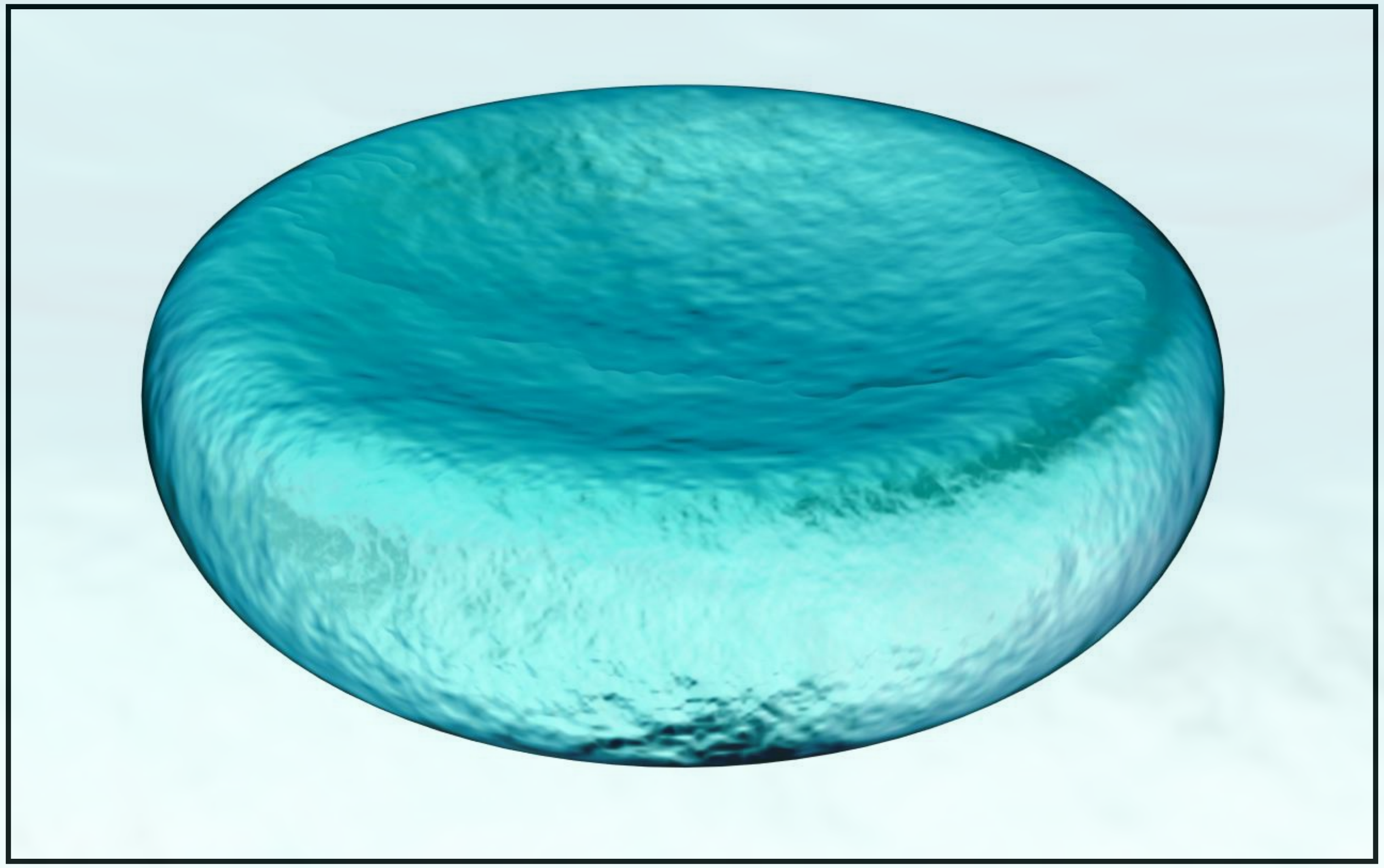
From black holes to fluid membranes via Newton-Cartan geometry



Mostly based on:

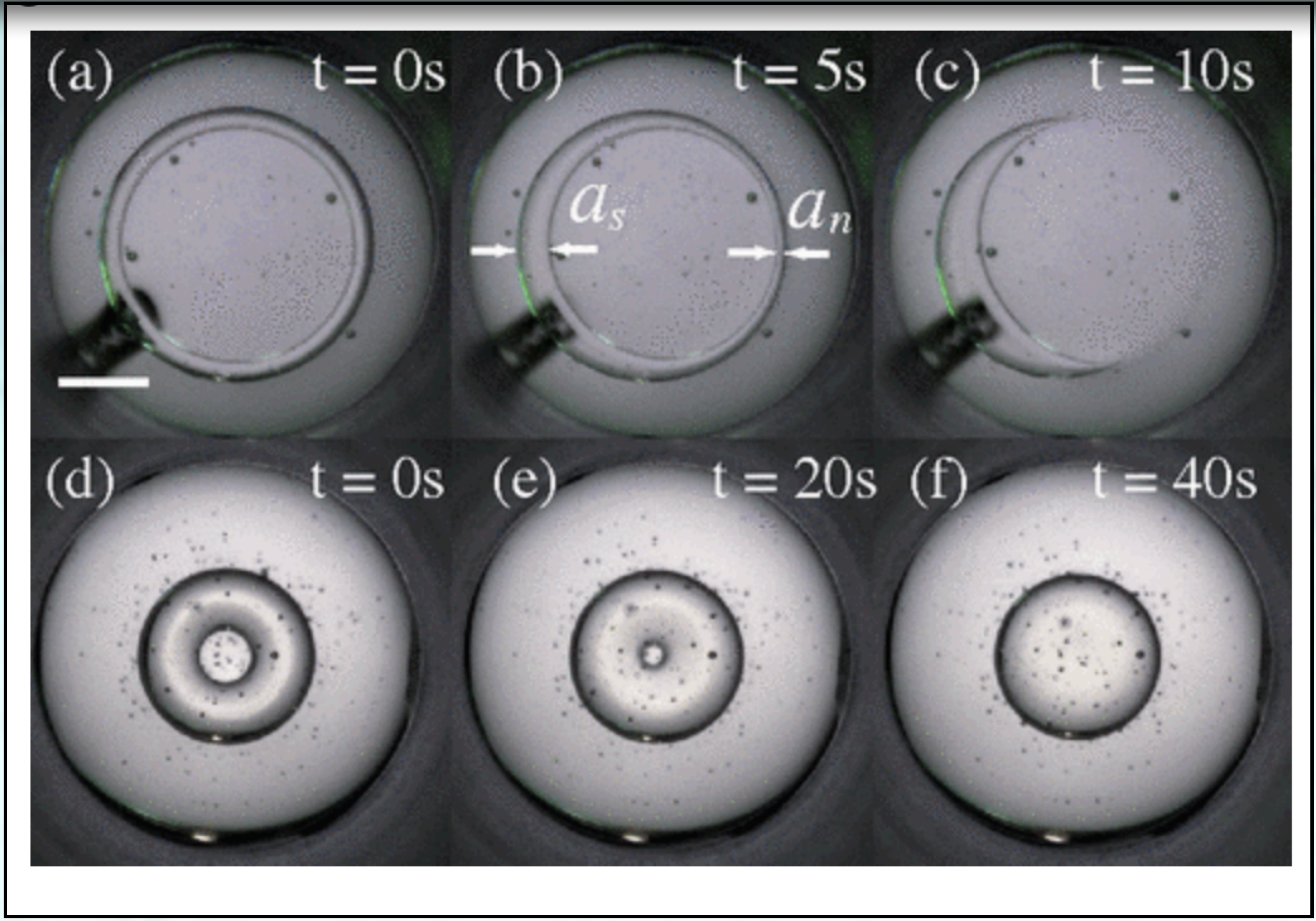
arXiv: 1912.01613, PRE, by JA, J. Hartong, E. Have, B. Nielsen & N. Obers
arXiv: 1304.7773 and 1312.0597 by J.A.

Jay Armas
University of Amsterdam
Dutch Institute for Emergent Phenomena

MOTIVATIONS

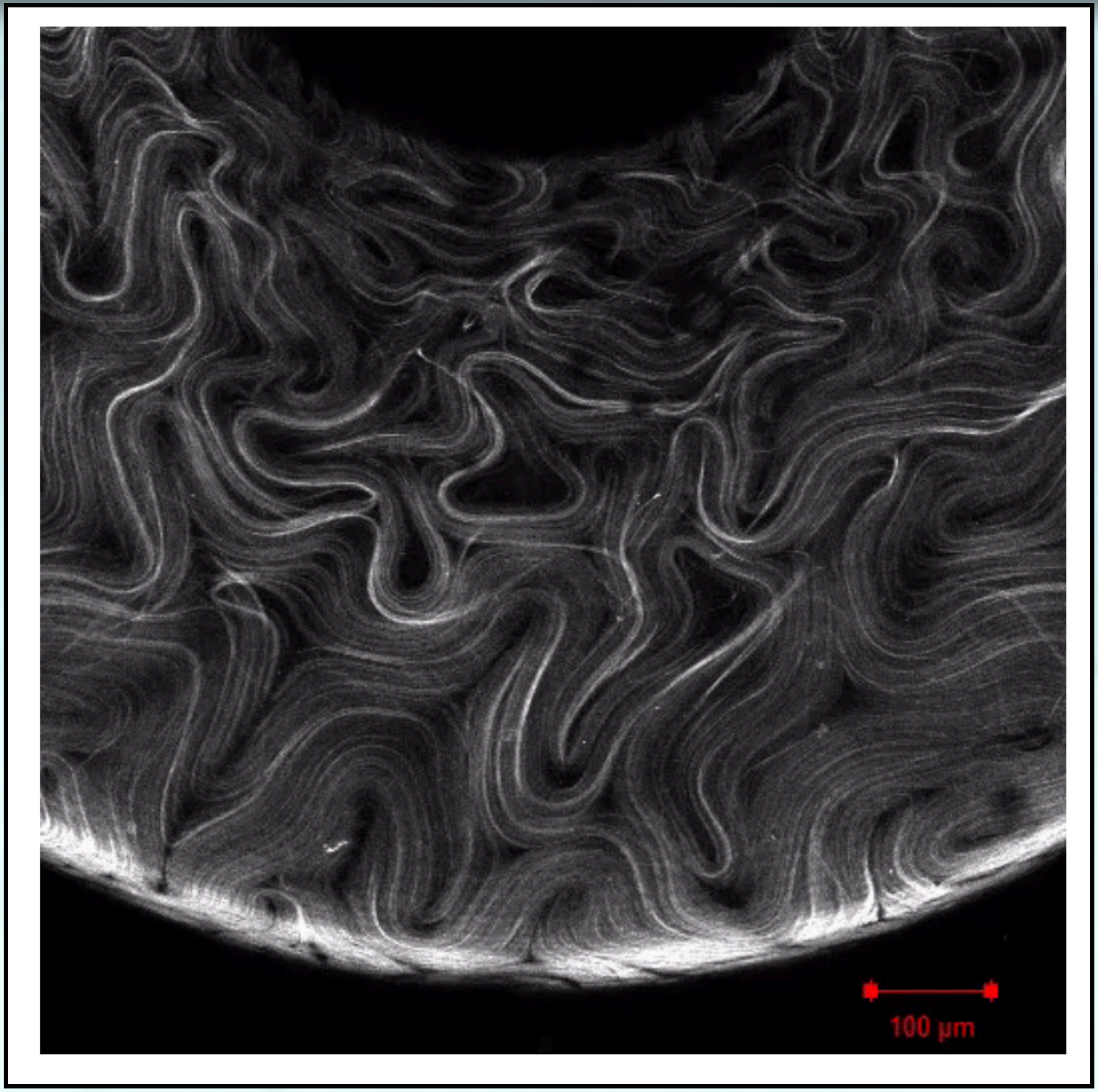
- Effective theory for modelling the response of continuous media at long wavelengths (e.g. Galilean invariant symmetries)
- World is curved. Specifically interested in media “living” on curved surfaces (e.g. surfaces waves,fluid membranes, cell migration, etc)
- Describe boundaries, branes, interfaces, defects in Galilean-invariant theories.

MOTIVATIONS



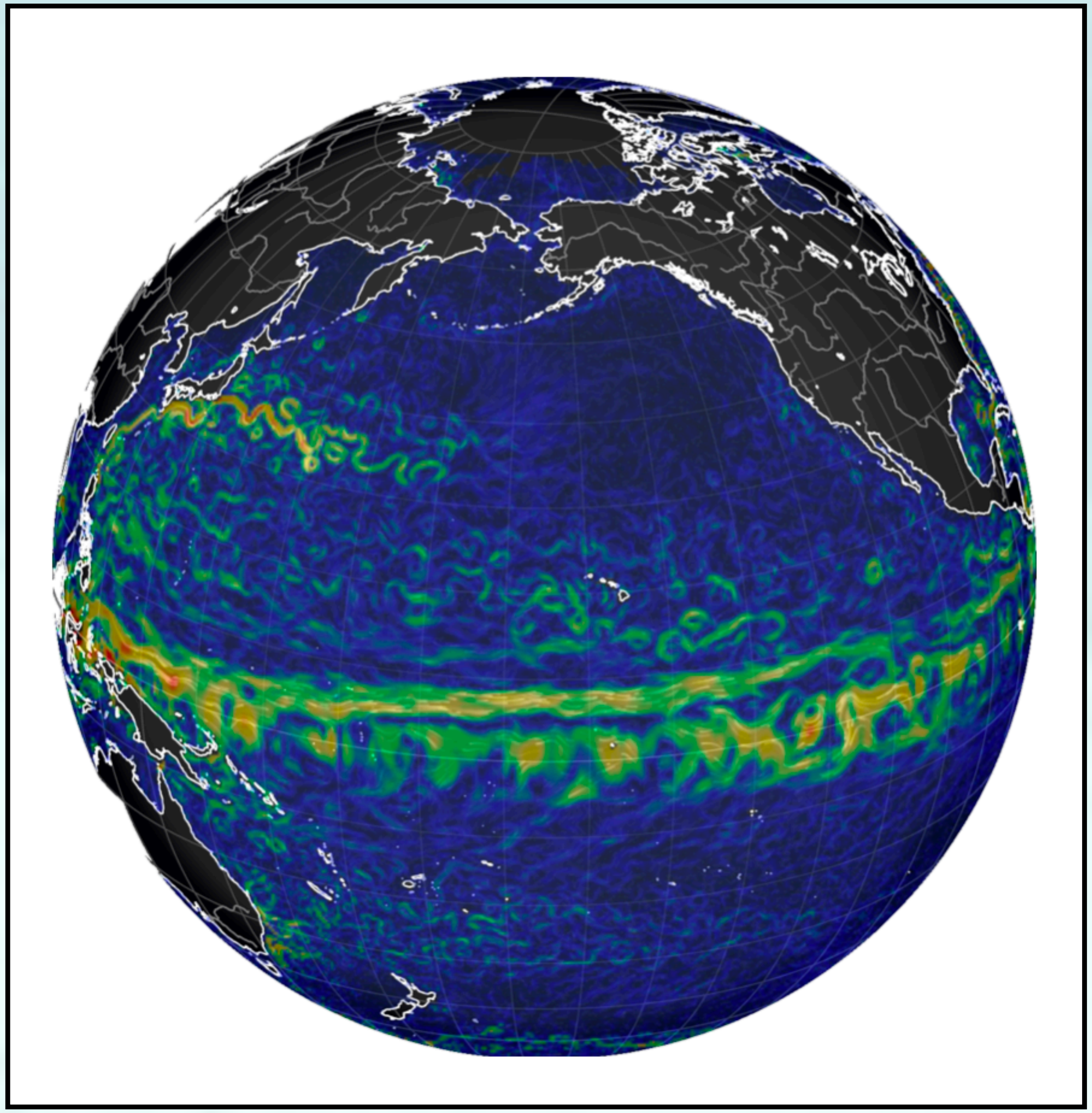
E. Pairam & A. Fernandez-Nieves, PRL, 2009

MOTIVATIONS



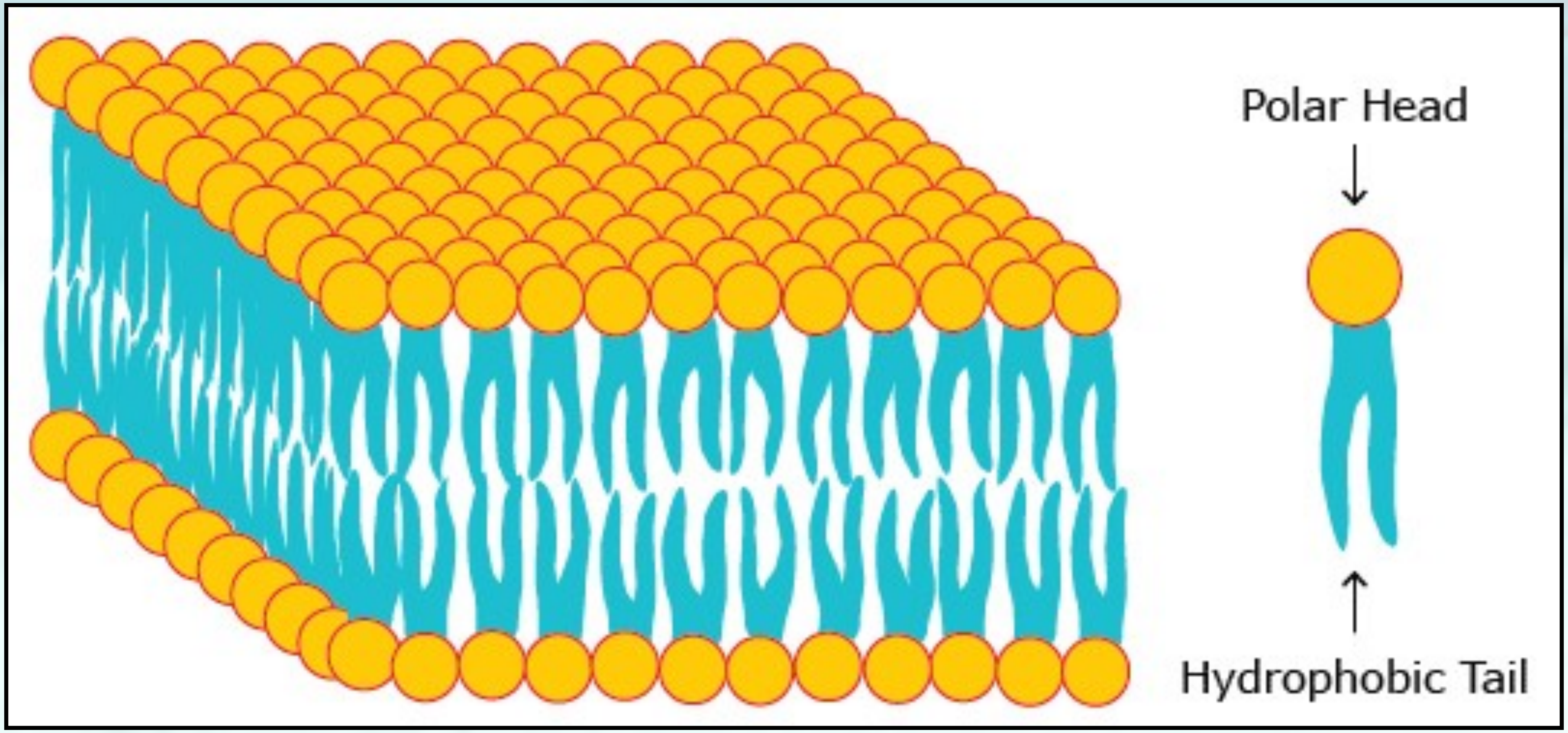
Perry W. Ellis et al., Nature Physics, 2017

MOTIVATIONS



Kelvin-Yanai waves on the equator

MOTIVATIONS

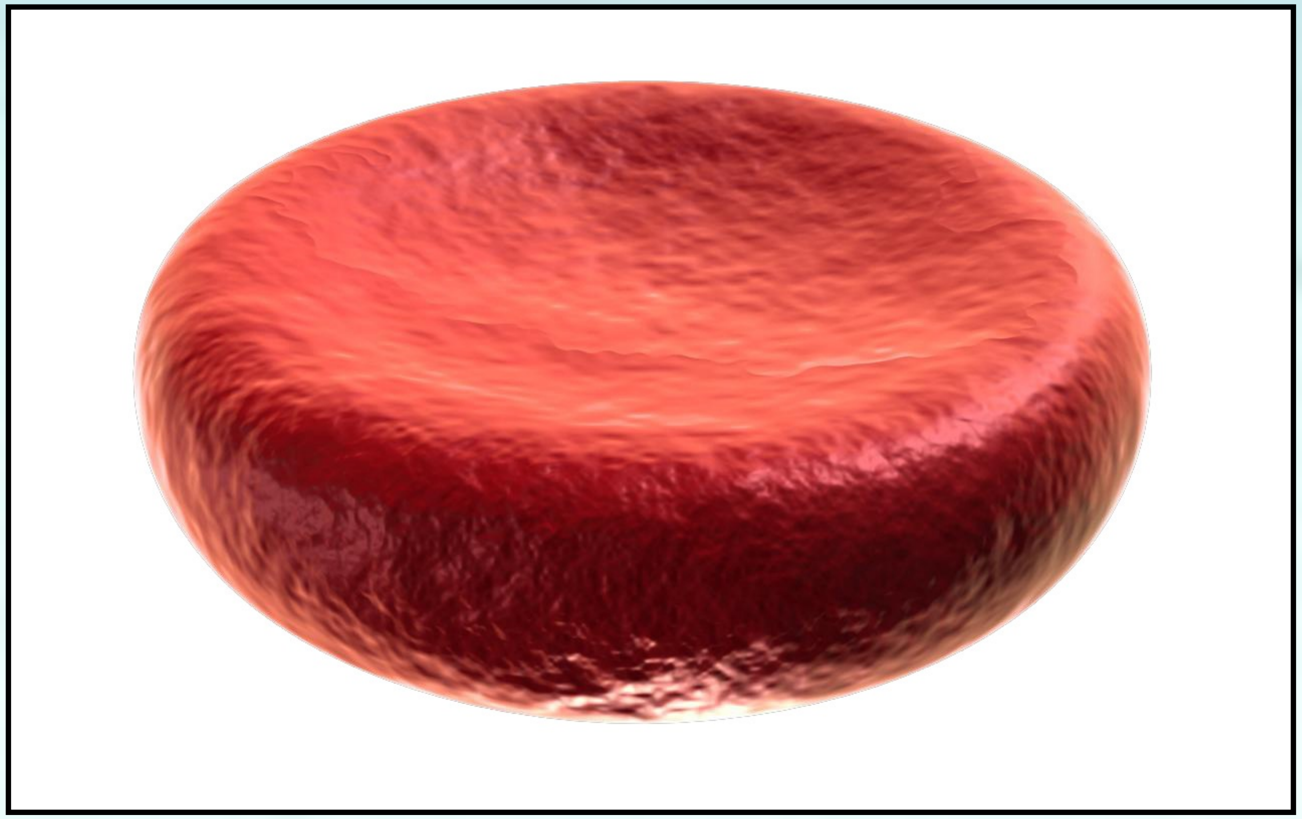


Lipid Bilayer

MOTIVATIONS

The Helfrich action for a lipid membrane:

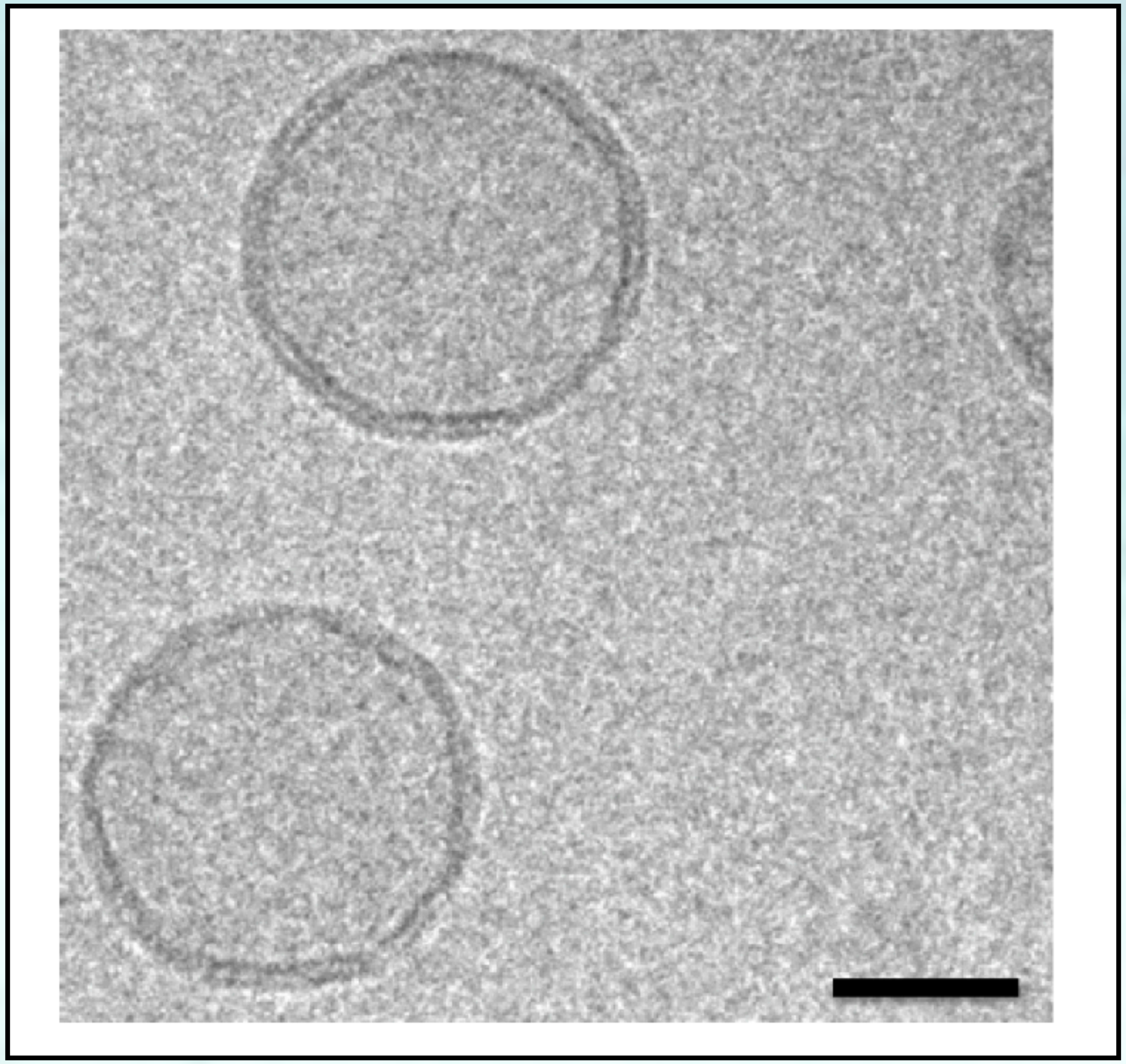
$$I[X^i] = \int_{\mathcal{W}} \left(\alpha + \alpha_1 \left(K + c_0 \right)^2 \right)$$



Helfrich, Canham 70's

MOTIVATIONS

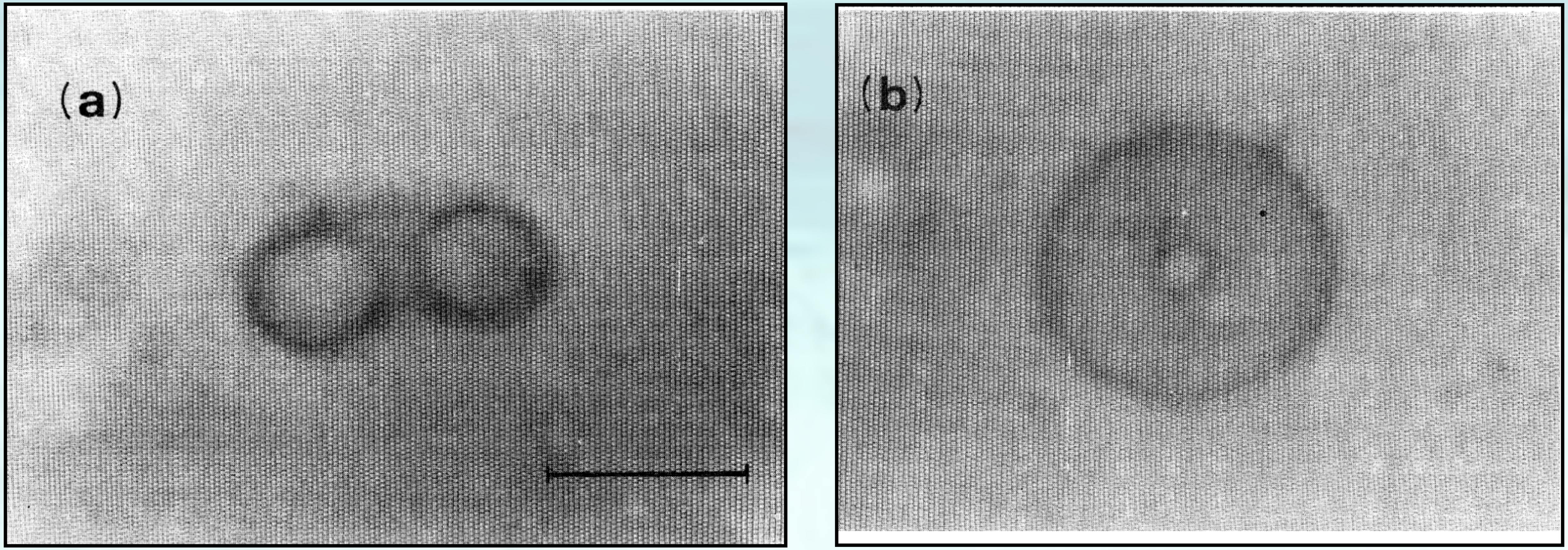
Spherical topology:



Liposome

MOTIVATIONS

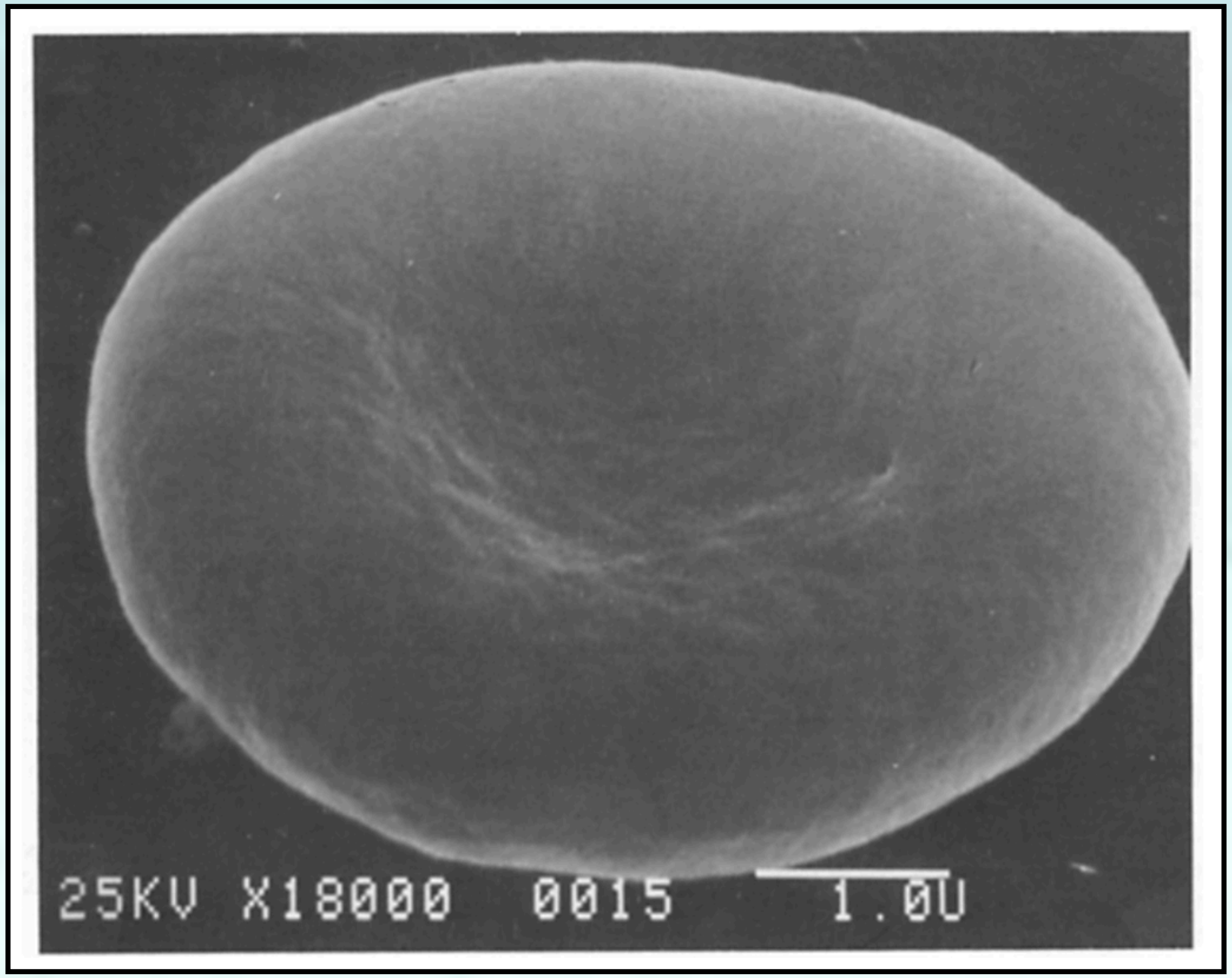
Torus:



Polymerized phospholipid membrane

MOTIVATIONS

Discoid:



Human erythrocyte

GOALS

Develop the framework in which symmetry organising principles can be used to model equilibrium Galilean invariant systems (e.g. fluid membranes).

Understand how such systems evolve as they depart from equilibrium.

PLAN

I - Model the analogue system with relativistic symmetries and “null reduction”

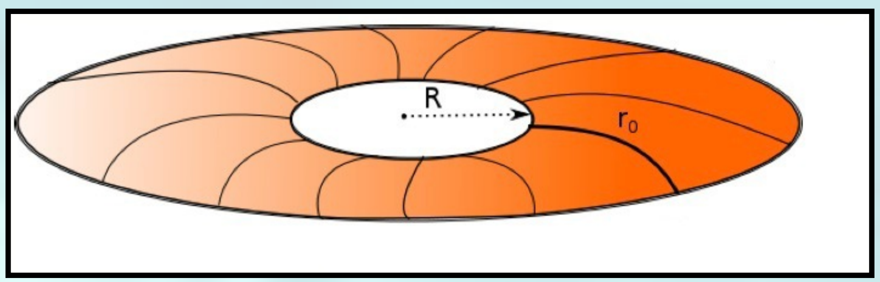
II - Construct from scratch: submanifolds and deformations in NC geometry

III - Eamples: Membranes, generalisation of the Helfrich bending energy

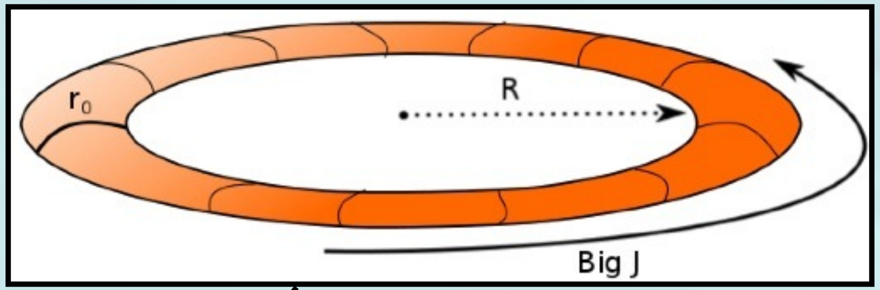
RELATIVISTIC COUNTERPART

PART I

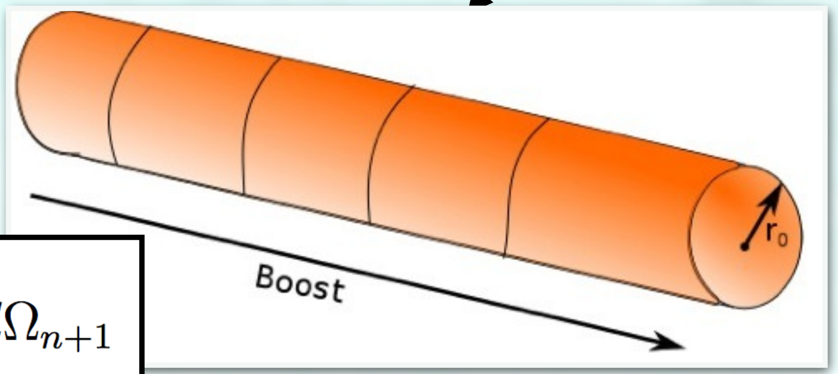
RELATIVISTIC COUNTERPART



increase J



zoom locally

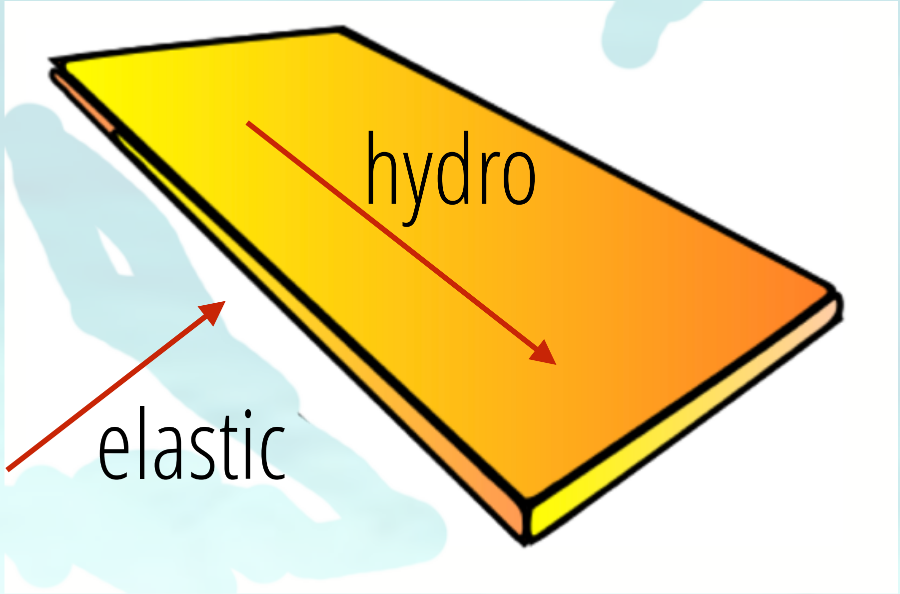


$$ds^2 = \left(\eta_{ab} + \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}$$

arXiv:0708.2181 by R. Emparan, T. Harmark, V. Niarchos, N.A. Obers & M.Rodriguez

arXiv:0910.1601 by R. Emparan, T. Harmark, V. Niarchos, N.A. Obers

RELATIVISTIC COUNTERPART



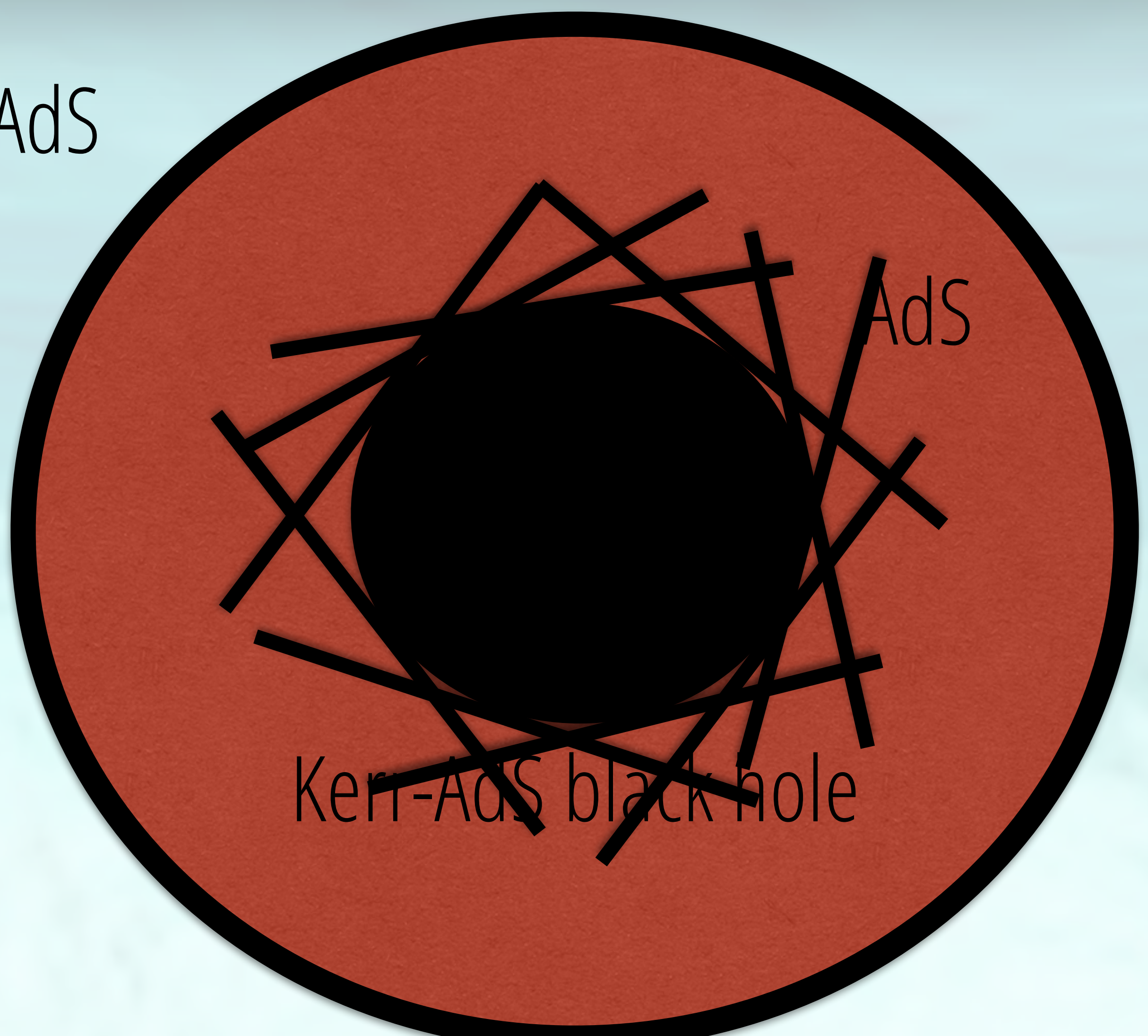
$$\hat{T}^{\mu\nu} = T^{\mu\nu}\delta(x) + D^{\mu\nu\rho}\partial_\rho\delta(x)$$

$$T^{\mu\nu} = (E+P)u^\mu u^\nu + Pg^{\mu\nu} - \eta\sigma^{\mu\nu} - \xi\theta P^{\mu\nu}$$

$$D^{\mu\nu\rho} = Y^{\mu\nu\lambda\sigma} K_{\lambda\sigma}{}^{\rho}$$

RELATIVISTIC COUNTERPART

boundary AdS



planar brane
is corrected

arXiv:0712.2456 by S. Bhattacharyya, V.E.Hubeny, S. Minwalla and M. Rangamani
arXiv:0708.1770 by S. Bhattacharyya, S. Lahiri, R. Loganayagam & S. Minwalla

RELATIVISTIC COUNTERPART

A simpler situation is to impose stationarity:

$$\theta = 0 \quad , \quad \sigma^{\mu\nu} = 0 \ \Rightarrow \ u^{\mu} = \frac{\mathbf{k}^{\mu}}{\mathbf{k}}$$

RELATIVISTIC COUNTERPART

When considering stationary fluids, one can build an equilibrium partition function (no boundaries):

$$I = \int_{\mathcal{M}} \left(P + \varpi_1 \mathfrak{a}^2 + \varpi_2 \omega^2 + \varpi_3 R + \ldots \right)$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta I}{\delta g_{\mu\nu}}$$

arXiv:1203.3544 by Bhattacharya et al.

RELATIVISTIC COUNTERPART

Partition function for equilibrium fluid droplets:

$$\mathcal{I} = \int_{\mathcal{M}} \sqrt{-\mathcal{G}} \left(P + \tilde{P}_1 \tilde{R} + \tilde{P}_2 \omega^2 + \tilde{P}_3 \mathfrak{a}^2 \right)$$

$$+ \int_{\partial\mathcal{M}} \sqrt{-\mathcal{H}} \left(\chi + \tilde{\mathcal{B}}_1 \mathfrak{a}^{\mu} n_{\mu} + \tilde{\mathcal{B}}_2 \ell^{\mu} n_{\mu} + \tilde{\mathcal{B}}_3 K \right)$$

arXiv:1512.08514 by J.A., J. Bhattacharya & N. Kundu

RELATIVISTIC COUNTERPART

The effective action for embedded fluids:

$$I[X^i] = \int_{\mathcal{W}} \sqrt{-\gamma}\Big(P + \upsilon_1 \mathfrak{a}^2 + \upsilon_2 \omega^2 + \upsilon_3 \mathcal{R} + \lambda_1 K^i K_i + \lambda_2 K^{abi} K_{abi} + \lambda_3 u^a u^b K_a{}^{ci} K_{bci} + \varpi_1 u^a \omega_a + \varpi_2 u^a \omega_a u^b \omega_b\Big)$$

$$\nabla_a \left(e^b u^\mu{}_b\right) = u^{\mu b} \gamma_{ab}{}^c e_c + n^\nu{}_i K_{ab}{}^i e^a \quad ,$$
$$\nabla_a \left(n^i n^\mu{}_i\right) = -u^{\mu b} K_{ab}{}^i n_i - n^\mu{}_j \omega_a{}^{ij} n_i$$

$$\omega_a = \epsilon_{ij} \omega_a{}^{ij}$$

arXiv:1304.7773 and arXiv:1312.0597 by JA
arXiv:1406.7813 by JA & T.Harmark

RELATIVISTIC COUNTERPART

Bending moment and spin current:

$$\mathcal{D}^{ab}{}_i = \frac{\partial I}{K_{ab}{}^i} = \mathcal{Y}^{abcd} K_{cdi}$$

$$\mathcal{S}^a{}_{ij} = \frac{\partial I}{\omega_a{}^{ij}}$$

Equations of motion:

$$\nabla_a T^{ab} - u_\mu{}^b \nabla_a \nabla_c \mathcal{D}^{ac\mu} + 2\,\mathcal{S}^a{}_{ij} K_{ac}{}^i K^{bcj} = \mathcal{D}^{aci} R^b{}_{aic} + \mathcal{S}^{aij} R^b{}_{aij} - \omega^{bij} \nabla_a \mathcal{S}^a{}_{ij}$$

$$T^{ab} K_{ab}{}^i = n^i{}_\rho \nabla_a \nabla_b \mathcal{D}^{ab\rho} - 2\, n^i{}_\rho \nabla_b \left(\mathcal{S}_a{}^{\rho j} K^{ab}{}_j \right) + \mathcal{D}^{abj} R^i{}_{ajb} + \mathcal{S}^{akj} R^i{}_{akj}$$

arXiv:1304.7773 and arXiv:1312.0597 by JA

RELATIVISTIC COUNTERPART

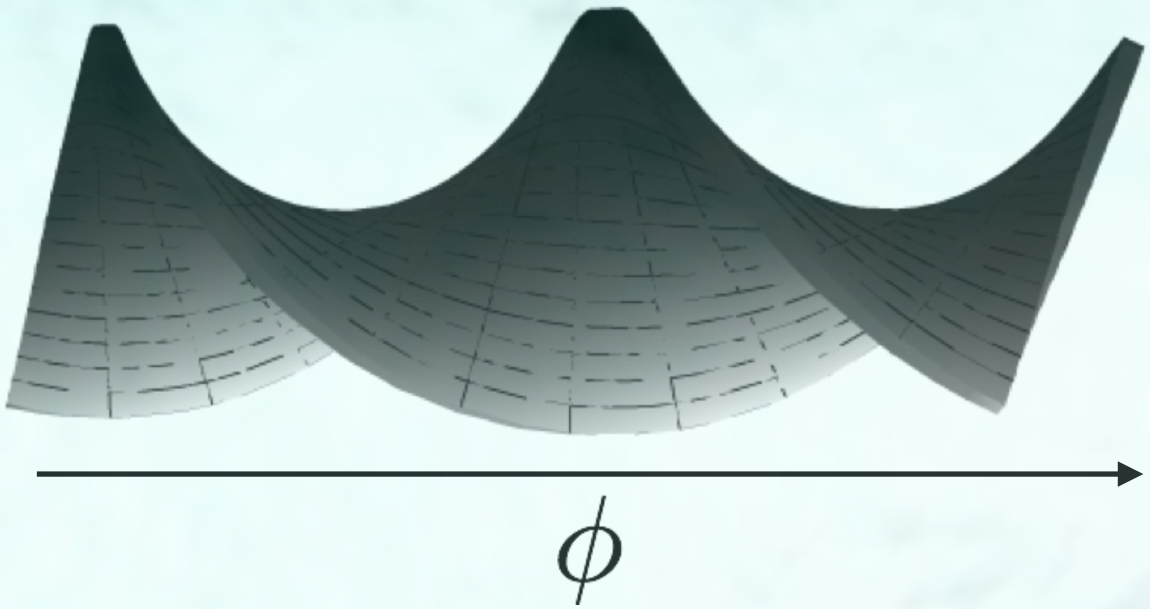
The helicoid:

$$\gamma_{ab}d\sigma^a d\sigma^b = -d\tau^2 + d\rho^2 + (\lambda^2 + \rho^2)d\phi^2$$

$$\mathbf{k}^2 = 1 - \Omega^2(\lambda^2 + \rho^2)$$

$$\mathcal{F}[R] = \frac{\Omega_{(n+1)}}{16\sqrt{\pi}G}\frac{r_+^n}{a\Omega}\int d\phi\lambda\Gamma\left(1+\frac{n}{2}\right)\left(1-\lambda^2\Omega^2\right)^{\frac{n+1}{2}}\ {}_2\tilde{F}_1\left(-\frac{1}{2},\frac{1}{2};\frac{n+3}{2};1-\frac{1}{\lambda^2\Omega^2}\right)$$

Topology: $\mathbb{R} \times \mathbb{S}^{D-3}$

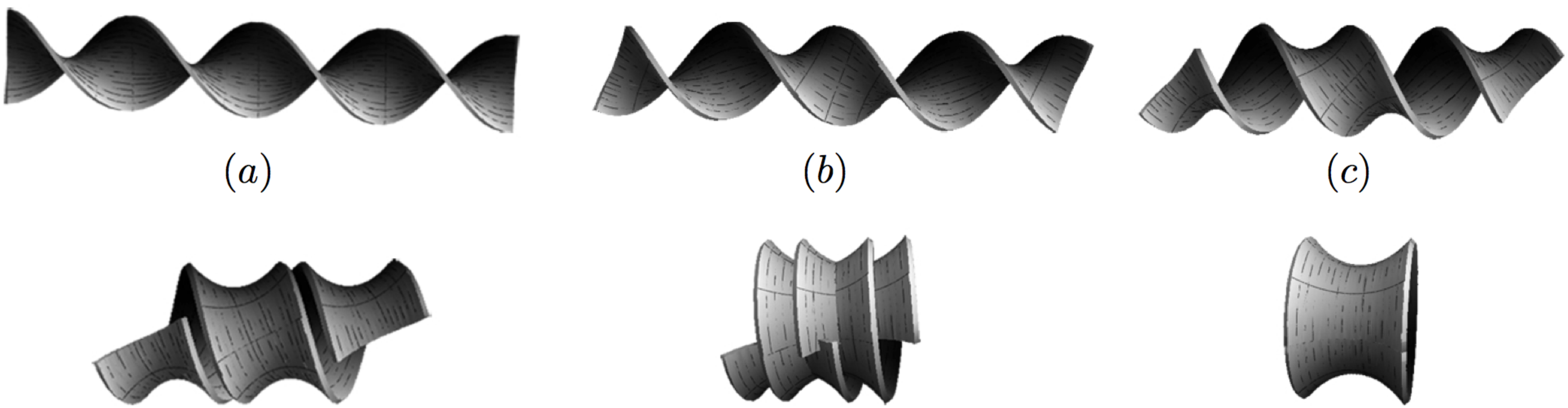


arXiv:1503.08834 by JA & M. Blau

RELATIVISTIC COUNTERPART

Black Scherk Surfaces/Catenoids in Plane Waves:

$$ds^2 = -(1 + A(x^q))dt^2 + (1 - A(x^q))dy^2 - 2A(x^q)dtdy + d\mathbb{E}^2_{(D-2)}(x^q)$$



(a) (b) (c) (d) (e) (f)

arXiv:1503.08834 by JA & M. Blau

RELATIVISTIC COUNTERPART

$$I[X^i] = \int_{\mathcal{W}} \sqrt{-\gamma}\Big(P + \upsilon_1 \mathfrak{a}^2 + \upsilon_2 \omega^2 + \upsilon_3 \mathcal{R} \\ + \lambda_1 K^i K_i + \lambda_2 K^{abi} K_{abi} + \lambda_3 u^a u^b K_a{}^{ci} K_{bci} \\ + \varpi_1 u^a \omega_a + \varpi_2 u^a \omega_a u^b \omega_b\Big)$$

↓ Null reduction

Galilean-invariant theory

NEWTON-CARTAN SUBMANIFOLDS

PART II

$\mathcal{M}_{d+1}$

NC structure: $\tau_\mu \,, h_{\mu\nu} \,, m_\mu \,, v^\mu \,, h^{\mu\nu}$

Completeness: $\delta^\mu_\nu = -v^\mu \tau_\nu + h^{\mu\rho} h_{\rho\nu}$ $\quad v^\mu \tau_\mu = -1$

Vielbeins: $h_{\mu\nu} = \delta_{\underline{a}\underline{b}} e^{\underline{a}}_\mu e^{\underline{b}}_\nu \,, \qquad h^{\mu\nu} = \delta^{\underline{a}\underline{b}} e^\mu_{\underline{a}} e^\nu_{\underline{b}} \,,$

$$v^\mu e^{\underline{a}}_\mu = 0 \,, \qquad \tau_\mu e^\mu_{\underline{a}} = 0 \,, \qquad e^\mu_{\underline{a}} e^{\underline{b}}_\mu = \delta^{\underline{b}}_{\underline{a}} \quad .$$

Boost invariants: $\bar{h}_{\mu\nu} = h_{\mu\nu} - 2\tau_{(\mu} m_{\nu)} \,, \qquad \hat{v}^\mu = v^\mu - h^{\mu\nu} m_\nu$

Geometry: $\nabla_\mu \tau_\nu = 0 \,, \qquad \nabla_\mu h^{\nu\rho} = 0$

$$\Gamma^\rho_{\mu\nu} = -\hat{v}^\rho \partial_\mu \tau_\nu + \frac{1}{2} h^{\rho\sigma} \left(\partial_\mu \bar{h}_{\nu\sigma} + \partial_\nu \bar{h}_{\mu\sigma} - \partial_\sigma \bar{h}_{\mu\nu} \right)$$

NEWTON-CARTAN SUBMANIFOLDS

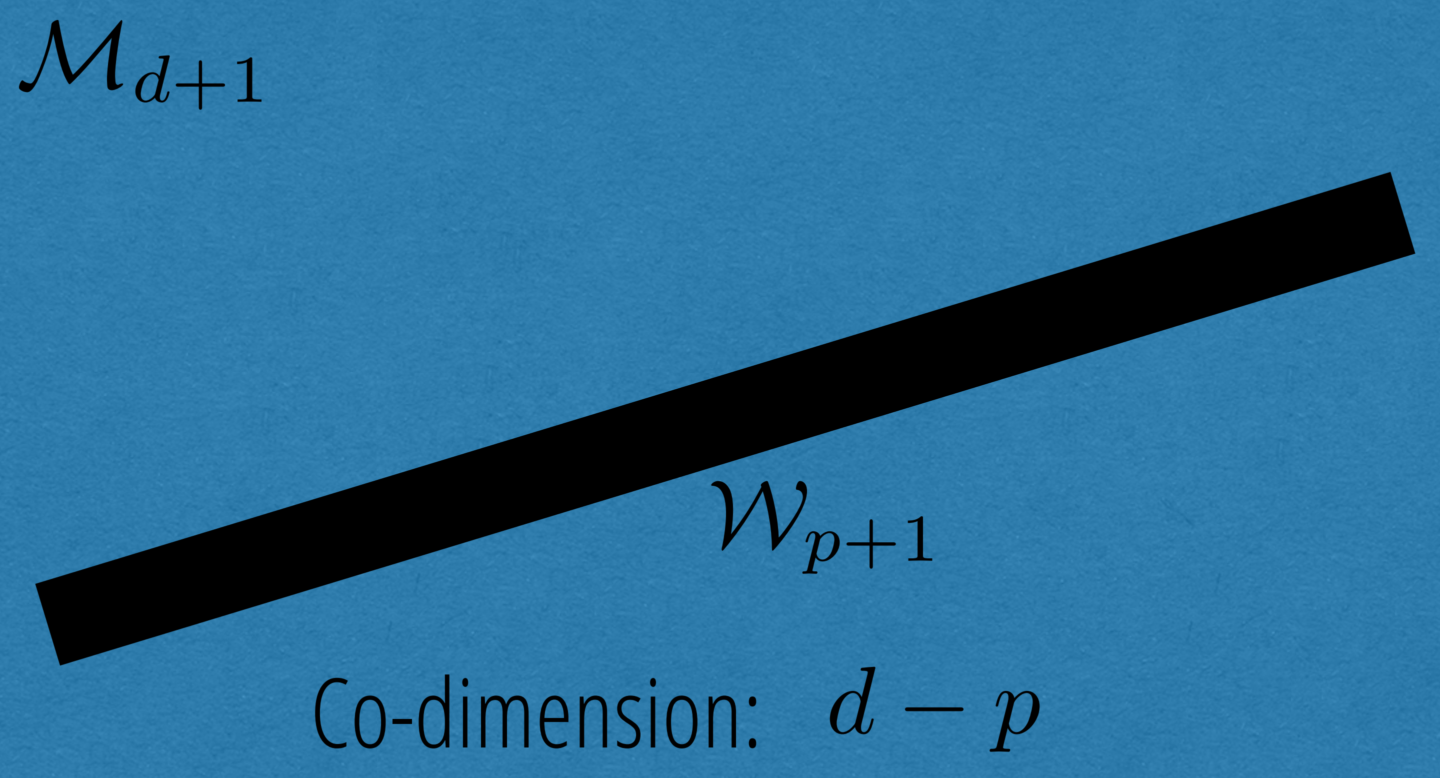
Transformations (diffeomorphisms, boost, rotations, gauge)

$$\delta\tau_\mu = \pounds_\xi \tau_\mu \,, \qquad \delta e^{\underline{a}}_\mu = \pounds_\xi e^{\underline{a}}_\mu + \lambda^{\underline{a}}{}_{\underline{b}} e^{\underline{b}}_\mu + \lambda^{\underline{a}} \tau_\mu \,, \qquad \delta m_\mu = \pounds_\xi m_\mu + \lambda_{\underline{a}} e^{\underline{a}}_\mu + \partial_\mu \sigma \,,$$
$$\delta v^\mu = \pounds_\xi v^\mu + \lambda^{\underline{a}} e^\mu_{\underline{a}} \,, \qquad \delta e^\mu_{\underline{a}} = \pounds_\xi e^\mu_{\underline{a}} + \lambda_{\underline{a}}{}^{\underline{b}} e^\mu_{\underline{b}} \quad .$$

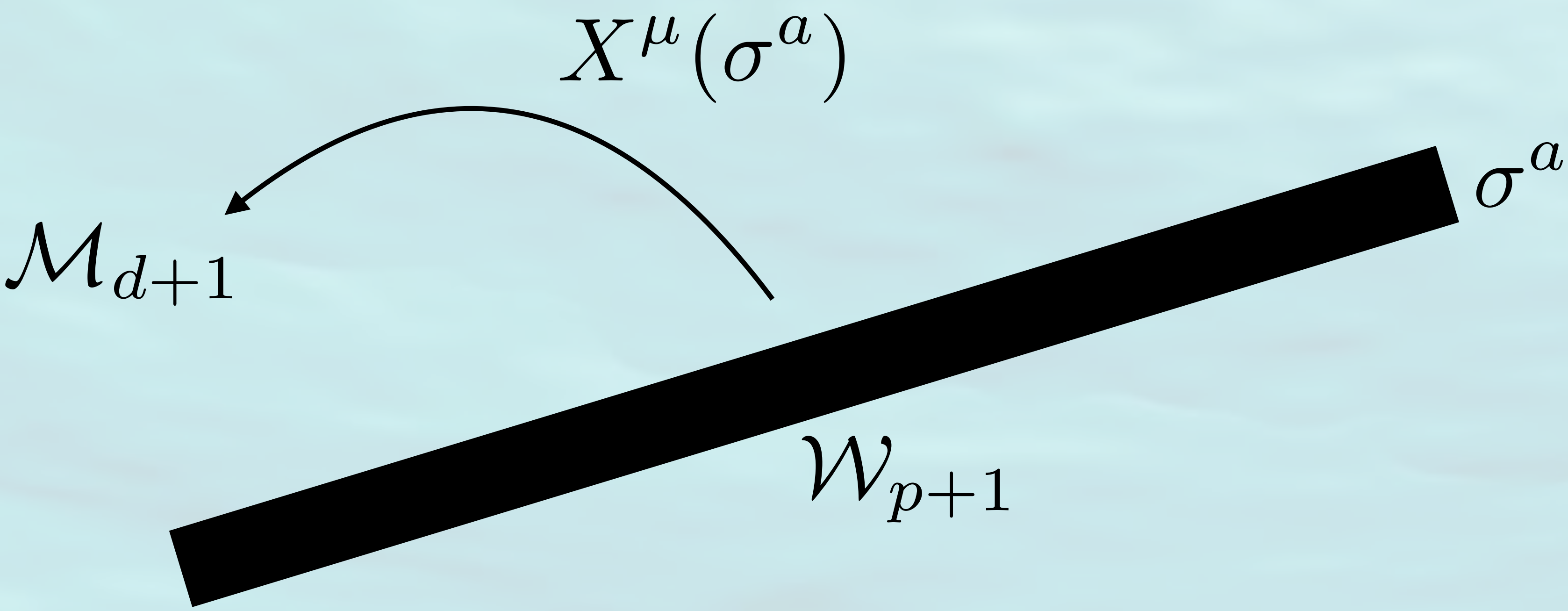
$$\delta h^{\mu\nu} = \pounds_\xi h^{\mu\nu} \quad , \quad \delta h_{\mu\nu} = \pounds_\xi h_{\mu\nu} + 2\lambda_{(\mu}\tau_{\nu)}$$

$$\delta \bar{h}_{\mu\nu} = \pounds_\xi \bar{h}_{\mu\nu} - 2\tau_{(\mu}\partial_{\nu)}\sigma \,, \qquad \delta\hat{v}^\mu = \pounds_\xi \hat{v}^\mu - h^{\mu\nu}\partial_\nu\sigma$$

$\mathcal{M}_{d+1}$


$\mathcal{M}_{d+1}$
$\mathcal{W}_{p+1}$
Co-dimension: $d-p$

NEWTON-CARTAN SUBMANIFOLDS



Explicit tangent vector: $u^\mu{}_a = \partial_a X^\mu$

Implicit normal vectors: $n^I_\mu u^\mu_a = 0 \;,\; h^{\mu\nu} n^I_\mu n^J_\nu = \delta^{IJ},\; I = 1, \ldots, d-p$

$n^I_\mu \to \mathcal{M}^I{}_J n^J_\mu$ $\quad SO(d-p)$

Completeness: $\delta^\mu_\nu = u^\mu_a u^a_\nu + n^I_\nu n^\mu_I$

$u^a_\mu n^\mu_I = 0 \;,\; u^\mu_a u^b_\mu = \delta^b_a \;,\; n^\mu_I n^J_\mu = \delta^J_I$

NEWTON-CARTAN SUBMANIFOLDS

Can one just pullback using $u^{\mu}{}_{a} = \partial_a X^{\mu}$

background structures $\tau_\mu, h_{\mu\nu}, m_\mu, v^\mu, h^{\mu\nu}$ to get an induced NC geometry?

Focus on timelike submanifolds

$$\tau_a = u^\mu_a \tau_\mu$$

$$\tau_I = n^\mu_I \tau_\mu = 0$$

$$n^{\mu I} = h^{\mu\nu} n^I_\nu$$

This implies that:

$$\begin{aligned}
h^{IJ} &= h^{\mu\nu} n^I_\mu n^J_\nu = \delta^{IJ} \,, \\
h^{aI} &= h^{\mu\nu} u^a_\mu n^I_\nu = u^a_\mu n^{\mu I} = 0 \,, \\
h_{IJ} &= h_{\mu\nu} n^\mu_I n^\nu_J = h_{\mu\nu} h^{\nu\rho} n^\mu_I n_{\rho J} = (\delta^\rho_\mu + v^\rho \tau_\mu) n^\mu_I n_{\rho J} = \delta_{IJ} \,, \\
h_{aI} &= h_{\mu\nu} u^\mu_a n^\nu_I = h_{\mu\nu} u^\mu_a h^{\nu\rho} n_{\rho I} = v_I \tau_a \,,
\end{aligned}$$

NEWTON-CARTAN SUBMANIFOLDS

Basic building blocks for the induced geometry:

$$h_{ab} = u^\mu_a u^\nu_b h_{\mu\nu}\,, \qquad v^a = u^a_\mu v^\mu\,, \qquad h^{ab} = u^a_\mu u^b_\nu h^{\mu\nu}\,, \qquad m_a = u^\mu_a m_\mu \quad , \quad v^I$$

The background satisfies: $\boxed{v^\mu h_{\mu\nu} = 0}$

But the induced structure does not: $\boxed{v^a h_{ab} = u^a_\mu v^\mu u^\rho_a u^\sigma_b h_{\rho\sigma} = -v^I h_{Ib} = -v^I v^I \tau_b}$

Need to define: $\check{h}_{ab} = h_{ab} - v^I v_I \tau_a \tau_b$

$$h^{ac}\check{h}_{cb} = \delta^a_b + \tau_b v^a \quad , \quad v^a \check{h}_{ab} = 0$$

NEWTON-CARTAN SUBMANIFOLDS

Can introduce boost invariants in which case the building blocks are:

$$\boxed{(\tau_a, \check{h}_{ab}, \check{m}_a) \text{ and } (v^a, h^{ab})}$$

Defined as:

$$\bar{h}_{ab} = u_a^\mu u_b^\nu \bar{h}_{\mu\nu} = \check{h}_{ab} - 2\tau_{(a}\check{m}_{b)} \quad , \quad \hat{v}^a = u_\mu^a \hat{v}^\mu = v^a - h^{ab}\check{m}_b$$

$$\check{m}_a = m_a - \frac{1}{2}v^I v_I \tau_a$$

Transformations:

$$\delta\hat{v}^a = \pounds_\zeta \hat{v}^a - h^{ab}\partial_b\sigma \quad , \quad \delta\bar{h}_{ab} = \pounds_\zeta \bar{h}_{ab} - 2\tau_{(a}\partial_{b)}\sigma$$

$$\delta\tau_a = \pounds_\zeta \tau_a \quad , \quad \delta\check{h}_{ab} = \pounds_\zeta \check{h}_{ab} + 2\check{\lambda}_{(a}\tau_{b)} \quad , \quad \delta\check{m}_a = \pounds_\zeta \check{m}_a + \check{\lambda}_a + \partial_a\sigma \quad ,$$

$$\delta v^a = \pounds_\zeta v^a + h^{ab}\check{\lambda}_b \quad , \quad \delta h^{ab} = \pounds_\zeta h^{ab} \quad ,$$

NEWTON-CARTAN SUBMANIFOLDS

Induced geometry:

$$D_a T^{b\mu} = \partial_a T^{b\mu} + \gamma^b_{ac} T^{c\mu} + u^\rho_a \Gamma^\mu_{\rho\lambda} T^{b\lambda}$$

$$\gamma^c_{ab} = -\hat{v}^c \partial_a \tau_b + \frac{1}{2} h^{cd} \left(\partial_a \bar{h}_{bd} + \partial_b \bar{h}_{ad} - \partial_d \bar{h}_{ab} \right)$$

Weingarten identities:

$$\mathfrak{D}_a n^I_\sigma = \partial_a n^I_\sigma - \Gamma^\lambda_{\mu\sigma} u^\mu_a n^I_\lambda - \omega_a{}^I{}_J n^J_\sigma = -u^b_\sigma K_{ab}{}^I + \frac{1}{2} u^b_\sigma \hat{v}^I \tau_{ab}$$

$$\mathfrak{D}_a u^\mu_b = D_a u^\mu_b = n^\mu_I K_{ab}{}^I - \frac{1}{2} n^\mu_I \hat{v}^I \tau_{ab} \quad ,$$

Extrinsic curvature:

$$K_{ab}{}^I = n^I_\mu D_a u^\mu_b + \frac{1}{2} \hat{v}^I \tau_{ab} = n^I_\mu \left(\partial_a u^\mu_b + u^\nu_a u^\rho_b \Gamma^\mu_{(\nu\rho)} \right) = -u^\mu_a u^\nu_b \nabla_{(\mu} n^I_{\nu)}$$

Spin connection:

$$\omega_a{}^I{}_J = n^\mu_J D_a n^I_\mu$$

NEWTON-CARTAN SUBMANIFOLDS

Integrability conditions.

Codazzi-Mainardi:

$$\mathfrak{D}_a \tilde{K}_{bc}{}^I - \mathfrak{D}_b \tilde{K}_{ac}{}^I = -R_{abc}{}^I + \hat{v}^d \tau_{ab} \tilde{K}_{dc}{}^I$$

$$\tilde{K}_{ab}{}^I = n^I_\mu D_a u^\mu_b = K_{ab}{}^I - \frac{1}{2} \hat{v}^I \tau_{ab}$$

Gauss-Codazzi:

$$\mathcal{R}_{abc}{}^d = \tilde{K}_{ac}{}^I \tilde{K}_b{}^d{}_I - \tilde{K}_{bc}{}^I \tilde{K}_a{}^d{}_I + R_{abc}{}^d$$

Ricci-Voss:

$$\Omega^I{}_{Jab} = R_{abJ}{}^I - 2h^{cd} \tilde{K}_{[a|c}{}^I \tilde{K}_{|b]dJ}$$

$$\Omega^I{}_{Jab} = 2\partial_{[a}\omega_{b]}{}^I{}_J - 2\omega_{[a|}{}^I{}_K \omega_{|b]}{}^K{}_J$$

NEWTON-CARTAN SUBMANIFOLDS

Embedding map variations:

$$\delta X^\mu(\sigma) = -\xi^\mu(\sigma)$$

Details:

$$\delta_X n^I_\mu = -n_{\mu J} n^{(I}_\rho n^{J)\nu} \nabla_\nu \xi^\rho - n_{\mu J} \hat{v}^{(I} n^{J)\nu} \tau_{\nu\rho} \xi^\rho + n^I_\rho \partial_\mu \xi^\rho + \widetilde{\lambda}^{IJ} n_{\mu J}$$

$$\delta_X \tau_\mu(X) = -\xi^\nu \partial_\nu \tau_\mu \quad , \quad \delta_X \bar{h}_{\mu\nu}(X) = -\xi^\rho \partial_\rho \bar{h}_{\mu\nu}$$

$$\delta_X \tau_a = -u^\mu_a \pounds_\xi \tau_\mu, \qquad \delta_X \bar{h}_{ab} = -u^\mu_a u^\nu_b \pounds_\xi \bar{h}_{\mu\nu}$$

NEWTON-CARTAN SUBMANIFOLDS

Effective action:

$$S = S[\tau_a, \bar{h}_{ab}, K_{ab}{}^I]$$

$$\delta S = \int_\Sigma \mathrm{d}^{p+1}\sigma e \left(\mathcal{T}^a \delta\tau_a + \frac{1}{2}\mathcal{T}^{ab}\delta\bar{h}_{ab} + \mathcal{D}^{ab}{}_I \delta K_{ab}{}^I \right)$$

Gauge invariance:

$$D_b\left(\mathcal{T}^{ab}\tau_a\right) = 0 \quad , \quad \mathcal{D}^{ab[I}K_{ab}{}^{J]} = 0$$

Momentum conservation:

$$D_a \mathcal{T}^{ad}_{\mathrm{m}} + 2D_a\left(\mathcal{D}^{b[a}{}_I h^{d]c} K_{bc}{}^I\right) - h^{cd}\mathcal{D}^{ab}{}_I D_c K_{ab}{}^I = 0$$

Dynamics:

$$\mathcal{T}^{ab}K_{ab}{}^I = \mathfrak{D}_a\mathfrak{D}_b\mathcal{D}^{abI} - \mathcal{D}^{ab}{}_J K_{ac}{}^I K_{bd}{}^J h^{cd} - \mathcal{D}^{ab}{}_J R_{Iab}{}^J \,,$$

EXAMPLES

PART III

EXAMPLES

Describing a thermal state in equilibrium: $K = (k^\mu, \lambda^K_\mu, \Lambda^K)$

Action on induced structure vanishes, implying:

$$\pounds_k \tau_a = 0 \quad , \quad \pounds_k \bar{h}_{ab} = 2\tau_{(a}\pounds_k \check{m}_{b)} + 2\tau_{(a}\partial_{b)}\Lambda^K \quad , \quad \pounds_k \check{m}_a + \check{\lambda}^K_a + \partial_a \Lambda^K = 0$$

Can define scalar boost invariants:

$$T = \frac{T_0}{k^a \tau_a} \quad , \quad \frac{\mu}{T} = \frac{\Lambda^K}{T_0} + \frac{1}{2T}\bar{h}_{ab}u^a u^b \quad , \quad u^b = \frac{k^b}{k^a \tau_a}$$

EXAMPLES

Simplest fluid membrane:

$$\mathcal{F} = \int_{\Sigma_s} d^p\sigma e_s\ \chi(T,\mu)$$

Thermodynamics:

$$s = \left(\frac{\partial\chi}{\partial T}\right)_\mu \quad , \quad n = \left(\frac{\partial\chi}{\partial\mu}\right)_T$$

$$\varepsilon + \chi = Ts + n\mu.$$

Currents:

$$\mathcal{T}^a = -\chi\hat{v}^a - \left(\varepsilon + \chi + \frac{n}{2}\bar{u}^2\right)u^a \quad , \quad \mathcal{T}^{ab} = \chi h^{ab} + nu^a u^b$$

Dynamics:

$$\mathcal{T}^{ab}K^I_{ab} = 0 \quad \Rightarrow \quad \chi K^I + nu^a u^b K^I_{ab} = 0$$

EXAMPLES

Flat 2+1 surface:

$$h_{ab} = \delta^i_a \delta^i_b \quad , \quad m_a = 0 \quad , \quad n_\mu = \delta^3_\mu$$

Deformations:

$$\delta_X \mathcal{T}^{ab} K_{ab} + \mathcal{T}^{ab} \delta_X K_{ab} = \left(\chi h^{ab} + n u^a u^b\right) \partial_a \partial_b \xi^\perp = 0$$

Waves:

$$\omega = \pm \sqrt{\frac{-\chi}{n}} k$$

EXAMPLES

Droplets:

$$S_{\text{bulk}} = \int_{\text{int}(\Sigma)} \mathrm{d}^{d+1}x \; e_{\text{b}} P_{\text{int}} + \int_{\text{ext}(\Sigma)} \mathrm{d}^{d+1}x \; e_{\text{b}} P_{\text{ext}}$$

Young-Laplace law:

$$\mathcal{T}^{ab} K_{ab} = \chi K + n u^a u^b K_{ab} = -\Delta p$$

$$\Delta p = P_{\text{ext}} - P_{\text{int}}$$

EXAMPLES

Generalised Helfrich energy:

$$\mathcal{F}_{\text{CH}} = \int_{\Sigma_s} \mathrm{d}^p\sigma e_s \left[a_0(T,\mu) + a_1(T,\mu)K + a_2(T,\mu)K^2 + a_3(T,\mu)K\cdot K\right]$$

Bending moment:

$$\mathcal{D}^{ab} = a_1 h^{ab} + \mathcal{Y}^{abcd}K_{cd} \quad , \quad \mathcal{Y}^{abcd} = 2a_2 h^{ab}h^{cd} + 2a_3 h^{a(c}h^{d)b}$$

Original Helfrich-energy:

$$\mathcal{F}_{\text{CH}} = \int_{\Sigma_s} \mathrm{d}^2\sigma \; e_s\left[\chi + \kappa(K + c_0)^2\right]$$

$$a_0 = \chi + \kappa c_0^2 \quad , \quad a_1 = 2\kappa c_0 \quad , \quad a_2 = \kappa$$

OUTLOOK

PART IV

OUTLOOK

- Find solutions to the generalised Helfrich energy (rotating fluids)
- Include fluid corrections in equilibrium
- Stability of fluid membranes
- Full dissipative effective action/dynamics (EFT)
- Boost-breaking membranes, other phases of matter (liquid crystals, viscoelasticity, etc).

THANK YOU