Enstrophy from symmetry

Natalia Pinzani Fokeeva - KU Leuven

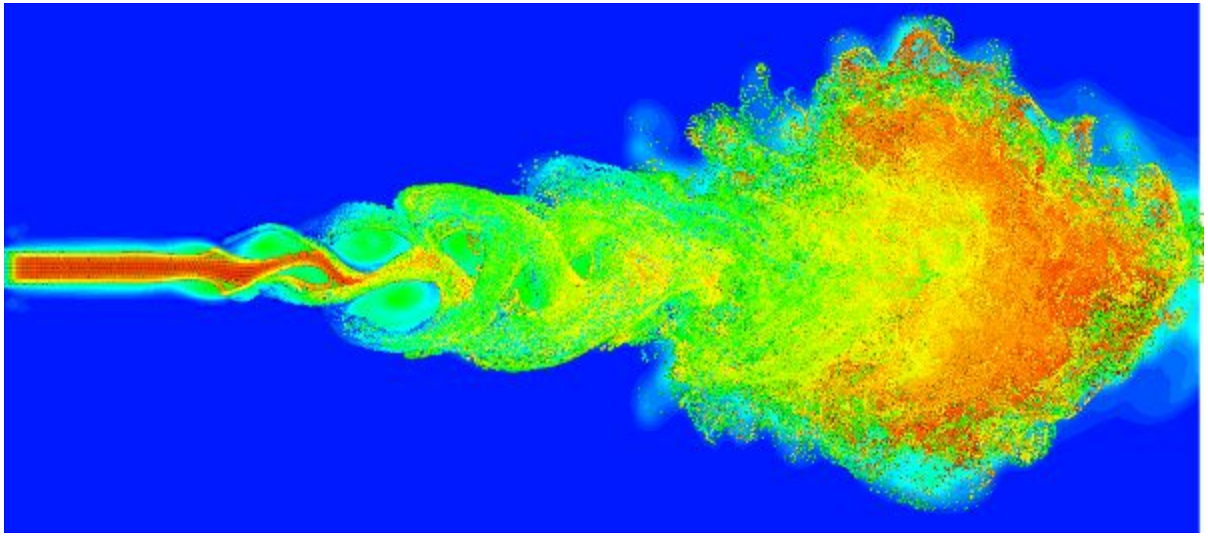
Non Lorentzian online zoom meeting - November 2020

with R. Marjieh, and A. Yarom [hep-th: 2009.03980]

+ w.i.p. with A. Yarom [hep-th: 2011.xxxxx]

Motivation

- 3+1 turbulence ⟶ **direct energy cascade** (from big to small scales)



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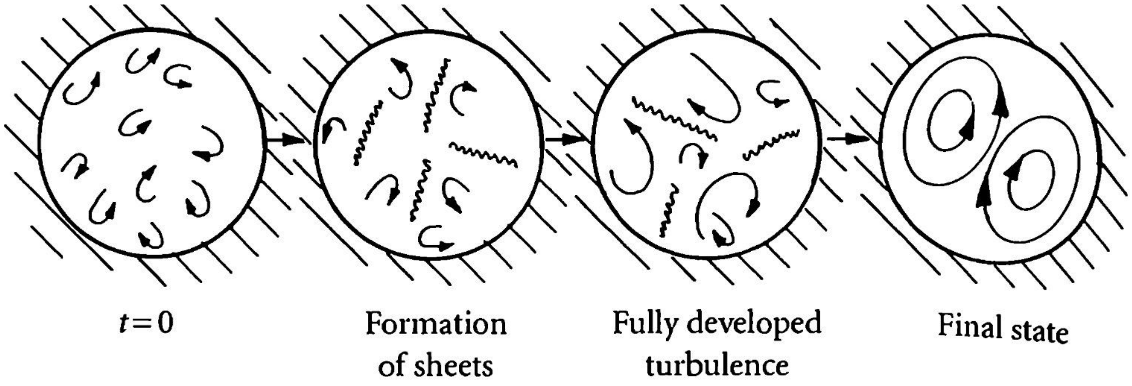
- 2+1 turbulence → **inverse energy cascade** (from small to big scales)

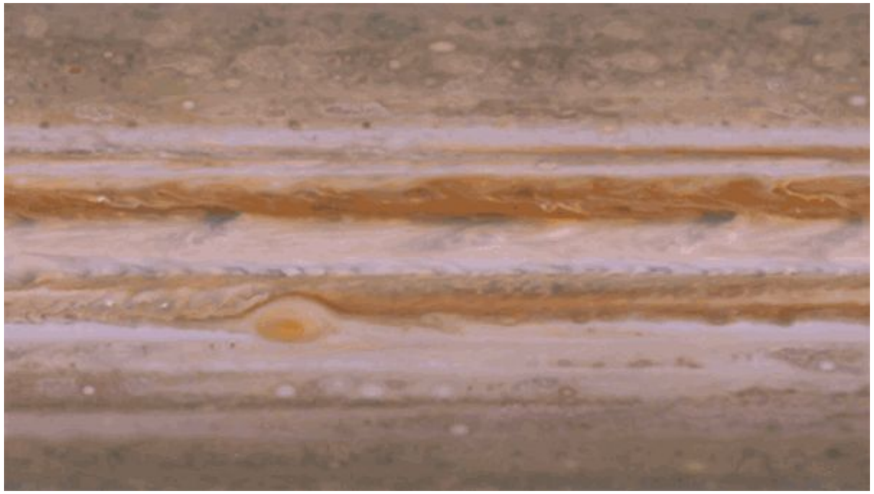


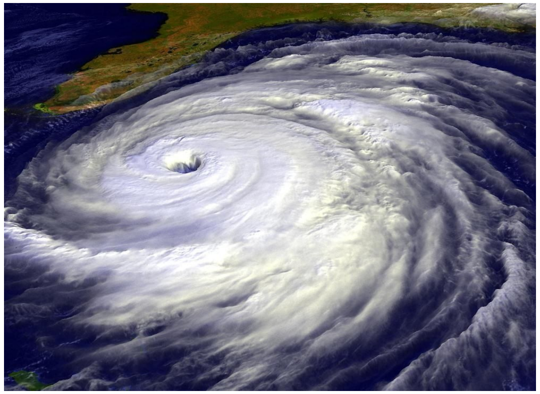
Motivation

- 2+1 turbulence → **inverse energy cascade** (from small to big scales)

- Large scale atmospheric flow

- In laboratory, a strong magnetic field or an intense rotation tend to suppress one component of motion





Motivation

- 2+1 turbulence → **inverse energy cascade**
- In **non relativistic** fluid flows, its origin can be traced back to the existence of an **approximately conserved enstrophy charge:**

$$\Omega = \int \omega^{ij}\omega_{ij} d^d x$$

$$\partial_t \Omega = \int \omega^{ij}\sigma^k_j \omega_{ki} \, d^d x - \frac{1}{R} P$$

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vorticity: $\omega_{ij} = \partial_i v_j - \partial_j v_i$

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shear:

$$\sigma_{ij} = \partial_i v_j + \partial_j v_i$$

Palinstrophy:

$$P = \int \partial_k \omega^{ij}\, \partial^k \omega_{ij}\, d^d x$$

Vortex-stretching term

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In d=2 spatial dimensions the Vortex-stretching term is vanishing

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$$\Omega = \int \omega^{ij}\omega_{ij} d^d x$$

$$\partial_t \Omega = -\frac{1}{R} P$$ in d=2

Reynolds number

→ $$\partial_t \Omega = 0$$ when $$R \to \infty$$

[Kraichnan 1967, Leith 1968, Batchelor 1969]

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[Kraichnan 1967, Leith 1968, Batchelor 1969]

- **Relativistic** generalization to uncharged, conformal fluid flows

[F. Carrasco, L. Lehner , Robert C. Myers, O. Reula, A. Singh 1210.6702]

1. Enstrophy conservation for fluids with other degrees of symmetry?

2. Can it be derived from a symmetry principle?

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- YES:

  [R. Marjieh, N.P.F., and A. Yarom [2009.03980]]

  - For generic **relativistic fluids** (relevant for heavy-ion collisions)
  - Covariant formulation for **Galilean fluids**

  [N.P.F., and A. Yarom [to appear]]

  - For **non-boost invariant fluids** (relevant for flocking behavior)
  - For **Carrollian fluids**

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An algorithm for enstrophy conservation:

Consider **relativistic fluids**

- If there exists a **closed two-form** $\Omega_{\mu\nu}$: $d\Omega = 0$
- that is **orthogonal to the velocity field** under the equations of motion: $\Omega_{\mu\nu}u^{\nu} = 0$
- and there exists $S^{\mu} = su^{\mu}$ that is conserved $\nabla_{\mu}S^{\mu} = 0$

→ The current

$$J^{\mu} = \frac{1}{s}\Omega^2 u^{\mu}$$

$$\Omega^2 = \Omega^{\alpha\beta}\Omega_{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}\Omega_{\mu\nu}\Omega_{\alpha\beta}$$

is conserved under the e.o.m. in 2+1 dimensions

$$\nabla_{\mu}J^{\mu} = -\frac{1}{s^2}\nabla_{\mu}(su^{\mu})\Omega^2 + \frac{2}{s}\Omega^{\alpha\beta}\nabla_{\mu}(u^{\mu}\Omega_{\alpha\beta})$$

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$$\mathcal{L}_u\Omega = u \cdot d\Omega + d(\Omega \cdot u) = 0$$

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$$\nabla_\mu J^\mu = -\frac{1}{s^2} \nabla_\mu (s u^\mu) \Omega^2 + \frac{2}{s} \Omega^{\mu\nu} \sigma^\alpha_\mu \Omega_{\alpha\nu} + \frac{2}{s} \left(1 - \frac{2}{d}\right) (\nabla_\mu u^\mu) \Omega^2$$

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$$\nabla_\mu J^\mu = 0$$

Enstrophy current conservation

Finding $\Omega_{\mu\nu}$:

- The **most general closed 2-form**:

$$\Omega_{\mu\nu} = \partial_\mu(Tf(T,\mu/T)u_\nu) - \partial_\nu(Tf(T,\mu/T)u_\mu) + cF_{\mu\nu}$$

- It must satisfy under the equations of motion: $\Omega_{\mu\nu}u^\nu = 0$

$$\Omega_{\alpha\beta}u^\beta = \left(f\,\frac{\rho T}{P+\epsilon} - \frac{\partial f}{\partial(\mu/T)}\right) T D^\perp_\alpha(\mu/T) - T\frac{\partial f}{\partial T} D^\perp_\alpha T - \left(f\,\frac{\rho T}{P+\epsilon} - c\right) F_{\alpha\beta}u^\beta$$

$$D^\perp_\alpha = (\delta^\mu_\alpha + u^\mu u_\alpha)\partial_\mu$$

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$$\frac{\partial f}{\partial T} = 0$$

In the absence of external sources: $P(T,\mu) = p(Tf(\mu/T))$

Otherwise: $f = c_0 + c\,\mu/T$

Enstrophy current for Galilean fluids:

Consider **Galilean fluids**

- Construct a two-form:
  - **closed**
  - **orthogonal to the velocity field** under the equations of motion
  - **covariant under Galilean boosts**
- The current

$$J^\mu_G = \frac{1}{s}\Omega^2 u^\mu_G$$

is conserved under the e.o.m. in 2+1 dimensions

where $\Omega^2 = \Omega^{\mu\nu}\Omega_{\mu\nu} = h^{\mu\alpha}h^{\nu\beta}\Omega_{\mu\nu}\Omega_{\alpha\beta}$

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Galilean covariant velocity field,

$u_G^\mu = (1, \vec{v})$

Galilean invariant metric, $h^{\mu\nu}n_\nu = 0$

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[K. Jensen [1408.6855, 1411.7024], J. Hartong, N. Obers [1504.07461]]

→ $$(\tilde{\nabla}_\mu - \tilde{\mathcal{G}}_\mu)J_G^\mu = 0$$

* Be careful about boundary terms
* Be careful about torsion
* Use a Galilean invariant connection

$\Omega_{\mu\nu}$ for Galilean fluids:

- The Newton-Cartan data: [we use K. Jensen [1408.6855]]

$$h^{\mu\nu}\,, \qquad n_\mu\,, \qquad \bar{n}^\mu\,, \qquad A_\mu$$

- Fluid dynamics in Newton-Cartan geometry:

$$u_G^\mu = (1, \vec{v})$$

- We ensure Galilean covariance via Milne boosts symmetry
- The **most general closed 2-form** that is **Milne and U(1) gauge invariant:**

$$\Omega_{\mu\nu} = \tilde{F}_{\mu\nu} + \partial_\mu(g n_\nu) - \partial_\nu(g n_\mu)$$

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$$\bar{h}_{\mu\nu}\bar{n}^\nu = 0$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$$

$$\tilde{A}_\mu = A_\mu + \bar{h}_{\mu\alpha} u_G^\alpha - \frac{1}{2} n_\mu v^2$$

Milne invariant

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$$\Omega_{\mu\nu} = \tilde{F}_{\mu\nu} + \partial_\mu(g n_\nu) - \partial_\nu(g n_\mu)$$

- It satisfies $\Omega_{\mu\nu} u_G^\nu = 0$ under the equations of motion iff:
  - In the absence of torsion, **barotropic e.o.s**: $P = P(n)$ and $g = -\mu + c_0(T)$
  - With torsion we require in addition: $n = n(\mu + c(T))$
  - **Incompressible** limit: always satisfied (in the absence of torsion)

Mini summary & comments

- Enstrophy current for relativistic/Galilean fluids
  - Relativistic fluids: $P(T,\mu) = p(Tf(\mu/T))$
  - Non relativistic: $P = P(n)$
  - Incompressible fluids
- Approximate conservation equation in 2d: $\nabla_\mu J^\mu = \mathcal{O}(\partial^4)$
- A family of conserved currents: $J_h^\mu = h(s/n, \Omega^2/s^2) s u^\mu$
- Enstrophy for Aristotelian fluids

2. Can it be derived from a symmetry principle?

- YES:

  [R. Marjieh, N.P.F., and A. Yarom [2009.03980]]

  - Using **effective actions** for fluid dynamics

The ideal fluid effective action

- The leading order effective action for (**relativistic**) fluid dynamics

$$S_{eff} = \int \sqrt{-g}\, P(T,\mu)\, d^{d+1}\sigma$$

- The invariants

$$T = \frac{1}{\sqrt{-\beta^i \beta^j g_{ij}}} \qquad \mu/T = \beta^i B_i + \Lambda_\beta$$

- The pullback sources

$$g_{ij}(\sigma) = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X(\sigma))$$

$$B_i(\sigma) = \partial_i X^\mu B_\mu(X(\sigma)) + \partial_i C(\sigma)$$

The ideal fluid effective action

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Initial state data

$$T = \frac{1}{\sqrt{-\beta^i \beta^j g_{ij}}} \qquad \mu/T = \beta^i B_i + \Lambda_\beta$$

- The pullback sources

External sources

$$g_{ij}(\sigma) = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X(\sigma))$$

$$B_i(\sigma) = \partial_i X^\mu B_\mu(X(\sigma)) + \partial_i C(\sigma)$$

Dynamical fields

Enstrophy from symmetry

- The leading order effective action for (**relativistic**) fluid dynamics

$$S_{eff} = \int \sqrt{-g}\, P(T,\mu)\, d^{d+1}\sigma$$

- The variation of the effective action

$$\delta_X S_{eff} = -\int d^{d+1}\sigma \sqrt{-|g_{ij}|}\left((\nabla_\mu T^\mu_\nu - F^\mu_\nu J_{c\mu} + A_\nu \nabla_\mu J^\mu_c)\delta X^\nu + \nabla_\mu J^\mu_c \delta C + \begin{pmatrix} \text{total} \\ \text{derivative} \end{pmatrix} \right)$$

$$(\nabla_\mu T^\mu_\nu - F^\mu_\nu J_{c\mu} + A_\nu \nabla_\mu J^\mu_c)\delta X^\nu + \nabla_\mu J^\mu_c \delta C = \nabla_\mu S^\mu$$

- The off-shell enstrophy conservation equation is

$$\nabla_\mu J^\mu = \frac{1}{s^2} E\Omega^2 - \frac{2}{sp'^2}(E^\alpha E_\alpha)\Theta + \frac{4}{sp'}\Omega_{\alpha\beta} a^\alpha E^\beta + \frac{4}{p'}\nabla_\alpha\left(\frac{1}{s}\Omega^{\alpha\beta}\right)E_\beta - 4\nabla_\alpha\left(\frac{\Omega^{\alpha\beta}E_\beta}{p's}\right)$$

Enstrophy from symmetry

- The leading order effective action for (**relativistic**) fluid dynamics

$$S_{eff} = \int \sqrt{-g}\, P(T,\mu)\, d^{d+1}\sigma$$

- The transformation of the dynamical fields

$$\delta X^\mu = \frac{1}{Ts^2}\Omega^2 u^\mu - \frac{2}{sp'^2}E^\mu\Theta - \frac{4}{sp'}P^{\mu\beta}\Omega_{\beta\alpha}a^\alpha + \frac{4}{p'}P^{\mu\beta}\nabla_\alpha\left(\frac{1}{s}\Omega^\alpha_\beta\right)$$

$$\delta C = \frac{\mu}{Ts^2}\Omega^2 - A_\alpha \delta X^\alpha$$

- Leads to the conserved Noether current

$$J'^\mu = \frac{\Omega^2}{s}u^\mu + \frac{4}{sp'}\Omega^{\mu\nu}E_\nu$$

Equations of motion

Galilean enstrophy from symmetry

- The leading order effective action for (**Galilean**) fluid dynamics

$$S_{eff} = \int d^{d+1}\sigma\sqrt{\gamma}P(T,\tilde{\mu})$$

- The invariants

$$T = \frac{1}{\beta^i n_i}\,, \qquad \mu = T\beta^i\tilde{A}_i + T\Lambda_\beta$$

- The pullback sources

$$n_i(\sigma) = \partial_i X^\mu n_\mu(X)$$

$$\tilde{A}_i(\sigma) = \partial_i X^\mu \tilde{A}_\mu(X) + \partial_i C$$

$$u_G^\mu(X) = T\beta^i\partial_i X$$

Milne invariant

$$\tilde{A}_\mu = A_\mu + \bar{h}_{\mu\alpha}u_G^\alpha - \frac{1}{2}n_\mu v^2$$

- It is possible to show that a symmetry of the effective action exists that leads to a conserved enstrophy current as a Noether current

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Outlook:

- $\nabla_\mu J^\mu \leq 0$?
- Does that imply **inverse energy cascade** in 2d?
- The corresponding **bulk symmetry** in AdS4?
- What is the corresponding **geometric quantity** that decreases?

Thank you!

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Inverse energy cascade

- Consider the energy and enstrophy spectrum

$$\frac{1}{2}\langle \vec{v}^2 \rangle = \int E(\vec{k}) d\vec{k} \qquad \frac{1}{2}\langle \omega^2 \rangle = \int |\vec{k}|^2 E(\vec{k}) d\vec{k}$$

- Equations of motion tell us

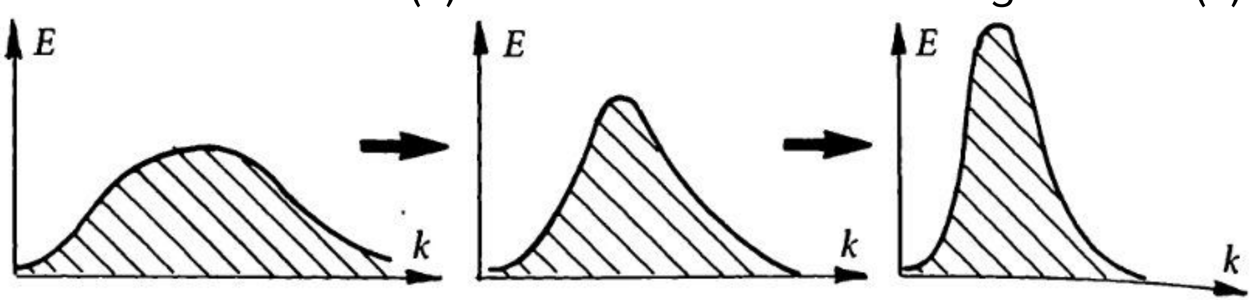
$$\frac{dE}{dt} = -\frac{1}{R}\Omega \qquad \partial_t \Omega = -\frac{1}{R} P$$

- Experimentally

$$\lim_{R\to\infty} \frac{P}{R} = const \quad \rightarrow \quad \frac{dE}{dt} = 0 \qquad \frac{d\Omega}{dt} < 0$$

- → The area under the curve E(k) is constant while the integral k^2 E(k) decreases.

- E(k) should Grow at small k and become depleted at large k



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