the symmetric tensor

$$\Sigma(X,Y) = \underbrace{h}_{\substack{\text{because} \\ \text{of } P\otimes\cdots}} (\underbrace{T^\nabla(\xi, X)}_{\substack{\text{because of} \\ \cdots\otimes\eta\wedge\cdots}}, Y)$$

or equivalently by the h-symmetric endomorphism S of $TM/\langle \xi \rangle$ defined by

$$S([X]) := T^\nabla(\xi, X).$$

Then the four types of carrollian structures correspond to $S = 0$, S scalar, S traceless, S none of the above.

Notice that since ξ is parallel, $T^\nabla(\xi, X) = \nabla_\xi X - [\xi, X]$ and since h is parallel

$$0 = (\nabla_X h)(Y,Z) = X \cdot h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

Put $X = \xi$:
$$\begin{aligned} 0 &= \xi \cdot h(Y,Z) - h(\nabla_\xi Y, Z) - h(Y, \nabla_\xi Z) \\ &= (\mathcal{L}_\xi h)(Y,Z) + h([\xi,Y],Z) + h(Y,[\xi,Z]) \\ &\qquad - h(\nabla_\xi Y, Z) - h(Y, \nabla_\xi Z) \\ &= (\mathcal{L}_\xi h)(Y,Z) - h(S(Y),Z) - h(Y,S(Z)) \end{aligned}$$

$$\therefore \quad \mathcal{L}_\xi h = 2\Sigma$$

If $S=0$, $\mathcal{L}_\xi h = 0$
If S is scalar, $\mathcal{L}_\xi h = f h \quad \exists f \in C^\infty(M)$
If S is h-traceless, then $\mathcal{L}_\xi \mu = 0$, where μ is the (perhaps only locally defined) volume form on M corresponding to the g-invariant tensor

$$\eta \wedge \pi^1 \wedge \cdots \wedge \pi^{n-1} \in \Lambda^n V^*$$

This is only G_0-invariant, where G_0 is the identity component of G. If the G-structure can be reduced to G_0 (e.g., M orientable or simply-connected), then $\mu \in \Omega^n(M)$ exists. Otherwise only locally, but in any case the condition $\mathcal{L}_\xi \mu = 0$ makes sense.
In summary,

Theorem 7

> There are 4 types of carrollian G-structures according to their intrinsic torsion:
>
> - $\mathcal{L}_\xi h = 0$ — ξ is h-Killing
> - $\mathcal{L}_\xi h = f h \quad (\exists 0 \neq f \in C^\infty(M))$ — ξ is h-conformal Killing
> - $\mathcal{L}_\xi \mu = 0$ — ξ is volume-preserving
> - none of the above — ξ is none of the above

Examples: symmetric carrollian spaces (C, dSC, $AdSC$) have $\mathcal{L}_\xi h = 0$
The carrollian lightcone has $\mathcal{L}_\xi h = 2h$.
I know no explicit homogeneous examples of the last two types.

Intrinsic torsion of an aristotelian structure

Rather than bore you with the detailed calculations, let me remark that here $G = O(n-1) < GL(V)$, and hence it sits inside $O(V)$ for some lorentzian or euclidean inner product on V. And hence by Lemma 3 $\ker \partial = 0$. Since G is reductive, V, V^* & $\operatorname{coker} \partial$ are fully reducible into irreducibles:

$$V = W \oplus \langle H \rangle \qquad V^* = W^* \oplus \langle \eta \rangle$$

where $W = \langle P_a \rangle$ & $W^* = \langle \pi^a \rangle$.

Then $\operatorname{coker} \partial \cong \mathbb{R} \oplus \odot^2_0 W \oplus \Lambda^2 W \oplus W$

and hence there are $2^4 = 16$ aristotelian geometries. To characterise them, we may use what we learnt from the galilean and carrollian cases:

$$\mathcal{L}_\xi h = \Sigma \qquad \mathcal{L}_\xi \mu = \operatorname{tr}(S) \mu \qquad d\tau = \tau \circ T^\nabla$$

where $S(X) = T^\nabla(\xi, X)$ and $\Sigma(X,Y) = h(S(X), Y)$

but we now also have an additional ingredient:

$$\begin{aligned} \mathcal{L}_\xi \tau &= \iota_\xi d\tau + d \overbrace{\iota_\xi \tau}^{1} \\ &= \iota_\xi (\tau \circ T^\nabla) \\ &= \tau \circ S \end{aligned}$$

$\Rightarrow$ if $d\tau = 0$ or $d\tau \wedge \tau = 0$, then $\mathcal{L}_\xi \tau = 0$ automatically.

if $d\tau \wedge \tau \neq 0$ then $\mathcal{L}_\xi \tau$ is not constrained.

For each of the four cases

$d\tau = 0$		we have	$\mathcal{L}_\xi h = 0$
$d\tau \wedge \tau = 0$			$\mathcal{L}_\xi h = f h$
$d\tau \wedge \tau \neq 0$	$\mathcal{L}_\xi \tau = 0$		$\mathcal{L}_\xi \mu = 0$
$d\tau \wedge \tau \neq 0$	$\mathcal{L}_\xi \tau \neq 0$		none of the above

giving a total of 16 aristotelian geometries.

Summary: Intrinsic torsion allows us to refine the classification of space-time geometries, while still keeping a sense of 'genericity' and a manageable small list:

3 types of galilean geometries

4 types of carrollian geometries

16 types of aristotelian geometries

which can be characterised in terms of the tensors defining the G-structure.

Contrary to what I claimed in the first lecture, the Bargmann case is substantially more complicated and I do not have a nice succinct statement in that case. That will have to wait.

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