FRACTONS, MULTIPOLE MOMENTS AND TENSOR GAUGE THEORIES

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Conservation of multipole moments

We would like to study theories where moments of charges are conserved. (I will spare you the CMT/QI motivation)

Say, we have a conserved charge (arbitrary dimension)

$$\dot{\rho} + \partial_i j^i = 0 \qquad\qquad \partial_0 Q = \partial_0 \int \rho = 0$$

If the current can be written as $j^i = \partial_j J^{ij}$ where J^{ij} is a local operator, **then** the dipole moment of ρ is conserved

$$\partial_0 D_i = \partial_0 \int x_i \rho = 0$$

Furthermore, **if** J^{ij} is traceless, **then** the trace of quadrupole moment is conserved

$$\partial_0 \mathrm{Tr}(Q_{ij}) = \partial_0 \int |x|^2 \rho = 0$$

Many references. CMT: Pretko

See also: Griffin Grosvenor Horava and Yan; Hinterbichler and Joyce

Example

Do such theories actually exist? If so, what kind of symmetry implies these conservation laws?

$$\mathcal{L} = \dot{\Phi}^\star \dot{\Phi} - m|\Phi|^2 - \lambda(\partial_i \Phi \partial_j \Phi - \Phi \partial_i \partial_j \Phi)(\partial_i \Phi^\star \partial_j \Phi^\star - \Phi^\star \partial_i \partial_j \Phi^\star)$$

Conserves **U(1)** charge and the dipole moment! Symmetry

$$\Phi' = e^{i(\alpha + \beta_i x_i)} \Phi$$

Forms a non-trivial algebra with the spatial symmetries! **Multipole algebra.**

$$[T_i, D_j] = Q\delta_{ij} \qquad [R_{ij}, D_k] = \delta_{ik} D_j - \delta_{jk} D_i$$

Definition of symmetries is not covariant. In curved space dipole is not conserved. Covariant formulation is not known.

Writing $\Phi = \sqrt{\rho} e^{i\theta}$ we get non-relativistic Goldstone theories, with polynomial shift symmetries

$$\mathcal{L} = \dot{\theta}\dot{\theta} - \lambda \partial_i \partial_j \theta \partial_i \partial_j \theta \qquad \delta\theta = \alpha + \beta_i x_i$$

Pretko

Generalizations

The story can be generalized very far

- Conservation of any set of multipole moments, while preserving rotational and translational symmetries
- Breaking down the rotational symmetry down to crystalline point group symmetry
- Breaking down some of the translation symmetries
- Including scale symmetries
- Allowing the charge density ρ_i to be a vector, rather than scalar. This involves having vector fields Φ_i
- Breaking the charge conservation from Z to Z_n. This leads to very non-trivial theories
- Combining the vector symmetries with 1-form symmetries
- Enhancing to subsystem symmetries

$$\mathcal{L} = \dot{\theta}\dot{\theta} + (\partial_x \partial_y \theta)^2 \qquad\qquad \delta\theta = f(x) + g(y)$$

- Probably many other things we have not yet considered

Gauging

This global symmetry can be gauged, leading to a **higher rank**, or **multipole** gauge theory.

$$\mathcal{L} = (\dot{\Phi}^\star + iA_0)(\dot{\Phi} - iA_0) - m|\Phi|^2 - \lambda(\partial_i\Phi\partial_j\Phi - \Phi\partial_i\partial_j\Phi - A_{ij})(\partial_i\Phi^\star\partial_j\Phi^\star - \Phi^\star\partial_i\partial_j\Phi^\star + iA_{ij})$$

$$\delta A_{ij} = \partial_i\partial_j\alpha \qquad \delta A_0 = \dot{\alpha}$$

The Lagrangian for the gauge field is a generalized Maxwell (in 3D)

$$\mathcal{L} = E_{ij}E_{ij} + B_{ij}B_{ij}$$

$$\partial_i\partial_j E_{ij} = \rho$$

$$E_{ij} = \partial_i\partial_j A_0 - \dot{A}_{ij} \qquad B_{ij} = \epsilon_{ikl}\partial_k A_{lj}$$

The gauge theory cannot be defined on arbitrary manifold: gauge symmetry breaks down when there is curvature.
can be defined on an Einstein manifold in 2D and 3D.

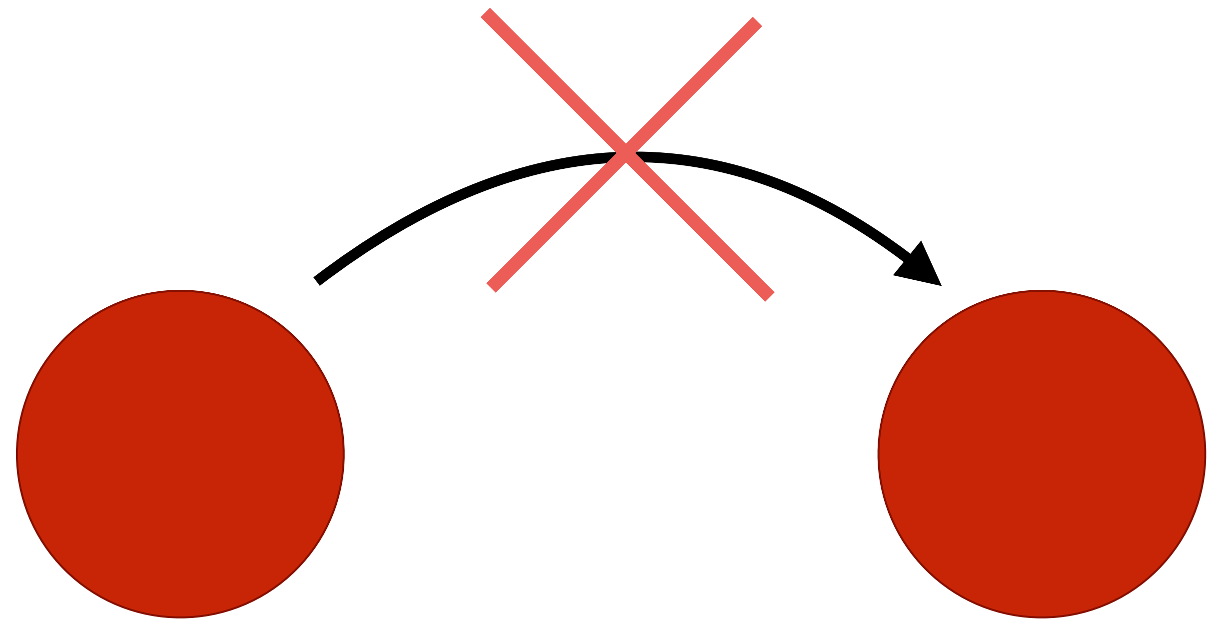
A variant of this theory, vector charge theory, is close to non-relativistic gravity.

$$\delta A_{ij} = \partial_i\alpha_j + \partial_j\alpha_i \qquad A_{0i} = \partial_0\alpha_i$$

$$E_{ij} = \partial_i A_{0j} + \partial_j A_{0i} - \dot{A}_{ij} \qquad B_{ij} = \epsilon_{ikl}\epsilon_{jrs}\partial_k\partial_r A_{ls}$$

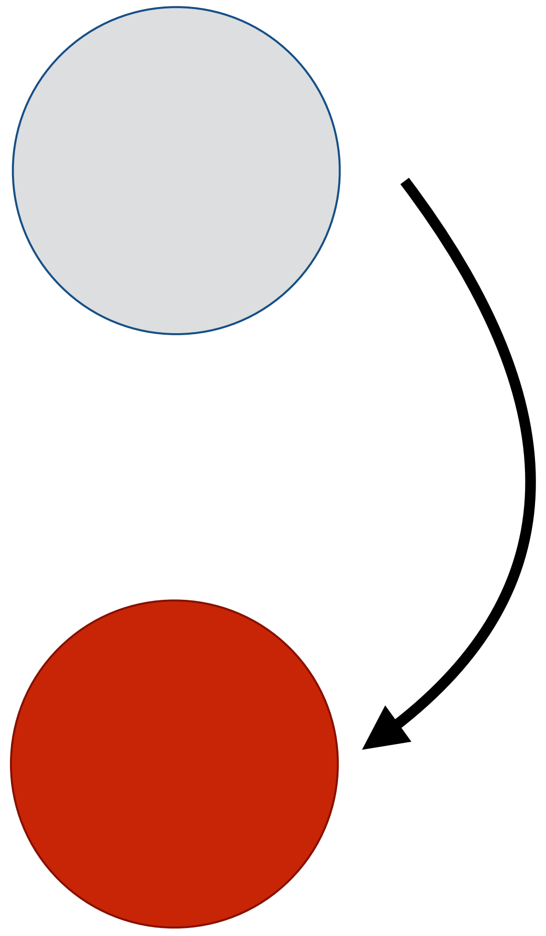
Who cares?

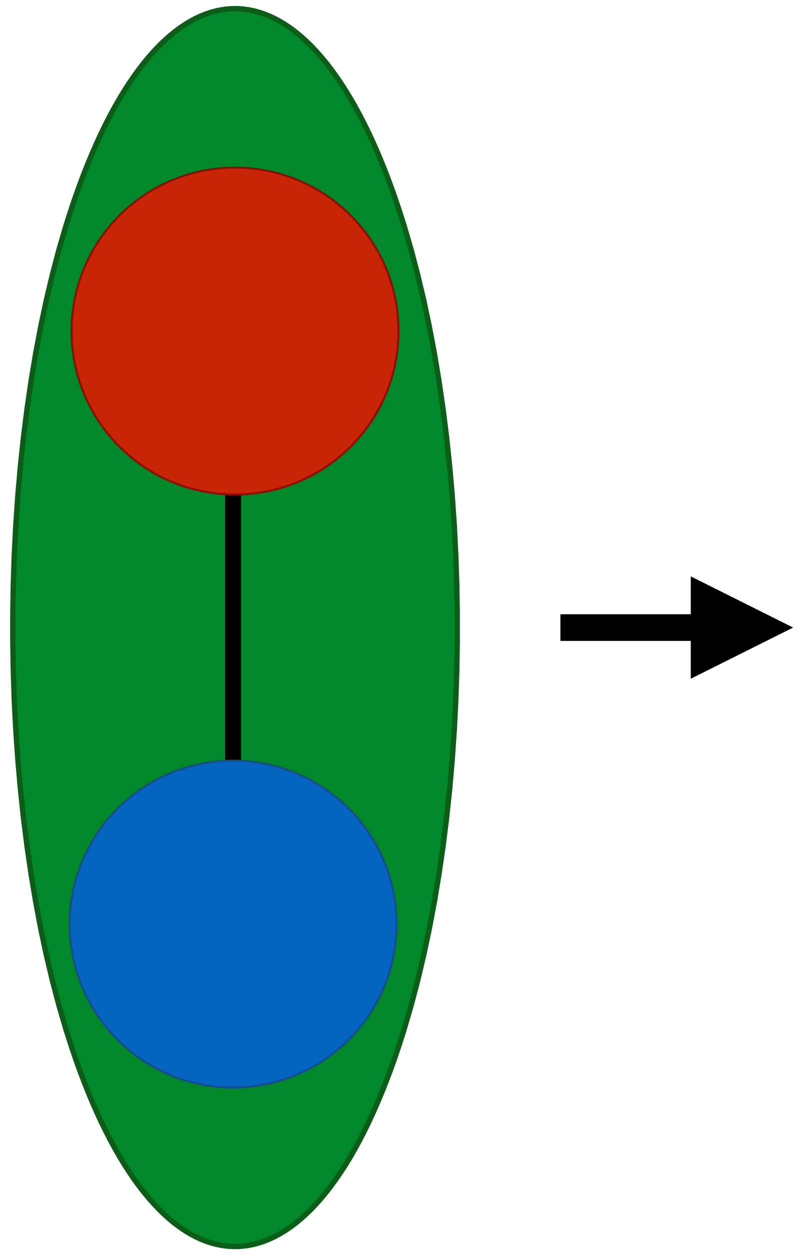
An isolated excitation cannot move if dipole is conserved!



Particles can hop by emitting dipoles. Depending on a theory

dipoles themselves are either mobile or must move perpendicular to their dipole moment or immobile.





In 3D this phenomenology appears in models of self-correcting quantum memory

Cute example: traceless scalar charge theory

Spatial symmetries

$$[T_i, T_j] = 0 \qquad [R_{ij}, T_k] = \delta_{k[i}T_{j]} \qquad [R_{ij}, R_{kl}] = \delta_{[k[i}R_{j]l]}$$

Dipole symmetries

$$[D_i, D_j] = 0 \qquad [T_i, D_j] = \delta_{ij}Q \qquad [R_{ij}, D_k] = \delta_{k[i}D_{j]}$$

$$[\Delta, T_i] = D_i$$

Δ corresponds to the trace of quadrupole moment

Upon identification

$D_i =$ generator of spatial translations

$T_i =$ generator of Galilean boosts

$\Delta =$ generator of time translations

$Q =$ generator of mass conservation

Becomes Bargmann algebra

Hydrodynamics with dipole symmetries

We do not know how to develop a theory with conservation of momentum **and** dipole moment.

We can study diffusion with conserved multipole moments. Continuity equation

$$\dot{\rho} + \partial_i \partial_j J^{ij} = 0$$

Is supplemented with constitutive relation

$$J_{ij} \propto D \partial_i \partial_j \rho$$

Leading to sub-diffusive behaviour

$$\dot{\rho} = -D \partial^4 \rho$$

We have looked at more complicated theories in the paper.

AG, Lucas, Nandkishore 2020. Many other papers by now.

Non-commutative classical mechanics

Consider a Hamiltonian system. This describes vortices and plasma in strong magnetic field in 2D. $z_\alpha(t)$ are complex positions

$$H = -2\pi \sum_{\alpha<\beta} \gamma_\alpha \gamma_\beta \ln |z_\alpha - z_\beta| \qquad \{z_\alpha, \bar{z}_\beta\} = i(\pi\gamma_\alpha)^{-1}\delta_{\alpha\beta}$$

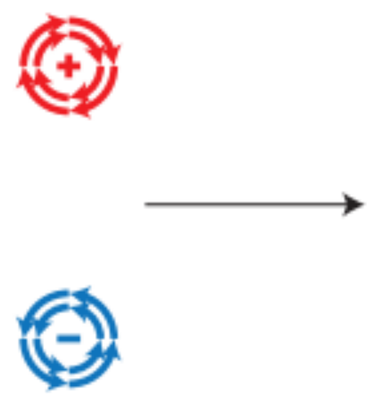
Hamiltonian is translation and rotation invariant. However the momentum and angular momentum are given by (in real coordinates)

$$P_i = -2\pi\epsilon_{ij} \sum_\alpha \gamma_\alpha x_j^\alpha \qquad L = 2\pi \sum_\alpha \gamma_\alpha x_i^\alpha x_j^\alpha \delta_{ij}$$

That is they are proportional to dipole and trace of quadrupole moment

$$P_i = -\epsilon_{ij} D_j \qquad L = \delta_{ij} Q_{ij}$$

So dipole and (trace of) quadrupole are conserved. This show in dynamics of vortices. Isolated vortex does not move, while the vortex dipole moves perpendicular to the dipole moment



Next level: Haah's code

Long held beliefs in CM:

- Topological order is described by TQFTs
- Phases of matter in TD limit are described by field theories.

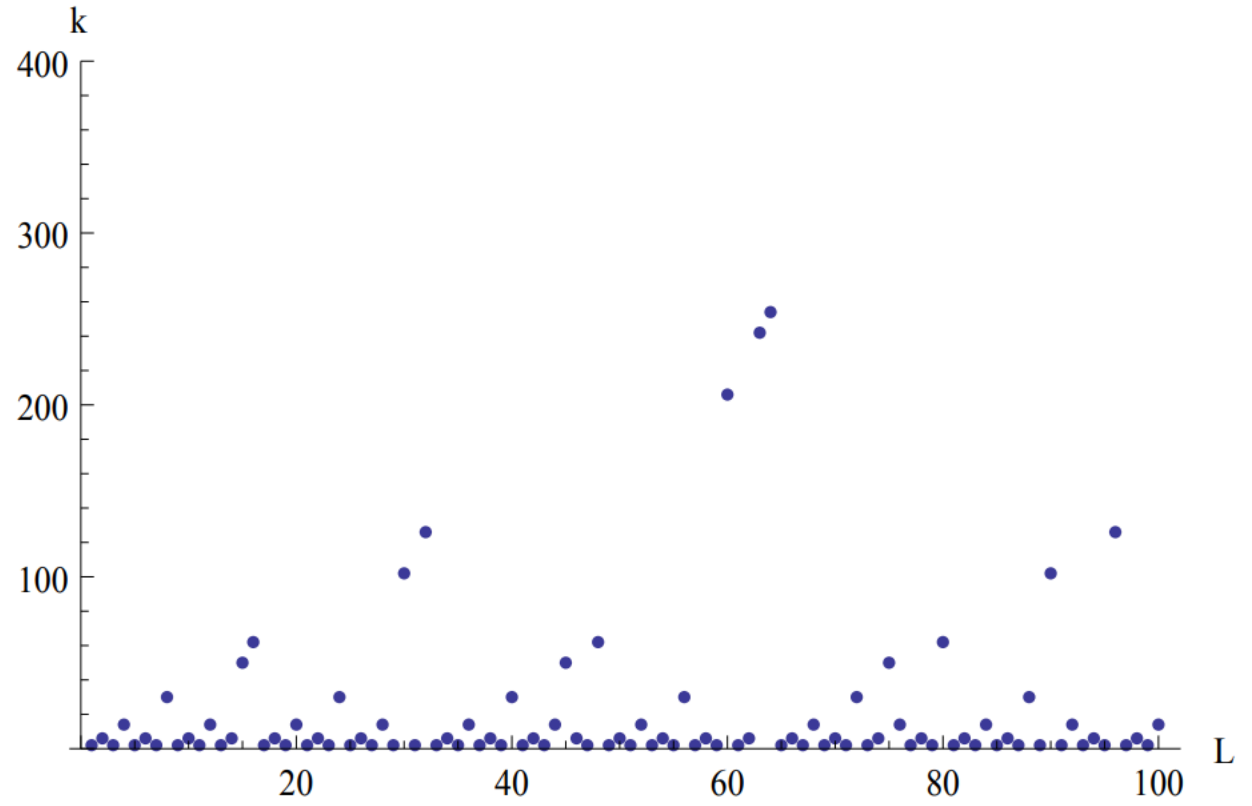
Haah code makes this belief far from obvious Haah's code is the first discovered fracton model. **All** excitations are immobile

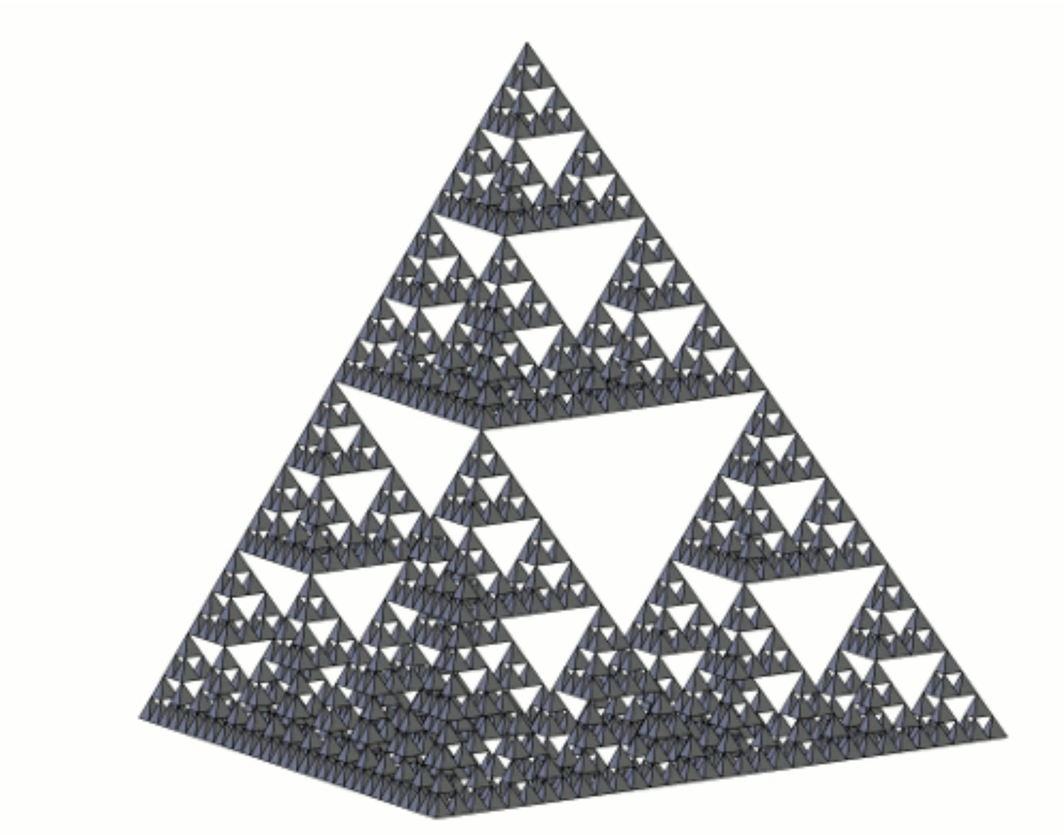
Excitations are $\mathbb{Z}_2$ charges created in quadruples at corners of a pyramid. Haah's model is topologically ordered, however it does not appear to admit a description in terms of a TQFT

Topological order is often quantified by degeneracy without symmetry. Haah's code has such degeneracy. It equals 2^k, where k is given by

$$\frac{k+2}{4} = \begin{cases} 1 & \textit{if } L = 2^p + 1, \\ L & \textit{if } L = 2^p, \\ L-2 & \textit{if } L = 4^p - 1, \\ 1 & \textit{if } L = 2^{2p+1} - 1. \end{cases}$$

L is a system size, and $p \in \mathbb{Z}$





GS degeneracy is determined by how many Sierpinski pyramid operators can be inscribed into **L** x **L** x **L** torus

Haah 2011

Fractons emerge in surprisingly many sub-fields