Unification of Riemannian and non-Riemannian geometries via Double Field Theory

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talk based on

2012.07766 w/ Chris Blair, Gerben Oling

2008.03084 w/ Shigeki Sugimoto

1909.10711 w/ Kyoungho Cho

1707.03713 w/ Kevin Morand

and some earlier works

Prologue

- Ever since the birth of General Relativity, Riemannian geometry has been the mathematical paradigm for modern physics. The metric, $g_{\mu\nu}$, is privileged to be the only fundamental variable that provides a concrete tool to address the notion of 'spacetime'.

- However, string theory suggests to put a two-form gauge potential, $B_{\mu\nu}$, and a scalar dilaton, ϕ, on an equal footing along with the metric: Forming the closed string massless sector, they are ubiquitous in all string theories, and are transformed to one another under T-duality.

- Postulating $\mathbf{O}(D, D)$ symmetry as the fundamental principle, Double Field Theory, initiated by Siegel 1993; Hull, Zwiebach 2009, augments GR including the Einstein field equations in an unambiguous manner, geometrising or gravitising the whole closed string massless sector:

  DFT = gravitational theory that string theory predicts

- Besides, formulated a priori in terms of $\mathbf{O}(D, D)$ covariant variables, (S)DFT as well as doubled (super)string action describe not only the conventional Riemannian geometry but also non-Riemannian ones where the notion of Riemannian metric ceases to exist.

  Essentially, it is a matter of how one parametrises the $\mathbf{O}(D, D)$ covariant variables in terms of either Riemannian $\{g, B, \phi\}$ or alternatively non-Riemannian component fields.

$\mathbf{O}(D,D)$ Symmetry Principle

- Working hypothesis is to view an $\mathbf{O}(D,D)$ invariant metric, $\mathcal{J}_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and an $\mathbf{O}(D,D)$ covariant generalised metric, $\mathcal{H}_{MN}$, as **fundamental entities.**
- The generalised metric should satisfy defining properties:

$$\mathcal{H}_{MN} = \mathcal{H}_{NM}\,, \qquad\qquad \mathcal{H}_{M}{}^{K}\mathcal{H}_{N}{}^{L}\mathcal{J}_{KL} = \mathcal{J}_{MN}\,.$$

- Combing the two, we have a pair of projectors (orthogonal and complete),

$$P_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} + \mathcal{H}_{MN})\,, \qquad\qquad \bar{P}_{MN} = \tfrac{1}{2}(\mathcal{J}_{MN} - \mathcal{H}_{MN})\,,$$

- Further, taking the 'square root' of each projector,

$$P_{MN} = V_{M}{}^{p}V_{N}{}^{q}\eta_{pq}\,, \qquad\qquad \bar{P}_{MN} = \bar{V}_{M}{}^{\bar{p}}\bar{V}_{N}{}^{\bar{q}}\bar{\eta}_{\bar{p}\bar{q}}\,,$$

we obtain a pair of DFT-vielbeins for twofold local Lorentz symmetries, $\mathbf{Spin}(1,D{-}1)\times\mathbf{Spin}(D{-}1,1)$,

$$V_{Mp}V^{M}{}_{q} = \eta_{pq}\,, \qquad\qquad \bar{V}_{M\bar{p}}\bar{V}^{M}{}_{\bar{q}} = \bar{\eta}_{\bar{p}\bar{q}}\,, \qquad\qquad V_{Mp}\bar{V}^{M}{}_{\bar{q}} = 0\,.$$

$\Rightarrow$ $\mathcal{J}_{MN}$ and $\mathcal{H}_{MN}$ are simultaneously diagonalisable as $\mathbf{diag}(\eta,\bar{\eta})$ and $\mathbf{diag}(\eta,-\bar{\eta})$.

- Besides, there is an $\mathbf{O}(D,D)$ singlet dilaton, d, giving the integral measure, e^{-2d}.

We shall see $\exists$ various ways of parametrising these $\mathbf{O}(D,D)$ covariant fields: Riemann *vs.* non-Riemann.

Semi-covariant formalism

w/ Imtak Jeon and Kanghoon Lee 2010, 2011

- In GR, the Christoffel symbol is the unique metric-compatible connection, $\nabla_\lambda g_{\mu\nu} = 0$, which satisfies either a torsionless condition, or an alternative condition that the metric is the only ingredient to form the connection.

- Similarly, the connection in DFT can be uniquely fixed

$$\Gamma_{LMN}=2\big(P\partial_L P\bar{P}\big)_{[MN]}+2\big(\bar{P}_{[M}{}^{J}\bar{P}_{N]}{}^{K}-P_{[M}{}^{J}P_{N]}{}^{K}\big)\partial_J P_{KL}-\tfrac{4}{D-1}\big(\bar{P}_{L[M}\bar{P}_{N]}{}^{K}+P_{L[M}P_{N]}{}^{K}\big)\big(\partial_K d+(P\partial^J P\bar{P})_{[JK]}\big)$$

while the compatibility holds,

$$\nabla_L \mathcal{J}_{MN} = 0\,, \qquad \nabla_L \mathcal{H}_{MN} = 0\,, \qquad \nabla_L d = -\tfrac{1}{2}e^{2d}\nabla_L\left(e^{-2d}\right) = 0\,.$$

- Further, spin connections for twofold local Lorentz symmetries can be determined

$$\Phi_{Mpq} = V^{N}{}_{p}\nabla_M V_{Nq}\,, \qquad \bar{\Phi}_{M\bar{p}\bar{q}} = \bar{V}^{N}{}_{\bar{p}}\nabla_M \bar{V}_{N\bar{q}}$$

by requiring that a master derivative,

$$\mathcal{D}_M = \partial_M + \Gamma_M + \Phi_M + \bar{\Phi}_M = \nabla_M + \Phi_M + \bar{\Phi}_M$$

should be compatible with the vielbeins,

$$\mathcal{D}_M V_{Np} = \nabla_M V_{Np} + \Phi_{Mp}{}^{q} V_{Nq} = 0\,, \qquad \mathcal{D}_M \bar{V}_{N\bar{p}} = \nabla_M \bar{V}_{N\bar{p}} + \bar{\Phi}_{M\bar{p}}{}^{\bar{q}}\bar{V}_{N\bar{q}} = 0\,.$$

These spin connections are essentially the ‘generalised fluxes’ à la Aldazabala, Marques, Nunez, and Grana.

Semi-covariant formalism

w/ Imtak Jeon and Kanghoon Lee 2010, 2011

- Semi-covariant Riemann curvature :

$$S_{KLMN}=S_{[KL][MN]}=S_{MNKL}:=\tfrac{1}{2}\left(R_{KLMN}+R_{MNKL}-\Gamma^{J}{}_{KL}\Gamma_{JMN}\right)\,,\qquad S_{[KLM]N}=0\,,$$

where R_{ABCD} denotes the ordinary "field strength", $R_{CDAB}{=}\partial_A\Gamma_{BCD}-\partial_B\Gamma_{ACD}+\Gamma_{AC}{}^{E}\Gamma_{BED}-\Gamma_{BC}{}^{E}\Gamma_{AED}$.

By construction, it varies as $\delta S_{ABCD}=\nabla_{[A}\delta\Gamma_{B]CD}+\nabla_{[C}\delta\Gamma_{D]AB}$, hence good for variational principle.

- Semi-covariance means, with $\partial_M\partial^M=0$ and $\mathcal{P}_{LMN}{}^{EFG}=P_L{}^{E}P_{[M}{}^{[F}P_{N]}{}^{G]}+\frac{2}{P_K{}^{K}-1}P_{L[M}P_{N]}{}^{[F}P^{G]E}$,

$$\delta_\xi\left(\nabla_L T_{M_1\cdots M_n}\right)=\hat{\mathcal{L}}_\xi\left(\nabla_L T_{M_1\cdots M_n}\right)+\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{LM_i}{}^{NEFG}\partial_E\partial_F\xi_G\,T_{M_1\cdots M_{i-1}NM_{i+1}\cdots M_n}$$

$$\delta_\xi S_{KLMN}=\hat{\mathcal{L}}_\xi S_{KLMN}+2\nabla_{[K}\left[(\mathcal{P}+\bar{\mathcal{P}})_{L][MN]}{}^{EFG}\partial_E\partial_F\xi_G\right]+2\nabla_{[M}\left[(\mathcal{P}+\bar{\mathcal{P}})_{N][KL]}{}^{EFG}\partial_E\partial_F\xi_G\right]$$

$$\delta_\xi\Gamma_{CAB}=\hat{\mathcal{L}}_\xi\Gamma_{CAB}+2\left[(\mathcal{P}+\bar{\mathcal{P}})_{CAB}{}^{FDE}-\delta_C^{F}\delta_A^{D}\delta_B^{E}\right]\partial_F\partial_{[D}\xi_{E]}$$

where $\hat{\mathcal{L}}_\xi T_{M_1\cdots M_n}=\xi^N\partial_N T_{M_1\cdots M_n}+\omega_T\,\partial_N\xi^N T_{M_1\cdots M_n}+\sum_{i=1}^{n}(\partial_{M_i}\xi_N-\partial_N\xi_{M_i})T_{M_1\cdots M_{i-1}}{}^{N}{}_{M_{i+1}\cdots M_n}$.

- The red-colored anomalies can be easily projected out to give fully covariant objects, *e.g.*

$$\mathcal{D}_p T_{\bar{q}}=\nabla_L T_M V^L{}_p\bar{V}^M{}_{\bar{q}}\,,\qquad S_{p\bar{q}}=S_{MN}V^M{}_p\bar{V}^N{}_{\bar{q}}\ \ (\text{Ricci})\,,\qquad S_{(0)}=S_{pq}{}^{pq}-S_{\bar{p}\bar{q}}{}^{\bar{p}\bar{q}}\ \ (\text{scalar})$$

$$\gamma^p\mathcal{D}_p\rho\,,\ \mathcal{D}_{\bar{p}}\rho\ (\text{Dirac})\,,\ \ \mathcal{D}_\pm\mathcal{C}=\gamma^p\mathcal{D}_p\mathcal{C}\pm\gamma^{(D+1)}\mathcal{D}_{\bar{p}}\mathcal{C}\bar{\gamma}^{\bar{p}}\,,\ \ (\mathcal{D}_\pm)^2=0\ \ \Rightarrow\ \mathcal{F}=\mathcal{D}_+\mathcal{C}\ \ (\text{bispinorial RR})$$

$\mathbf{O}(D,D)$ symmetric 'minimal' coupling

- $D=10$, $\mathcal{N}=2$ SDFT (full order 32 SUSY) w/ Imtak Jeon, Kanghoon Lee, Yoonji Suh 1210.5078

$$\mathcal{L}_{\rm type\,II}=e^{-2d}\Big[\,\tfrac{1}{8}S_{(0)}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})+i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'$$
$$-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}}+i\bar{\psi}'^{p}\mathcal{D}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\,\Big]$$

which unifies IIA and IIB SUGRAs (Riemannian/non-Riemannian) as different solution sectors.

- $D=4$ DFT minimally coupled to the Standard Model w/ Kangsin Choi 1506.05277

$$\mathcal{L}_{\rm SM}\;=\;e^{-2d}\left[\begin{array}{l}\frac{1}{16\pi G_N}S_{(0)}+\sum_{A}{\rm Tr}(F_{p\bar{q}}F^{p\bar{q}})+\sum_{\psi}\bar{\psi}\gamma^{p}\mathcal{D}_{p}\psi+\sum_{\psi'}\bar{\psi}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'\\ -\mathcal{H}^{MN}(\mathcal{D}_{M}\phi)^{\dagger}\mathcal{D}_{N}\phi\;-\;V(\phi)\;+y_d\,\bar{q}\cdot\phi\,d+y_u\,\bar{q}\cdot\tilde{\phi}\,u+y_e\,\bar{l}'\cdot\phi\,e'\end{array}\right]$$

– Every single term above is completely covariant, w.r.t. $\mathbf{O}(D,D)$, diffeomorphisms, and twofold local Lorentz symmetries.

Einstein Double Field Equations

w/ Stephen Angus, Kyungho Cho 1804.00964

- Let us consider a DFT action coupled to generic matter, Υ_a,

$$\text{Action} = \int_\Sigma e^{-2d}\Big[\,\tfrac{1}{16\pi G}S_{(0)} + L_{\text{matter}}(\Upsilon_a, \mathcal{D}_M\Upsilon_b)\,\Big]$$

and its arbitrary variation by all the fields, $\delta d, \delta V_{Mp}, \delta\bar{V}_{M\bar{p}}, \delta\Upsilon_a$,

$$\delta\text{Action} = \int_\Sigma e^{-2d}\left[\tfrac{1}{4\pi G}\bar{V}^{M\bar{q}}\delta V_M{}^p(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{8\pi G}\delta d(S_{(0)} - 8\pi G T_{(0)}) + \delta\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right]$$

where we naturally set

$$K_{p\bar{q}} := \tfrac{1}{2}\left(V_{Mp}\frac{\delta L_{\text{matter}}}{\delta\bar{V}_M{}^{\bar{q}}} - \bar{V}_{M\bar{q}}\frac{\delta L_{\text{matter}}}{\delta V_M{}^p}\right) = -2V_{Mp}\bar{V}_{N\bar{q}}\frac{\delta L_{\text{matter}}}{\delta\mathcal{H}_{MN}}\,, \qquad T_{(0)} := e^{2d}\times\frac{\delta\left(e^{-2d}L_{\text{matter}}\right)}{\delta d}$$

- Like the General Covariance in GR, the diffeomorphic invariance of the DFT action,

$$0 = \int_\Sigma e^{-2d}\left[\tfrac{1}{8\pi G}\xi^N\mathcal{D}^M\left\{4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}(S_{p\bar{q}} - 8\pi G K_{p\bar{q}}) - \tfrac{1}{2}\mathcal{J}_{MN}(S_{(0)} - 8\pi G T_{(0)})\right\} + \delta_\xi\Upsilon_a\frac{\delta L_{\text{matter}}}{\delta\Upsilon_a}\right]$$

then guides us to identify the Einstein curvature, w/ S. Rey, W. Rim, Y. Sakatani 2015

$$G_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}S_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}S_{(0)}\,, \qquad \nabla_M G^{MN} = 0 \qquad \text{(off-shell)}$$

and the Energy-Momentum tensor,

$$T_{MN} := 4V_{[M}{}^p\bar{V}_{N]}{}^{\bar{q}}K_{p\bar{q}} - \tfrac{1}{2}\mathcal{J}_{MN}T_{(0)}\,, \qquad \nabla_M T^{MN} = 0 \qquad \text{(on-shell)}$$

- **Equating them, we obtain the Einstein equations in DFT:** $G_{MN} = 8\pi G T_{MN}$

Question: Is DFT a mere reformulation of SUGRA in an **O**(D, D) manifest manner?

The answer would be (and had been) yes, if we employ a well-known parametrisation, Giveon, Rabinovici, Veneziano '89, Duff '90

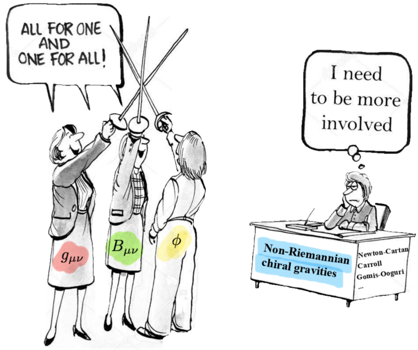
$$\mathcal{H}_{MN} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \qquad e^{-2d} = \sqrt{|g|}e^{-2\phi}$$

Upon this parametrisation, EDFEs, $G_{MN} = 8\pi G T_{MN}$, unify

$$R_{\mu\nu} + 2\nabla_\mu(\partial_\nu\phi) - \tfrac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma} = 8\pi G K_{(\mu\nu)}$$

$$e^{2\phi}\nabla^\rho\left(e^{-2\phi}H_{\rho\mu\nu}\right) = 16\pi G K_{[\mu\nu]}$$

$$R + 4\Box\phi - 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} = 8\pi G T_{(0)}$$



- However, the truth is that, DFT works perfectly fine with any generalised metric that satisfies the defining properties: $\mathcal{H}_{MN} = \mathcal{H}_{NM}$, $\mathcal{H}_M{}^K\mathcal{H}_N{}^L\mathcal{J}_{KL} = \mathcal{J}_{MN}$ (or the DFT-vielbeins for SDFT). And the above famous parametrisation is not the most general solution to them.

  Hence the answer to the question can be **negative.**

- Early non-Riemannian examples, followed by a complete classification, include

  i) $\mathcal{H} = \pm\mathcal{J} = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *ii)* T-dual of F1 over the two longitudinal directions w/ K. Lee 1307.8377

  iii) Gomis–Ooguri non-relativistic string flat background w/ S. Ko, C. Melby-Thompson, R. Meyer 1508.01121

Classification of DFT geometries

w/ Kevin Morand 1707.03713

The most general parametrisations of the generalised metric, $\mathcal{H}_{MN}=\mathcal{H}_{NM}$, $\mathcal{H}_M{}^K\mathcal{H}_N{}^L\mathcal{J}_{KL}=\mathcal{J}_{MN}$,

can be classified by two non-negative integers, $(n,\bar{n})$, $0\leq n+\bar{n}\leq D$:

$$\mathcal{H}_{MN} = \begin{pmatrix} H^{\mu\nu} & -H^{\mu\sigma}B_{\sigma\lambda}+Y_i^{\mu}X^i_{\lambda}-\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}^{\bar{\imath}}_{\lambda} \\ B_{\kappa\rho}H^{\rho\nu}+X^i_{\kappa}Y_i^{\nu}-\bar{X}^{\bar{\imath}}_{\kappa}\bar{Y}_{\bar{\imath}}^{\nu} & K_{\kappa\lambda}-B_{\kappa\rho}H^{\rho\sigma}B_{\sigma\lambda}+2X^i_{(\kappa}B_{\lambda)\rho}Y_i^{\rho}-2\bar{X}^{\bar{\imath}}_{(\kappa}B_{\lambda)\rho}\bar{Y}_{\bar{\imath}}^{\rho} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} H & Y_i(X^i)^T-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T \\ X^i(Y_i)^T-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T & K \end{pmatrix}\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}$$

i) Symmetric and skew-symmetric fields: $H^{\mu\nu}=H^{\nu\mu}$, $K_{\mu\nu}=K_{\nu\mu}$, $B_{\mu\nu}=-B_{\nu\mu}$;

ii) Two kinds of zero eigenvectors: with $i,j=1,2,\cdots,n$ and $\bar{\imath},\bar{\jmath}=1,2,\cdots,\bar{n}$,

$$H^{\mu\nu}X^i_{\nu}=0=H^{\mu\nu}\bar{X}^{\bar{\imath}}_{\nu}\,, \qquad K_{\mu\nu}Y_j^{\nu}=0=K_{\mu\nu}\bar{Y}_{\bar{\jmath}}^{\nu}\;;$$

iii) Completeness relation: $H^{\mu\rho}K_{\rho\nu}+Y_i^{\mu}X^i_{\nu}+\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}^{\bar{\imath}}_{\nu}=\delta^{\mu}{}_{\nu}$.

It follows that $Y_i^{\mu}X^j_{\mu}=\delta_i{}^j$, $\bar{Y}_{\bar{\imath}}^{\mu}\bar{X}^{\bar{\jmath}}_{\mu}=\delta_{\bar{\imath}}{}^{\bar{\jmath}}$, $Y_i^{\mu}\bar{X}^{\bar{\jmath}}_{\mu}=0=\bar{Y}_{\bar{\imath}}^{\mu}X^j_{\mu}$, $HKH=H$, and $KHK=K$.

Classification of DFT geometries

w/ Kevin Morand 1707.03713

$$\mathcal{H}_{(n,\bar{n})}=\begin{pmatrix}1&0\\B&1\end{pmatrix}\begin{pmatrix}H&Y_i(X^i)^T-\bar{Y}_{\bar{\imath}}(\bar{X}^{\bar{\imath}})^T\\X^i(Y_i)^T-\bar{X}^{\bar{\imath}}(\bar{Y}_{\bar{\imath}})^T&K\end{pmatrix}\begin{pmatrix}1&-B\\0&1\end{pmatrix},$$

$$H^{\mu\nu}X^i_\nu=0=H^{\mu\nu}\bar{X}^{\bar{\imath}}_\nu\,,$$
$$K_{\mu\nu}Y^\nu_j=0=K_{\mu\nu}\bar{Y}^\nu_{\bar{\jmath}}\,,$$
$$H^{\mu\rho}K_{\rho\nu}+Y^\mu_iX^i_\nu+\bar{Y}^\mu_{\bar{\imath}}\bar{X}^{\bar{\imath}}_\nu=\delta^\mu{}_\nu\,.$$

- $\mathcal{H}_{(n,\bar{n})}$ is invariant under

  i) local $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ rotations: with $R\in\mathbf{GL}(n)$ and $\bar{R}\in\mathbf{GL}(\bar{n})$,

$$X^i_\mu\;\rightarrow\;R^i{}_jX^j_\mu\,,\qquad Y^\mu_i\;\rightarrow\;Y^\mu_j\,R^{-1\,j}{}_i\,,\qquad \bar{X}^{\bar{\imath}}_\mu\;\rightarrow\;\bar{R}^{\bar{\imath}}{}_{\bar{\jmath}}\bar{X}^{\bar{\jmath}}_\mu\,,\qquad \bar{Y}^\mu_{\bar{\imath}}\;\rightarrow\;\bar{Y}^\mu_{\bar{\jmath}}\,\bar{R}^{-1\,\bar{\jmath}}{}_{\bar{\imath}}$$

  ii) 'Milne-shift' symmetries: with local parameters, $V_{\mu i},\,\bar{V}_{\nu\bar{\imath}}$,

$$Y^\mu_i\;\rightarrow\;Y^\mu_i+H^{\mu\nu}V_{\nu i}\,,\qquad \bar{Y}^\mu_{\bar{\imath}}\;\rightarrow\;\bar{Y}^\mu_{\bar{\imath}}+H^{\mu\nu}\bar{V}_{\nu\bar{\imath}}\,,$$

$$K_{\mu\nu}\;\rightarrow\;K_{\mu\nu}-2X^i_{(\mu}K_{\nu)\rho}H^{\rho\sigma}V_{\sigma i}-2\bar{X}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}+(X^i_\mu V_{\rho i}+\bar{X}^{\bar{\imath}}_\mu\bar{V}_{\rho\bar{\imath}})H^{\rho\sigma}(X^i_\nu V_{\sigma i}+\bar{X}^{\bar{\imath}}_\nu\bar{V}_{\sigma\bar{\imath}})\,,$$

$$B_{\mu\nu}\;\rightarrow\;B_{\mu\nu}-2X^i_{[\mu}V_{\nu]i}+2\bar{X}^{\bar{\imath}}_{[\mu}\bar{V}_{\nu]\bar{\imath}}+2X^i_{[\mu}\bar{X}^{\bar{\imath}}_{\nu]}\left(Y^\rho_i\bar{V}_{\rho\bar{\imath}}+\bar{Y}^\rho_{\bar{\imath}}V_{\rho i}+V_{\rho i}H^{\rho\sigma}\bar{V}_{\sigma\bar{\imath}}\right).$$

- The corresponding DFT-vielbeins, $\{V_{Mp},\bar{V}_{M\bar{p}}\}$, can also be easily obtained with the twofold local Lorentz symmetries identified as $\mathbf{O}(t{+}n,s{+}n)\times\mathbf{O}(s{+}\bar{n},t{+}\bar{n})$, for which $H^{\mu\nu}$ and $K_{\mu\nu}$ have the signature, $(t,s,n+\bar{n})$, for time, space, and non-Riemannian dimensions.
- In fact, $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ and the Milne-shift symmetries are parts of $\mathbf{O}(t{+}n,s{+}n)\times\mathbf{O}(s{+}\bar{n},t{+}\bar{n})$.
- The trace is given by $\mathcal{H}_M{}^M=2(n{-}\bar{n})$ which the $\mathbf{O}(D,D)$ rotations cannot change.

Classification of DFT geometries

w/ Kevin Morand 1707.03713

- The underlying coset is $\frac{\mathbf{O}(D,D)}{\mathbf{O}(t+n,s+n)\times\mathbf{O}(s+\bar{n},t+\bar{n})}$ with dimensions $D^2-(n-\bar{n})^2$

  Berman, Blair, and Otsuki 2019; w/ K. Cho 2019

- As we shall see later, string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions:

$$X^{i}_{\mu}\,\partial_{+}x^{\mu}(\tau,\sigma)=0\,,\qquad\qquad \bar{X}^{\bar{\imath}}_{\mu}\,\partial_{-}x^{\mu}(\tau,\sigma)=0\,.$$

I. $(n,\bar{n})=(0,0)$ corresponds to the Riemannian case or Generalized Geometry à la Hitchin.

II. $(n,\bar{n})\neq(0,0)$: Non-Riemannian, e.g.

- $(1,0)$ Newton–Cartan gravity, $\mathrm{d}s^2=-c^2\mathrm{d}t^2+\mathrm{d}\mathbf{x}^2$, $\lim\limits_{c\rightarrow\infty} g^{-1}$ is finite & degenerate
- $(D{-}1,0)$ ultra-relativistic Carroll gravity, $\mathrm{d}\tau^2=\mathrm{d}t^2-c^{-2}\mathrm{d}\mathbf{x}^2$, $\lim\limits_{c\rightarrow 0} g^{-1}$ is finite & degenerate
- $(1,1)$ Stringy/torsional Newton–Cartan including Gomis–Ooguri, $\lim\limits_{c\rightarrow\infty}\mathcal{H}_{(0,0)}=\mathcal{H}_{(1,1)}$

  Andringa, Bergshoeff, Gomis, de Roo 2012; Harmark, Hartong, Obers 2017 and many NL audiences; w/ Melby-Thompson, Meyer, Ko 2015; Blair 2019. DFT suggests $\mathbf{GL}(1)\times\mathbf{GL}(1)$, $\mathbf{Spin}(1,9)\times\mathbf{Spin}(9,1)$, and also explains $\lim_{c\rightarrow\infty}\mathrm{NS\text{-}NS}$ *a la* Bergshoeff, Lahnsteiner, Romano, Rosseel, Simsek 2021.

- $(D,0)$ and $(0,D)$ are uniquely given as $\mathcal{H}=\pm\mathcal{J}$ with the trivial coset, $\frac{\mathbf{O}(D,D)}{\mathbf{O}(D,D)}$.

  These two are the perfectly $\mathbf{O}(D,D)$-symmetric vacua of DFT with no moduli.

"$(0,0)$ *spacetime emerges after SSB of* $\mathbf{O}(D,D)$, *identifying* $\{g,B\}$ *as Nambu–Goldstone boson moduli.*"

Berman, Blair, and Otsuki 2019

Non-Riemannian parametrisations of DFT

w/ Kyungho Cho 1909.10711

- In principle, $G_{MN} = 8\pi G T_{MN}$ should govern all the dynamics of various non-Riemannian geometries. What remains to be done is to insert the $(n, \bar{n})$ parametrisations and to organise the expressions. Here, based on the (semi-)covariant formalism of DFT, we propose an undoubled upper-indexed covariant derivative, w.r.t. diffeomorphisms and $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$,

$$\mathbb{D}^{\mu} = H^{\mu\rho}\partial_{\rho} + \Omega^{\mu} + \Upsilon^{\mu} + \bar{\Upsilon}^{\mu} \,,$$

  which satisfies generalised compatibility relations,

$$\mathbb{D}^{\lambda}H^{\mu\nu} + 2Y_i^{(\mu}H^{\nu)\rho}\mathbb{D}^{\lambda}X_{\rho}^{i} + 2\bar{Y}_{\bar{\imath}}^{(\mu}H^{\nu)\rho}\mathbb{D}^{\lambda}\bar{X}_{\rho}^{\bar{\imath}} = 0\,, \qquad Y_i^{\rho}\mathbb{D}^{\mu}X_{\rho}^{j} = 0\,,$$

$$\mathbb{D}^{\lambda}K_{\mu\nu} + 2X^{i}_{(\mu}K_{\nu)\rho}\mathbb{D}^{\lambda}Y_i^{\rho} + 2\bar{X}^{\bar{\imath}}_{(\mu}K_{\nu)\rho}\mathbb{D}^{\lambda}\bar{Y}_{\bar{\imath}}^{\rho} = 0\,, \qquad \bar{Y}_{\bar{\imath}}^{\rho}\mathbb{D}^{\mu}\bar{X}_{\rho}^{\bar{\jmath}} = 0$$

  and enables us to express the DFT action:

$$\int e^{-2d}S_{(0)}\Big|_{(n,\bar{n})} = \int e^{-2d}\Big[R - \tfrac{1}{12}H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}\mathbb{H}_{\lambda\mu\nu}\mathbb{H}_{\rho\sigma\tau} - \mathbb{H}_{\lambda\mu\nu}H^{\lambda\rho}\big(Y_i^{\mu}\mathbb{D}^{\nu}X_{\rho}^{i} - \bar{Y}_{\bar{\imath}}^{\mu}\mathbb{D}^{\nu}\bar{X}_{\rho}^{\bar{\imath}}\big) + 4K_{\mu\nu}\mathbb{D}^{\mu}d\,\mathbb{D}^{\nu}d\Big]$$

  c.f. the usual *i.e.* Riemannian NS-NS sugra and also D. Gallegos, U. Gürsoy, S. Verma, N. Zinnato 2020

  We also identify a diffeomorphism covariant, $\mathbf{GL}(n)\times\mathbf{GL}(\bar{n})$ and Milne-shift invariant $\mathbb{H}$-flux,

$$\widehat{\mathbb{H}}^{\lambda\mu\nu} := H^{\lambda\rho}H^{\mu\sigma}H^{\nu\tau}\mathbb{H}_{\rho\sigma\tau} + 6H^{\rho[\lambda}Y_i^{\mu}\mathbb{D}^{\nu]}X_{\rho}^{i} - 6H^{\rho[\lambda}\bar{Y}_{\bar{\imath}}^{\mu}\mathbb{D}^{\nu]}\bar{X}_{\rho}^{\bar{\imath}}\,.$$

- However, analysis of infinitesimal variations $\delta\mathcal{H}_{MN}$ around a generic $(n, \bar{n})$ background shows that $\delta\mathcal{H}_{MN}$'s include $n\times\bar{n}$ number of degrees which can decrease the 'non-Riemannianity', e.g. $(n, \bar{n}) \rightarrow (n-1, \bar{n}-1)$, allowing Riemannian spacetime to emerge. If we keep $(n, \bar{n})$ fixed, $n\times\bar{n}$ number of EDFEs will be missing. *c.f.* Bergshoeff, Lahnsteiner, Romano, Rosseel, Simsek 2021
  This seems to suggest that, various non-Riemannian gravities with $n\times\bar{n} \neq 0$ should better be identified as different solution sectors of DFT rather than viewed as independent theories.

Non-Riemannian isometries

w/ Chris Blair and Gerben Oling 2012.07766

- Analysing the DFT Killing equations, $\hat{\mathcal{L}}_\xi \mathcal{H}_{MN} = 8\bar{P}_{(M}{}^{[K} P_{N)}{}^{L]} \nabla_K \xi_L = 0$, we may address the notion of Non-Riemannian isometries. A constant $(n, \bar{n})$ generalised metric is generically given by a direct product of $\mathcal{H}_{(0,0)}$ and $\mathbf{O}(n,n)$, $\mathbf{O}(\bar{n},\bar{n})$ invariant metrics, $\mathcal{H} = \pm\mathcal{J}$.

$$\mathcal{H}_{AB} = \begin{pmatrix} \eta^{ab} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta^i{}_j & 0 \\ 0 & 0 & 0 & 0 & 0 & -\delta^{\bar{\imath}}{}_{\bar{\jmath}} \\ 0 & 0 & 0 & \eta_{ab} & 0 & 0 \\ 0 & \delta_i{}^j & 0 & 0 & 0 & 0 \\ 0 & 0 & -\delta_{\bar{\imath}}{}^{\bar{\jmath}} & 0 & 0 & 0 \end{pmatrix}$$

<u>Killing vector, $\xi_M = (\xi^\mu, \lambda_\nu)$</u>

$$\xi^a = w^a{}_b x^b + \zeta^a(x^j) + \bar{\zeta}^a(\bar{x}^{\bar{\jmath}}),$$

$$\xi^i = \zeta^i(x^j), \qquad \bar{\xi}^{\bar{\imath}} = \bar{\zeta}^{\bar{\imath}}(\bar{x}^{\bar{\jmath}}),$$

$$\lambda_a = \zeta_a(x^j) - \bar{\zeta}_a(\bar{x}^{\bar{\jmath}}),$$

$$\lambda_i = \rho_i(x^j), \qquad \bar{\lambda}_{\bar{\imath}} = \bar{\rho}_{\bar{\imath}}(\bar{x}^{\bar{\jmath}}).$$

where we have set the coordinates to read $x^\mu = (x^a, x^i, \bar{x}^{\bar{\imath}})$. The appearance of the arbitrary functions of x^j or $\bar{x}^{\bar{\jmath}}$ means the supertranslational nature of the non-Riemannian isometries.

Duvel 1993; Batlle, Gomis, and Not 2016; Bergshoeff, Gomis, Rosseel, Simsek, and Yan 2019

- For consistency, the Killing spinors in SDFT also depend arbitrarily on the non-Riemannian directions, leading to 'supersupersymmetries' that square to the above supertranslations.

Section condition = Doubled-yet-gauged

JHP 1304.5946

- DFT necessarily imposes the section condition for $x^M = (\tilde{x}_\mu, x^\nu)$,

$$\partial_M \partial^M = \partial_\mu \tilde{\partial}^\mu + \tilde{\partial}^\mu \partial_\mu = 0$$

which can be generically solved by letting $\tilde{\partial}^\mu = 0$, up to global $\mathbf{O}(D,D)$ rotations.

- The section condition is mathematically equivalent to a certain translational invariance:

$$\Phi_s(x) = \Phi_s(x + \Delta)\,, \qquad \Delta^M = \Phi_t \partial^M \Phi_u\,,$$

where $\Phi_s, \Phi_t, \Phi_u \in \{\, d, \mathcal{H}_{MN}\,, \xi^M\,, \cdots \}$, arbitrary functions appearing in DFT, and Δ^M is said to be 'derivative-index-valued'.

  - ► 'Physics' should be invariant under such a shift of the doubled coordinates, suggesting

> The doubled coordinates are gauged by derivative-index-valued shifts, satisfying $\Delta^M \partial_M = 0$,
>
> $$x^M \;\sim\; x^M + \Delta^M(x) \quad : \quad \text{Coordinate Gauge Symmetry}$$
>
> Each equivalence class, or gauge orbit in $\mathbb{R}^{D+D}$, corresponds to a single physical point in $\mathbb{R}^D$.

- With $\tilde{\partial}^\mu = 0$ and $\Delta^M = c_\mu \partial^M x^\mu$, we note $(\tilde{x}_\mu\,,\, x^\nu) \;\sim\; (\tilde{x}_\mu + c_\mu\,,\, x^\nu)$.
  $\mathbf{O}(D,D)$ then rotates the gauged directions and hence the section.

c.f. Alfonsi 2019, 2020 for formal discussion

Section condition = Doubled-yet-gauged

JHP 1304.5946

- In DFT, the usual coordinate basis of one-forms, $\mathrm{d}x^{A}$, is not covariant:
  - Neither diffeomorphic covariant,

$$\delta x^{M}=\xi^{M}\,,\qquad\qquad \delta(\mathrm{d}x^{M})=\mathrm{d}x^{N}\partial_{N}\xi^{M}\ \neq\ \mathrm{d}x^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})$$

  - Nor invariant under the coordinate gauge symmetry,

$$\mathrm{d}x^{M}\quad\longrightarrow\quad\mathrm{d}\big(x^{M}+\Delta^{M}\big)\ \neq\ \mathrm{d}x^{M}\,.$$

► The naive contraction, $\mathrm{d}x^{M}\mathrm{d}x^{N}\mathcal{H}_{MN}$, is not an invariant scalar nor proper length.

- These problems can be all cured by gauging the one-forms, $\mathrm{d}x^{A}$, explicitly,

$$Dx^{M}:=\mathrm{d}x^{M}-\mathcal{A}^{M}\,,\qquad\qquad \mathcal{A}^{M}\partial_{M}=0\quad\text{(derivative-index-valued)}\,.$$

Dx^{M} is covariant:

$$\delta x^{M}=\Delta^{M}\,,\quad \delta\mathcal{A}^{M}=\mathrm{d}\Delta^{M}\qquad\Longrightarrow\qquad \delta(Dx^{M})=0\,;$$

$$\delta x^{M}=\xi^{M}\,,\quad \delta\mathcal{A}^{M}=\partial^{M}\xi_{N}(\mathrm{d}x^{N}-\mathcal{A}^{N})\quad\Longrightarrow\quad \delta(Dx^{M})=Dx^{N}(\partial_{N}\xi^{M}-\partial^{M}\xi_{N})\,.$$

– Concretely, setting $\tilde{\partial}^{\mu}=0$ and $\mathcal{A}^{M}=A_{\lambda}\partial^{M}x^{\lambda}=(A_{\mu},0)$, we get $Dx^{M}=(\mathrm{d}\tilde{x}_{\mu}-A_{\mu},\mathrm{d}x^{\nu})\,.$

Proper Length

w/ S. Ko, M. Suh 2016 and w/ T. Basile, E. Joung 2019

- With $Dx^M = \mathrm{d}x^M - \mathcal{A}^M$, we may define a proper length in DFT, through a path integral,

$$\textbf{Proper Length} := -\ln\left[\int \mathcal{D}\mathcal{A}\ \exp\left(-\int\sqrt{Dx^M Dx^N \mathcal{H}_{MN}}\right)\right].$$

– With $\tilde{\partial}^\mu = 0$, $\mathcal{A}^M = (A_\mu, 0)$, and the decomposition, $A_\mu = (KH + X^i Y_i + \bar{X}^{\bar{\imath}} \bar{Y}_{\bar{\imath}})_\mu{}^\nu A_\nu$,

$$Dx^M Dx^N \mathcal{H}_{MN} = \mathrm{d}x^\mu \mathrm{d}x^\nu K_{\mu\nu} + \left[\mathrm{d}\tilde{x}_\mu - B_{\mu\kappa}\mathrm{d}x^\kappa - (KHA)_\mu\right]\left[\mathrm{d}\tilde{x}_\nu - B_{\nu\lambda}\mathrm{d}x^\lambda - (KHA)_\nu\right] H^{\mu\nu}$$
$$+ 2X^i_\mu \mathrm{d}x^\mu\left[\mathrm{d}\tilde{x}_\nu - B_{\nu\rho}\mathrm{d}x^\rho - (X\cdot YA)_\nu\right] Y_i^\nu - 2\bar{X}^{\bar{\imath}}_\mu \mathrm{d}x^\mu\left[\mathrm{d}\tilde{x}_\nu - B_{\nu\rho}\mathrm{d}x^\rho - (\bar{X}\cdot\bar{Y}A)_\nu\right]\bar{Y}_{\bar{\imath}}^\nu$$

– Essentially, $(KHA)_\mu$ leads to Gaussian integral, while $(X\cdot YA)_\nu$ and $(\bar{X}\cdot\bar{Y}A)_\mu$ are Lagrange multipliers to freeze the non-Riemannian dimensions: $X^i_\mu \mathrm{d}x^\mu = 0$, $\bar{X}^{\bar{\imath}}_\mu \mathrm{d}x^\mu = 0$

The **Proper Length** then reduces to a rather familiar form, $\int\sqrt{\mathrm{d}x^\mu \mathrm{d}x^\nu K_{\mu\nu}(x)}$, which is independent of $\tilde{x}_\mu$. Hence, it measures the distance between two gauge orbits, as desired.

- This line of thought readily leads to an **O**(D,D) symmetric particle action (Faddeev–Popov),

$$S_{\mathrm{particle}} = \int \mathrm{d}\tau\ \tfrac{1}{2}e^{-1}D_\tau x^M D_\tau x^N \mathcal{H}_{MN}(x) - \tfrac{1}{2}m^2 e + k_M\mathcal{A}^M + k(e-1) + \tfrac{1}{2}\theta_M\dot{\theta}^M + \sum_{a=1}^{2}\tfrac{1}{2}\vartheta_a\dot{\vartheta}^a$$

where $\theta^M = (C_\mu, B^\nu)$ and $\vartheta^a = (c, b)$. This is a constrained system, and the relevant Dirac bracket coincides with the graded Poisson bracket introduced by Deser and Sämann 2016.

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$$Dx^M Dx^N \mathcal{H}_{MN} = \mathrm{d}x^\mu \mathrm{d}x^\nu K_{\mu\nu} + \left[\mathrm{d}\tilde{x}_\mu - B_{\mu\kappa}\mathrm{d}x^\kappa - (KHA)_\mu\right]\left[\mathrm{d}\tilde{x}_\nu - B_{\nu\lambda}\mathrm{d}x^\lambda - (KHA)_\nu\right] H^{\mu\nu}$$
$$+\, 2X^i_\mu \mathrm{d}x^\mu \left[\mathrm{d}\tilde{x}_\nu - B_{\nu\rho}\mathrm{d}x^\rho - (X\cdot YA)_\nu\right] Y_i^\nu - 2\bar{X}^{\bar{\imath}}_\mu \mathrm{d}x^\mu \left[\mathrm{d}\tilde{x}_\nu - B_{\nu\rho}\mathrm{d}x^\rho - (\bar{X}\cdot\bar{Y}A)_\nu\right] \bar{Y}_{\bar{\imath}}^\nu$$

– Essentially, $(KHA)_\mu$ leads to Gaussian integral, while $(X\cdot YA)_\nu$ and $(\bar{X}\cdot\bar{Y}A)_\mu$ are Lagrange multipliers to freeze the non-Riemannian dimensions: $X^i_\mu \mathrm{d}x^\mu = 0$, $\bar{X}^{\bar{\imath}}_\mu \mathrm{d}x^\mu = 0$

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- This line of thought readily leads to an $\mathbf{O}(D,D)$ symmetric particle action (Faddeev–Popov),

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where $\theta^M = (C_\mu, B^\nu)$ and $\vartheta^a = (c, b)$. This is a constrained system, and the relevant Dirac bracket coincides with the graded Poisson bracket introduced by Deser and Sämann 2016.

Doubled-yet-gauged (super)string

- The formalism extends to string: Chris Hull 2006; w/ Kanghoon Lee 2013

$$S_{\rm string} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}D_\alpha x^M D_\beta x^N \mathcal{H}_{MN}(x) - \epsilon^{\alpha\beta}D_\alpha x^M \mathcal{A}_{\beta M}$$

which is manifestly $\mathbf{O}(D,D)$ symmetric, worldsheet diffeomorphism invariant, the coordinate gauge symmetry invariant, and doubled target spacetime diffeomorphism covariant as

$$\delta x^M = \xi^M\,,\quad \delta\mathcal{A}_{\alpha M} = D_\alpha x^N \partial^M \xi_N \quad\Longrightarrow\quad \delta S_{\rm string} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}D_\alpha x^M D_\beta x^N \hat{\mathcal{L}}_\xi \mathcal{H}_{MN}$$

Thus, any (supertranslational) Killing vectors induce (infinitely many) Noether symmetries.

Classically, upon a generic $(n,\bar{n})$ non-Riemannian backgrounds, after integrating out the auxiliary gauge potential —quadratic in $(KHA)_\mu$ and linear in $(X\cdot YA)_\mu, (\bar{X}\cdot\bar{Y}A)_\mu$—

$$S_{\rm string} \ \Rightarrow\ \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu K_{\mu\nu} + \tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu B_{\mu\nu} + \tfrac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha \tilde{x}_\mu \partial_\beta x^\mu$$

and string becomes chiral and anti-chiral over the n and $\bar{n}$ dimensions respectively,

$$X^i_\mu\left(\partial_\alpha x^\mu + \frac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right) = 0\,, \qquad \bar{X}^{\bar{\imath}}_\mu\left(\partial_\alpha x^\mu - \frac{1}{\sqrt{-h}}\epsilon_\alpha{}^\beta\partial_\beta x^\mu\right) = 0\,.$$

- Extension to κ-symmetric Green–Schwarz superstring unifies IIA & IIB JHP 1609.04265

$$\mathcal{S}_{\rm GS} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \ -\tfrac{1}{2}\sqrt{-h}h^{\alpha\beta}\Pi^M_\alpha \Pi^N_\beta \mathcal{H}_{MN} - \epsilon^{\alpha\beta}D_\alpha x^M\left(\mathcal{A}_{\beta M} - i\Sigma_{\beta M}\right)$$

where $\Pi^M_\alpha = D_\alpha x^M - i\Sigma^M_\alpha$, $\Sigma^M_\alpha = \bar{\theta}\gamma^M\partial_\alpha\theta + \bar{\theta}'\bar{\gamma}^M\partial_\alpha\theta'$. See also Chris Blair 1908.00074 for RNS

String quantization

w/ Shigeki Sugimoto 2008.03084

- BRST quantization on a constant $(n,\bar{n})$ background boils down to n pairs of chiral $\{\beta_i,\gamma^i\}$, $\bar{n}$ pairs of anti-chiral $\{\bar{\beta}_{\bar{\imath}},\bar{\gamma}^{\bar{\imath}}\}$, and ordinary (left-right combined) $D{-}n{-}\bar{n}$ number of x^a, s.t.

$$\mathbf{c}_{\mathbf{L}/\mathbf{R}}=D\pm(n-\bar{n})-26\quad\text{(bosonic string)}\,;\qquad \mathbf{c}_{\mathbf{L}/\mathbf{R}}=D\pm(n-\bar{n})-10\quad\text{(superstring)}$$

  These central charges should vanish. Thus, necessarily we require $n=\bar{n}$ and $D=26$ or 10.

- Furthermore, the BRST string spectrum agrees with the linearised EDFEs, $G_{MN}=0$.
  - Concretely for $n{+}\bar{n}=D$ (maximally non-Riemannian), the physical states consist of four sectors only:

$$\delta\mathcal{H}_{i\bar{\imath}}\,\gamma^i_{-1}|k_j\!\downarrow\rangle\otimes\bar{\gamma}^{\bar{\imath}}_{-1}|k_{\bar{\jmath}}\!\downarrow\rangle\,,\qquad \delta\mathcal{H}_i{}^{\bar{\imath}}\,\gamma^i_{-1}|k_j\!\downarrow\rangle\otimes\bar{\beta}_{-1\bar{\imath}}|k_{\bar{\jmath}}\!\downarrow\rangle$$
$$\delta\mathcal{H}^i{}_{\bar{\imath}}\,\beta_{-1i}|k_j\!\downarrow\rangle\otimes\bar{\gamma}^{\bar{\imath}}_{-1}|k_{\bar{\jmath}}\!\downarrow\rangle\,,\qquad \delta\mathcal{H}^{i\bar{\imath}}\,\beta_{-1i}|k_j\!\downarrow\rangle\otimes\bar{\beta}_{-1\bar{\imath}}|k_{\bar{\jmath}}\!\downarrow\rangle$$

    which should satisfy on-shell relations for $Q_{\mathbf{B}}$-closedness :

$$k_i\delta\mathcal{H}^i{}_{\bar{\imath}}=0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}_i{}^{\bar{\imath}}=0\,,\qquad k_i\delta\mathcal{H}^{i\bar{\imath}}=0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}^{i\bar{\imath}}=0$$

    and equivalence relations for $Q_{\mathbf{B}}$-exactness :

$$\delta\mathcal{H}^i{}_{\bar{\imath}}\sim\delta\mathcal{H}^i{}_{\bar{\imath}}-k_{\bar{\imath}}\xi^i\,,\qquad \delta\mathcal{H}_i{}^{\bar{\imath}}\sim\delta\mathcal{H}_i{}^{\bar{\imath}}+k_i\xi^{\bar{\imath}}\,,\qquad \delta\mathcal{H}_{i\bar{\imath}}\sim\delta\mathcal{H}_{i\bar{\imath}}+k_i\lambda_{\bar{\imath}}-k_{\bar{\imath}}\lambda_i$$

  - Remarkably, the $4n\bar{n}$ of $\{\delta\mathcal{H}_{i\bar{\imath}},\delta\mathcal{H}_i{}^{\bar{\imath}},\delta\mathcal{H}^i{}_{\bar{\imath}},\delta\mathcal{H}^{i\bar{\imath}}\}$ are precisely the moduli of the assumed maximally non-Riemannian generalised metric, while the $Q_{\mathbf{B}}$-closedness and the $Q_{\mathbf{B}}$-exactness match with the linearised EDFEs and the DFT-diffeomorphisms, *i.e.* $\hat{\mathcal{L}}_\xi\mathcal{H}_{MN}$, respectively.

In view of the supertranslational Killing isometries and also a classical intuition for chiral strings, $x^i(\tau,\sigma)=x^i(0,\tau+\sigma)$, namely that they are fixed in space and hardly interact, it would be worthwhile to investigate non-Riemannian geometries as an alternative to string compactifications, which might enlarge the string theory landscape far beyond the Riemannian paradigm.

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w/ Shigeki Sugimoto 2008.03084

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$$k_i\delta\mathcal{H}^{i}{}_{\bar{\imath}}=0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}_{i}{}^{\bar{\imath}}=0\,,\qquad k_i\delta\mathcal{H}^{i\bar{\imath}}=0\,,\qquad k_{\bar{\imath}}\delta\mathcal{H}^{i\bar{\imath}}=0$$

    and equivalence relations for $Q_{\mathbf{B}}$-exactness :

$$\delta\mathcal{H}^{i}{}_{\bar{\imath}}\sim\delta\mathcal{H}^{i}{}_{\bar{\imath}}-k_{\bar{\imath}}\xi^{i}\,,\qquad \delta\mathcal{H}_{i}{}^{\bar{\imath}}\sim\delta\mathcal{H}_{i}{}^{\bar{\imath}}+k_i\xi^{\bar{\imath}}\,,\qquad \delta\mathcal{H}_{i\bar{\imath}}\sim\delta\mathcal{H}_{i\bar{\imath}}+k_i\lambda_{\bar{\imath}}-k_{\bar{\imath}}\lambda_i$$

  - Remarkably, the $4n\bar{n}$ of $\{\delta\mathcal{H}_{i\bar{\imath}},\delta\mathcal{H}_{i}{}^{\bar{\imath}},\delta\mathcal{H}^{i}{}_{\bar{\imath}},\delta\mathcal{H}^{i\bar{\imath}}\}$ are precisely the moduli of the assumed maximally non-Riemannian generalised metric, while the $Q_{\mathbf{B}}$-closedness and the $Q_{\mathbf{B}}$-exactness match with the linearised EDFEs and the DFT-diffeomorphisms, *i.e.* $\hat{\mathcal{L}}_{\xi}\mathcal{H}_{MN}$, respectively.

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Thank you.