Double Field Theory and Geometric Quantisation based on 2101.12155 [hep-th] with Luigi Alfonsi

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Motivation and context

- We are living in a post-duality era. That is we are now aware, in so many different contexts of duality related perspectives and equivalences.
- Various formalisms reflecting duality symmetry have been developed. In supersymmetric quantum field theories the (0,2) 6d theory associated with the M-theory fivebrane is one example where there is a geometric construction of duality through dimensional reduction of a chiral theory.
- Double Field theory and Exceptional field theories are the supergravity versions of such a theory where there is a *generalised* geometric construction of duality in supergravity through dimensional reduction.
- There is much longer history of developing duality manifest formalisms, most notabley with quantum mechanics.

- Classical systems are invariant under canonical transformations.
- In the Hamiltonian formulation of mechanics this is manifest. It is also pleasantly geometric in that canonical transformation are symplectomorphisms of phase space.
- However, using the Hamiltonian form breaks Lorentz symmetry and so we are usual presented with a choice between manifest Lorentz symmetry or manifest canonical symmetry.
- It is worth remarking that duality symmetries: electromagnetic duality, T-duality etc. are all canonical symmetries!
- The development of Quantum mechanics even using the Hamiltonian form still *hid* canonical symmetries.
- Much work was done to try and restore this: Weyl and Moyal developed quantum mechanics on noncommutative phase space from which emerged deformation quantisation.

Geometric Quantisation is the attempt to put the “usual” approach to quantisation on a mathematical footing but also to bring out (and in fact extend) the symplectic structure. This is the work of Kostant and Souriau in the 1970s. I highly recommend the books by Woodhouse and Bates and Weinstein.

Plan:
Review geometric quantisation
Apply these techniques to the string
Compare with double field theory
Construct some interesting things along the way!

Geometric Quantisation

In Hamiltonian mechanics, a *classical system* $(\mathcal{P}, \omega, H)$ is defined by a symplectic manifold $(\mathcal{P}, \omega)$, describing the *phase space* of the system, with $\omega \in \Omega^2(M)$ a closed non-degenerate 2-form, called the symplectic form, and a smooth function $H \in \mathcal{C}^\infty(\mathcal{P})$, called the *Hamiltonian*.

Locally, $\omega = \mathrm{d}\theta$, where the local 1-form $\theta \in \Omega^1(U)$ is called Liouville potential. (The definition of the Liouville potential is gauge dependent) Now, given a path $\gamma : \mathbb{R} \to \mathcal{P}$ on the phase space, we define the Lagrangian $L_H \in \Omega^1(\mathbb{R})$ by

$$L_H \;=\; \gamma^*\theta - H\mathrm{d}\tau\,. \tag{0.1}$$

Where we denote the pull-back of the Liouville one-form θ to the curve γ by $\gamma^*\theta$. The action $S_H[\gamma(\tau)]$ associated to such a Lagrangian will be given by

$$S_H[\gamma(\tau)] \;=\; \int_{\mathbb{R}} (\gamma^*\theta - H\mathrm{d}\tau)\,. \tag{0.2}$$

Prequantum Bundle

Consider the Lie group $U(1)_\hbar := \mathbb{R}/2\pi\hbar\mathbb{Z}$. The *prequantum bundle* $\mathcal{Q} \twoheadrightarrow \mathcal{P}$ is defined as the principal $U(1)_\hbar$-bundle, whose first Chern class $\mathrm{c}_1(\mathcal{Q}) \in H^2(M,\mathbb{Z})$ is the image of the element $[\omega] \in H^2(M,\mathbb{R})$ of the de Rham cohomology group. We can now define the associated bundle $\mathcal{E} \twoheadrightarrow \mathcal{P}$ to the prequantum bundle with fibre $\mathbb{C}$, i.e.

$$\mathcal{E} \;:=\; \mathcal{Q} \times_{U(1)_\hbar} \mathbb{C}, \tag{0.3}$$

where the natural action $U(1)_\hbar \times \mathbb{C} \to \mathbb{C}$ is given by the map $(\phi, z) \mapsto e^{\frac{i}{\hbar}\phi} z$. Now, the *prequantum Hilbert space* of the system is defined by

$$\mathbf{H}_{\mathrm{pre}} \;:=\; \mathrm{L}^2(\mathcal{P},\mathcal{E}), \tag{0.4}$$

i.e. the Hilbert space of L^2-integrable sections of the bundle $\mathcal{E}$ on the base manifold $\mathcal{P}$. Whenever the first Chern class of $\mathcal{Q}$ is trivial, then the bundle $\mathcal{E} = \mathcal{P} \times \mathbb{C}$ is trivial and the prequantum Hilbert space reduces to $\mathbf{H}_{\mathrm{pre}} = \mathrm{L}^2(\mathcal{P};\mathbb{C})$.

Quantum bundle

Denote the tangent bundle of phase space by $T\mathcal{P}$.
A *polarisation* of the phase space $(\mathcal{P}, \omega)$ is a Lagrangian subbundle $L \subset T\mathcal{P}$, i.e. an n-dimensional subbundle of $T\mathcal{P}$ such that:

$$\omega|_L = 0$$

and $[V, W] \subset L$ for any pair of vectors $V, W \in L$.
Let us consider the square root bundle $\sqrt{\det(L)}$.
Then the *quantum Hilbert space* is defined by the space of sections:

$$\mathbf{H} \;:=\; \Big\{ \psi \in \mathrm{L}^2\big(\mathcal{P},\, \mathcal{E} \otimes \sqrt{\det(L)}\big) \;\Big|\; \nabla_V \psi = 0 \;\; \forall\, V \in L \Big\}. \quad (0.5)$$

If the Lagrangian subbundle L is integrable, we can write $L = T\mathcal{M}$ for some n-dimensional submanifold $\mathcal{M} \subset \mathcal{P}$ of the phase space.
Then, quantum states $|\psi\rangle \in \mathbf{H}$ can be uniquely chosen of the form

$$|\psi\rangle = \psi \otimes \sqrt{\mathrm{vol}_{\mathcal{M}}}, \quad (0.6)$$

To illustrate the ideas in this section lets look at a simple example with $(M, \omega) = (\mathbb{R}^{2n}, \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu)$, take $\theta = p_\mu \mathrm{d}x^\mu$ for the Liouville potential. We have two perpendicular polarisations defined by the Lagrangian fibrations $L_p := \mathrm{Span}\left(\frac{\partial}{\partial p_\mu}\right)$ and $L_x := \mathrm{Span}\left(\frac{\partial}{\partial x^\mu}\right)$. The covariant derivative is related to the Liouville potential which gives:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x^\mu}} &= \frac{\partial}{\partial x^\mu} - \frac{i}{\hbar} p_\mu \\ \nabla_{\frac{\partial}{\partial p_\mu}} &= \frac{\partial}{\partial p_\mu}. \end{aligned} \tag{0.7}$$

Therefore, for the polarisation L_p and L_x, we obtain respectively the sections

$$\begin{aligned} |\psi\rangle &= \psi(p) e^{-ip_\mu x^\mu} \otimes \sqrt{\mathrm{d}^n p} \\ |\psi\rangle &= \psi(x) \otimes \sqrt{\mathrm{d}^n x}, \end{aligned} \tag{0.8}$$

where $\sqrt{\mathrm{d}^n p}$ is the half form such that $\sqrt{\mathrm{d}^n p} \otimes \sqrt{\mathrm{d}^n p} = \mathrm{d}^n p$ and analogously for $\sqrt{\mathrm{d}^n x}$.

Canonical tranformations

A *symplectomorphism* between two manifolds $(\mathcal{P}, \omega) \xrightarrow{f} (\mathcal{P}', \omega')$ is a diffeomorphism $f : \mathcal{P} \to \mathcal{P}'$ which maps the symplectic form of the first manifold into the symplectic form of the second one, i.e. such that it satisfies $\omega = f^*\omega'$. What in Hamiltonian physics is known under the name of *canonical transformation* with generating function F is equivalently a symplectomorphism $f : (\mathcal{P}, \omega) \to (\mathcal{P}', \omega')$ such that the Liouville potential is gauge-transformed by $\theta - f^*\theta' = \mathrm{d}F$.

There exists a powerful way to formalise a canonical transformation by using the notion of *Lagrangian correspondence*. To define a Lagrangian correspondence we first need to introduce the *graph* of a symplectomorphism $f:(\mathcal{P},\omega)\to(\mathcal{P}',\omega')$, which is the submanifold of the product space $\mathcal{P}\times\mathcal{P}'$ given by

$$\Gamma_f \;:=\; \big\{(a,b)\in\mathcal{P}\times\mathcal{P}' \;\big|\; b=f(a)\big\}. \tag{0.9}$$

Let us call $\iota:\Gamma_f\hookrightarrow\mathcal{P}\times\mathcal{P}'$ the inclusion in the product space. The submanifold $\Gamma_f\subset\mathcal{P}\times\mathcal{P}'$ can be immediately recognised as a Lagrangian submanifold of $(\mathcal{P}\times\mathcal{P}',\,\pi^*\omega-\pi'^*\omega')$, i.e. the total symplectic form vanishes when restricted on Γ_f.

$$\iota^*\big(\pi^*\omega-\pi'^*\omega'\big) \;=\; 0. \tag{0.10}$$

To formalise a canonical transformation, we need to add another condition: the correspondence space $(\mathcal{P}\times\mathcal{P}',\,\pi^*\omega-\pi'^*\omega')$ must be symplectomorphic to a symplectic manifold $(T^*\mathcal{M},\,\omega_{\mathrm{can}})$. This implies that we can write the combination of Liouville potentials $\pi^*\theta-\pi'^*\theta'$ as the Liouville 1-form on $\mathcal{P}\times\mathcal{P}'\cong T^*\mathcal{M}$.

Consider a simple example. Let us start from symplectic manifolds which are cotangent bundles of configuration spaces, i.e. $\mathcal{P} = T^*M$ and $\mathcal{P}' = T^*M'$. Write the Liouville potential as

$$p_\mu \mathrm{d}x^\mu - p'_\mu \mathrm{d}x'^\mu \;=\; \mathrm{d}F \tag{0.11}$$

in local coordinates on the correspondence space $\mathcal{P} \times \mathcal{P}' = T^*(M \times M')$. We immediately notice that, in the notation of the previous paragraph, we have $\mathcal{M} := M \times M'$. Now the generating function $F = F(x, x')$ of the canonical transformation can be properly seen as the pullback of a function of the product manifold $M \times M'$.

$$p_\mu \;=\; \frac{\partial F}{\partial x^\mu}, \qquad p'_\mu \;=\; -\frac{\partial F}{\partial x'^\mu}. \tag{0.12}$$

In particular, If we choose $M, M' = \mathbb{R}^d$ and $F(x, x') = \delta_{\mu\nu} x^\mu x'^\nu$, we recover the symplectic linear transformation $(x, p) \mapsto f(x, p) = (p, -x)$.

Canonical transformation on the Hilbert space

We need to show how these symplectomorphisms give rise to isomorphisms of the corresponding quantum Hilbert spaces.
Let us call $\mathbf{H}_L$ and $\mathbf{H}_{L'}$ the quantum Hilbert spaces corresponding respectively to the L and L' polarisations of the phase space.
We can lift sections $\psi \in \mathbf{H}_L$ and $\psi' \in \mathbf{H}_{L'}$ to the Hilbert space $\mathbf{H}_{T\Gamma_f}$ and consider their products $\langle \pi^*\psi \,|\, \pi'^*\psi' \rangle$ in this space. This is then naturally defines a pairing $(\!(\cdot \,,\, \cdot)\!) : \mathbf{H}_L \times \mathbf{H}_{L'} \to \mathbb{C}$ between the two polarised Hilbert spaces given by

$$(\!(\cdot \,,\, \cdot)\!) \;:=\; \big\langle \pi^* \cdot \,\big|\, \pi'^* \cdot \big\rangle \tag{0.13}$$

such a pairing is equivalently a linear isomorphism $f^* : \mathbf{H}_{L'} \xrightarrow{\cong} \mathbf{H}_L$ such that

$$(\!(\cdot \,,\, \cdot)\!) \;=\; \langle \cdot \,|\, f^* \cdot \rangle \tag{0.14}$$

Looking at this in more mundane langauge gives:
The pairing given by

$$(\!(\psi, \psi')\!) \;=\; \int_{\mathcal{M}} \mathrm{d}^n x \, \mathrm{d}^n x' \, \psi^\dagger(x) \psi'(x') e^{-\frac{i}{\hbar} F(x,x')} \tag{0.15}$$

where we called $\mathcal{M}$ the manifold such that $T^*\mathcal{M} \cong \mathcal{P} \times \mathcal{P}$. Finally the isomorphism $f^* : \mathbf{H}_L \to \mathbf{H}_{L'}$ induced by the diffeomorphism f will be given in coordinates by

$$(f^*\psi')(x) \;=\; \int_{M'} \mathrm{d}^n x \, \psi'(x') e^{-\frac{i}{\hbar} F(x,x')} \tag{0.16}$$

For example, choose $M, M' = \mathbb{R}^n$ and let the symplectomorphism $f : (\mathbb{R}^{2n}, \mathrm{d}p_\mu \wedge \mathrm{d}x^\mu) \to (\mathbb{R}^{2n}, \mathrm{d}p'_\mu \wedge \mathrm{d}x'^\mu)$ be the linear transformation $f(x, p) = (p, -x)$. This is generated by generating function $F(x, x') = \delta_{\mu\nu} x^\mu x'^\nu$. Thus, if we substitute $(x', p') = f(x, p) = (p, -x)$, we recover that $(f^*)^{-1}$ is exactly the Fourier transformation of wave-functions:

$$\begin{aligned} (f^*\psi')(x) &= \int_{M'} \mathrm{d}^n p \, \psi'(p) e^{-\frac{i}{\hbar} p_\mu x^\mu} \\ \big((f^*)^{-1}\psi\big)(p) &= \int_M \mathrm{d}^n x \, \psi(x) e^{\frac{i}{\hbar} p_\mu x^\mu} \end{aligned} \tag{0.17}$$

Thus the same quantum state $|\psi\rangle \in \mathbf{H}$ can be represented as a wave-function $\langle x \,|\, \psi \rangle = \psi(x)$ or as its Fourier transform $\langle p \,|\, \psi \rangle = \psi(p)$ in the two basis $\{ \langle x| \}_{x \in M}$ and $\{ \langle p| \}_{p \in M'}$ given by the Lagrangian correspondence.

The String

Consider a surface of the form $\Sigma \simeq \mathbb{R} \times S^1$ with coordinates $\sigma \in [0, 2\pi)$ and $\tau \in \mathbb{R}$. The fields $X^\mu(\sigma, \tau)$ of the σ-model can now be seen as curves $\mathcal{C}^\infty(\mathbb{R}, \mathcal{L}M)$ on the *loop space* $\mathcal{L}M := \mathcal{C}^\infty(S^1, M)$ of the original manifold M. This will be denoted as follows:

$$\begin{aligned} \mathbb{R} &\hookrightarrow \mathcal{L}M \\ \tau &\longmapsto X^\mu(\sigma, \tau) \end{aligned} \tag{0.18}$$

where $X^\mu(\sigma, \tau)$ is a loop for any fixed $\tau \in \mathbb{R}$. In other words we have

$$\mathcal{C}^\infty(\mathbb{R}, \mathcal{L}M) \cong \mathcal{C}^\infty(\Sigma, M) \tag{0.19}$$

This is why the configuration space for the closed string can be identified with the free loop space $\mathcal{L}M$ of the spacetime manifold M.

The phase space of the closed string.

The phase space of a string on spacetime M will be the free loop space of T^*M. By definition, this can be used as a definition of the cotangent bundle of $\mathcal{L}M$, i.e.

$$T^*\mathcal{L}M \;:=\; \mathcal{L}(T^*M) \tag{0.20}$$

i.e. the smooth space of loops $(X(\sigma), P(\sigma))$ in the cotangent bundle of T^*M. This space comes equipped with a canonical symplectic form:

$$\Omega \;:=\; \oint \mathrm{d}\sigma\, \delta P_\mu(\sigma) \wedge \delta X^\mu(\sigma) \;\; \in \; \Omega^2(T^*\mathcal{L}M) \tag{0.21}$$

We can now define a Liouville potential Θ such that its derivative is the canonical symplectic form $\Omega \in \Omega^2(T^*\mathcal{L}M)$. Thus we have

$$\Theta \;:=\; \oint \mathrm{d}\sigma\, P_\mu(\sigma)\, \delta X^\mu(\sigma) \;\; \in \; \Omega^1(T^*\mathcal{L}U) \tag{0.22}$$

The Hamiltonian

Formally pack together the momentum $P(\sigma)$ and the derivative $X'(\sigma)$ in the following doubled vector:

$$\mathbb{P}^M(\sigma) := \begin{pmatrix} X'^{\mu}(\sigma) \\ P_\mu(\sigma) \end{pmatrix} \tag{0.23}$$

with $M = 1, \ldots, 2n$. Notice that $\mathbb{P}^M(\sigma)$ is uniquely defined at any given loop $(X(\sigma), P(\sigma))$ in the phase space.

Thus, we can rewrite the Hamiltonian of the string as

$$H[X(\sigma), P(\sigma)] = \oint \mathrm{d}\sigma \, \frac{1}{2} \mathbb{P}^M(\sigma) \, \mathcal{H}_{MN}(X(\sigma)) \, \mathbb{P}^N(\sigma) \tag{0.24}$$

where the matrix $\mathcal{H}_{MN}$ is defined by

$$\mathcal{H}_{\mathrm{MN}} := \begin{pmatrix} g_{\mu\nu} - B_{\mu\lambda} g^{\lambda\rho} B_{\rho\nu} & B_{\mu\lambda} g^{\mu\nu} \\ -g^{\mu\lambda} B_{\lambda\mu} & g^{\mu\nu} \end{pmatrix}. \tag{0.25}$$

Background gauge fields

Consider an ordinary particle, in presence of an electromagnetic field with a minimally coupled 1-form potential A, the canonical momentum p_μ which is defined from the Lagrangian perspective by $p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}$ is given by: $p_\mu = k_\mu + eA_\mu$. The Liouville potential is $\theta = k_\mu \mathrm{d}x^\mu + eA$, and so the symplectic form is $\omega = \mathrm{d}k_\mu \wedge \mathrm{d}x^\mu + eF$.

Notice that, in canonical coordinates, we have a Hamiltonian $H = g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu)$ and the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}_\mu, \hat{x}^\nu] = i\hbar\delta_\mu^\nu, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0. \tag{0.26}$$

On the other hand, in terms of the kinetic non-canonical coordinates, we have the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{k}_\mu, \hat{x}^\nu] = i\hbar\delta_\mu^\nu, \quad [\hat{k}_\mu, \hat{k}_\nu] = i\hbar eF_{\mu\nu}. \tag{0.27}$$

String background fields

Similarly to the charged particle, for a string, we find that the symplectic form can be expressed by

$$\Omega \;=\; \oint \mathrm{d}\sigma\, \delta\Big(K_\nu(\sigma) + B_{\mu\nu}\big(X(\sigma)\big)X'^{\mu}(\sigma)\Big) \wedge \delta X^\nu(\sigma) \qquad (0.28)$$

where $K_\nu(\sigma) := P_\nu(\sigma) - B_{\mu\nu}\big(X(\sigma)\big)X'^{\mu}(\sigma)$ is the *non-canonical momentum* of the string and $P_\mu(\sigma)$ is its canonical momentum. This we lead to a deformation of the algebra.

$$\begin{aligned}
\big[\hat{K}_\mu(\sigma),\, \hat{X}^\nu(\sigma')\big] \;&=\; 2\pi i\hbar\delta_\mu^{\;\nu}\,\delta(\sigma-\sigma'), \\
\big[\hat{X}^\mu(\sigma),\, \hat{X}^\nu(\sigma')\big] \;&=\; 0, \\
\big[\hat{K}_\mu(\sigma),\, \hat{K}_\nu(\sigma')\big] \;&=\; H_{\mu\nu\lambda}\big(X(\sigma)\big)X'^{\lambda}(\sigma)\,\delta(\sigma-\sigma').
\end{aligned} \qquad (0.29)$$

T-duality as a symplectomorphism.

We now interpret T-duality as a symplectomorphism of phase spaces of two closed strings of the form

$$\begin{aligned} f: \quad (T^*\mathcal{L}M,\, \Omega) &\longrightarrow (T^*\mathcal{L}\widetilde{M},\, \widetilde{\Omega}) \\ \big(X^\mu(\sigma), P_\mu(\sigma)\big) &\longmapsto \big(\widetilde{X}_\mu(\sigma), \widetilde{P}^\mu(\sigma)\big). \end{aligned} \tag{0.30}$$

By considering the generating functional

$$F[X(\sigma), \widetilde{X}(\sigma)] = \frac{1}{2}\oint \mathrm{d}\sigma \big(X'^\mu(\sigma)\widetilde{X}_\mu(\sigma) - X^\mu(\sigma)\widetilde{X}'_\mu(\sigma)\big) \tag{0.31}$$

we obtain exactly T-duality on the phase space:

$$P_\mu(\sigma) \;=\; \widetilde{X}'_\mu(\sigma), \qquad \widetilde{P}^\mu(\sigma) \;=\; X'^\mu(\sigma). \tag{0.32}$$

The Lagrangian correspondence space is then the loop space of the doubled space of DFT. We can notice that, in this simple case, the doubled space can be identified with the correspondence space of a topological T-duality over a base point.

T-duality as change of basis on the Hilbert space.

The Lagrangian correspondence induces a map of quantum Hilbert spaces

The expansions in different basses will be then related by the Fourier-like transformation $(f^*)^{-1}$ of string wave-functionals, given by

$$\widetilde{\Psi}[\widetilde{X}(\sigma)] = \int_{\mathcal{L}M} \mathcal{D}X(\sigma)\, e^{\frac{i}{\hbar}F[X(\sigma),\widetilde{X}(\sigma)]}\, \Psi[X(\sigma)] \tag{0.33}$$

We can also explicitly write the matrix of the change of basis on $\mathbf{H}$ by

$$\langle X(\sigma)|\widetilde{X}(\sigma)\rangle = e^{\frac{i}{\hbar}F[X(\sigma),\widetilde{X}(\sigma)]} \tag{0.34}$$

Interestingly, this isomorphism is naturally defined by lifting the polarised wave functionals $\Psi[X(\sigma)] \in \mathbf{H}_L$ and $\widetilde{\Psi}[\widetilde{X}(\sigma)] \in \mathbf{H}_{\widetilde{L}}$ to wave-functionals $\Psi[\mathbb{X}(\sigma)]$ on the doubled space and by considering their Hermitian product in the Hilbert space of the doubled space.

The phase space and the doubled space.

To describe doubled strings, we introduce new coordinates $\widetilde{X}_\mu(\sigma)$ which satisfy the equation $P_\mu(\sigma) = \widetilde{X}'_\mu(\sigma)$. Let us define the following doubled coordinates:

$$\mathbb{X}^M(\sigma) \;:=\; \begin{pmatrix} X^\mu(\sigma) \\ \widetilde{X}_\mu(\sigma) \end{pmatrix}. \tag{0.35}$$

Therefore, for a doubled string the doubled momentum $\mathbb{P}^M(\sigma)$ coincides with the derivative along the circle of the doubled position vector $\mathbb{X}^M(\sigma)$. Thus, instead of encoding the σ-model of the closed string by an embedding $(X^\mu(\sigma),\, P_\mu(\sigma))$ into the phase space, we can encode it by an embedding $\mathbb{X}^M(\sigma) = (X^\mu(\sigma),\, \widetilde{X}_\mu(\sigma))$ into a doubled position space. Our objective is, then, be able to reformulate a string wave-functional $\Psi[X^\mu(\sigma), P_\mu(\sigma)]$ in terms of doubled fields as a wave-functional of the form $\Psi\big[\mathbb{X}^M(\sigma)\big]$.

Let us define the following zero-modes of the doubled loop-space vectors:

$$\mathbb{x}^M := \frac{1}{2\pi}\oint \mathrm{d}\sigma\, \mathbb{X}^M(\sigma), \qquad \mathbb{p}^M := \frac{1}{2\pi\alpha'}\oint \mathrm{d}\sigma\, \mathbb{X}'^M(\sigma), \quad (0.36)$$

which, in components, read

$$\mathbb{x}^M = \begin{pmatrix} x^\mu \\ \tilde{x}_\mu \end{pmatrix}, \qquad \mathbb{p}^M = \begin{pmatrix} \tilde{p}^\mu \\ p_\mu \end{pmatrix} \equiv \begin{pmatrix} w^\mu \\ \tilde{w}_\mu \end{pmatrix} \quad (0.37)$$

By using the new coordinate $\widetilde{X}_\mu$, we can rewrite the action of a closed string by

$$S_{\mathrm{string}}[X(\sigma,\tau), P(\sigma,\tau)] = \frac{1}{2\pi\alpha'}\int \mathrm{d}\tau \oint \mathrm{d}\sigma \left(\dot{X}^\mu \widetilde{X}'_\mu - \frac{1}{2}\mathbb{X}'^M \mathcal{H}_{MN}\, \mathbb{X}'^N \right) \quad (0.38)$$

Let us use the following notation for the derivatives

$$\dot{\mathbb{X}}(\sigma,\tau) := \frac{\partial \mathbb{X}(\sigma,\tau)}{\partial \tau} \qquad \mathbb{X}'(\sigma,\tau) := \frac{\partial \mathbb{X}(\sigma,\tau)}{\partial \sigma} \tag{0.39}$$

Since in the action of the closed string the field $\mathbb{X}^M(\sigma)$ never appears, but only its derivatives $\mathbb{X}'^M(\sigma)$, we only need to require that the latter are periodic, i.e.

$$\mathbb{X}'^M(\sigma + 2\pi) = \mathbb{X}'^M(\sigma) \tag{0.40}$$

This implies that the generalised boundary conditions are

$$\mathbb{X}^M(\sigma + 2\pi, \tau) = \mathbb{X}^M(\sigma,\tau) + 2\pi\alpha' \mathbb{p}^M(\tau), \tag{0.41}$$

where the quasi-period $\mathbb{p}^M(\tau)$ can, in general, be dynamical and depend on proper time.

Let us define the *quasi-loop space* $\mathcal{L}_{\mathrm{Q}}\mathcal{M}$ of a manifold $\mathcal{M}$ as it follows:

$$\mathcal{L}_{\mathrm{Q}}\mathcal{M} \;:=\; \big\{\, \mathbb{X} : [0, 2\pi) \to \mathcal{M} \;\big|\; \mathrm{d}\mathbb{X}(2\pi) = \mathrm{d}\mathbb{X}(0) \,\big\} \qquad (0.42)$$

The phase space of the doubled string will be a symplectic manifold $(\mathcal{L}_{\mathrm{Q}}\mathcal{M}, \mathbb{\Omega})$ where the symplectic form $\mathbb{\Omega} \in \Omega^2(\mathcal{L}_{\mathrm{Q}}\mathcal{M})$ will be determined.

The duality augmented abbreviated action is:

$$S_{abb} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \oint \mathrm{d}\sigma \left(\dot{X}^\mu P_\mu + \dot{\tilde{X}}_\mu \widetilde{P}^\mu \right) \tag{0.43}$$

which becomes

$$S_{abb} = \int \mathrm{d}\tau \oint \mathrm{d}\sigma \, \frac{1}{4\pi\alpha'} \left(\dot{\mathbb{X}}^M \eta_{MN} \mathbb{X}'^N \right) \ . \tag{0.44}$$

Combining this with the Hamiltonian to produce the total action $S = S_{abb} - H$ gives the Tseytlin action:

$$S_{\mathrm{Tsey}}[\mathbb{X}(\sigma,\tau)] \ = \ \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \oint \mathrm{d}\sigma \left(\dot{\mathbb{X}}^M \eta_{MN} \mathbb{X}'^N - \mathbb{X}'^M \mathcal{H}_{MN} \mathbb{X}'^N \right) \ . \tag{0.45}$$

Now, one can ask about the role of the quasi periodic boundary conditions, in fact being careful with boundary peices gives rise to an extra term (see paper for the discussion on this):

$$S_{\mathrm{abb}} \ = \ S_{\mathrm{Tsey,abb}} + \int \mathrm{d}\tau \, \frac{\pi\alpha'}{2} \dot{\mathbb{p}}^M \omega_{MN} \mathbb{p}^N \, . \tag{0.46}$$

Note that, $\dot{\mathbb{p}}^M = 0$ on toroidal backgrounds so its absence would not be spotted in usual toroidal T-duality.

To get the symplectic form of the string we use:

$$\mathbb{S}[\mathbb{X}(\sigma,\tau)] + \int \mathrm{d}\tau H[\mathbb{X}(\sigma,\tau)] \;=\; \int \mathrm{d}\tau \, \iota_{V_H} \Theta \qquad (0.47)$$

to determine the Liouville potential Θ on the phase space of the doubled string and, hence, the its symplectic structure.

$$\mathbb{S}[\mathbb{X}(\sigma,\tau)] + \int \mathrm{d}\tau H[\mathbb{X}(\sigma,\tau)]$$
$$= \int \mathrm{d}\tau \left(\mathbb{p}_M \dot{\mathbb{x}}^M - \frac{\pi\alpha'}{2}\omega^{MN}\mathbb{p}_M\dot{\mathbb{p}}_N + i\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\,\omega_{MN}\,\dot{\alpha}^M_{-n}\alpha^N_n \right) \qquad (0.48)$$

So:

$$\mathbb{\Theta} \;=\; \mathbb{p}_M \mathrm{d}\mathbb{x}^M - \frac{\pi\alpha'}{2}\omega^{MN}\mathbb{p}_M \mathrm{d}\mathbb{p}_N + i\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n}\,\omega_{MN}\,\mathrm{d}\mathbb{\alpha}^M_{-n}\mathbb{\alpha}^N_n\,. \tag{0.49}$$

By calculating the differential $\mathbb{\Omega} = \delta\mathbb{\Theta}$, we finally obtain the symplectic form

$$\mathbb{\Omega} \;=\; \mathrm{d}\mathbb{p}_M\wedge\mathrm{d}\mathbb{x}^M - \frac{\pi\alpha'}{2}\omega^{MN}\mathrm{d}\mathbb{p}_M\wedge\mathrm{d}\mathbb{p}_N + i\sum_{n\in\mathbb{N}\setminus\{0\}} \frac{1}{n}\,\omega_{MN}\,\mathrm{d}\mathbb{\alpha}^M_{-n}\wedge\mathrm{d}\mathbb{\alpha}^N_n\,. \tag{0.50}$$

This is, therefore, the symplectic form of the phase space $(\mathcal{L}_{\mathrm{Q}}\mathcal{M},\, \mathbb{\Omega})$ of the doubled string.

From this we can get the algebra of observables:

$$\left[\hat{\mathbb{X}}^M(\sigma)\,,\hat{\mathbb{X}}^N(\sigma')\right] \;=\; i\pi\hbar\alpha'\omega^{MN} - i\hbar\eta^{MN}\varepsilon(\sigma-\sigma') \qquad (0.51)$$

where the function $\varepsilon(\sigma)$ is the quasi-periodic function defined by

$$\varepsilon(\sigma) \;:=\; \sigma - i\sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{e^{in\sigma}}{n} \qquad (0.52)$$

and it satisfies the following properties: firstly, its derivative $\varepsilon'(\sigma)=\delta(\sigma)$ is the Dirac comb; secondly, it satisfies the boundary condition $\varepsilon(\sigma+2\pi n)=\varepsilon(\sigma)+2\pi n$ and, finally, it is an odd function, i.e. $\varepsilon(-\sigma)=-\varepsilon(\sigma)$.

The zero-modes of a doubled string $\mathbb{X}^M(\sigma,\tau)$ can be thought as a particle in a doubled phase space $(\mathbb{x}^M(\tau), \mathbb{p}_M(\tau))$. Similarly we expect that the wave-functional $\Psi[\mathbb{X}(\sigma)]$ at zero modes is just a wave-function $\psi(\mathbb{x},\mathbb{p})$ on the doubled phase space of zero modes:

$$\Psi[\mathbb{X}(\sigma)] \xrightarrow{\text{0 modes}} \psi(\mathbb{x},\mathbb{p}), \qquad \Omega \xrightarrow{\text{0 modes}} \omega \tag{0.53}$$

The phase space of the zero-modes of a doubled string is, therefore, a $4n$-dimensional symplectic manifold $(\mathcal{P}, \omega)$ with symplectic form

$$\omega \;=\; \eta_{MN}\, \mathrm{d}\mathbb{p}^M \wedge \mathrm{d}\mathbb{x}^N - \frac{\pi\alpha'}{2}\omega_{MN}\, \mathrm{d}\mathbb{p}^M \wedge \mathrm{d}\mathbb{p}^N \tag{0.54}$$

and underlying smooth manifold $\mathcal{P} = \mathbb{R}^{4n}$.
Now, we can apply the machinery of geometric quantisation to this symplectic manifold $(\mathcal{P}, \omega)$ to quantise the zero-modes of a doubled string.

The algebra of zero modes

Explicitly, in undoubled notation, we have the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{x}^\mu, \hat{\tilde{x}}_\nu] = \pi i\hbar\alpha' \delta^\mu_\nu, \quad [\hat{\tilde{x}}_\mu, \hat{\tilde{x}}_\nu] = -2\pi i\hbar\alpha' B_{\mu\nu},$$
$$[\hat{k}_\mu, \hat{k}_\nu] = 0, \quad [\hat{k}_\mu, \hat{\tilde{k}}^\nu] = 0, \quad [\hat{\tilde{k}}^\mu, \hat{\tilde{k}}^\nu] = 0,$$
$$[\hat{x}^\mu, \hat{\tilde{k}}^\nu] = [\hat{\tilde{x}}_\mu, \hat{k}_\nu] = 0, \quad [\hat{x}^\mu, \hat{k}_\nu] = i\hbar\delta^\mu_\nu, \quad [\hat{\tilde{x}}_\mu, \hat{\tilde{k}}^\nu] = i\hbar\delta_\mu^{\ \nu}. \tag{0.55}$$

Examining this algebra from the perspective of the limits we discussed earlier we see that $\hbar$ controls the noncommutativity of the position with the momentum and $\hbar\alpha'$ the noncommutativity of the coordinates and their duals. Finally, $\alpha' B$ the noncommutativity of the spacetime coordinates.

Following standard text book techniques we can immediately show that any position coordinate x^μ and its dual $\tilde{x}_\mu$ satisfy the following uncertainty relation:

$$\Delta x \, \Delta \tilde{x} \;\geq\; \frac{\pi \hbar}{2} \alpha'. \tag{0.56}$$

This means that x^μ and $\tilde{x}_\mu$ cannot be measured with absolute precision at the same time, but there will be always a minimum uncertainty proportional to the area $\hbar\alpha'$. This provides support to the intuition of a minimal distance scale in string theory. The standard lore is that for small distances one goes to the T-dual frame and the distances will always be larger than the string scale. In addition, both the couples (x, p) and $(\tilde{x}, \tilde{p})$ satisfy the usual uncertainty relation between position and momentum:

$$\Delta x \, \Delta p \;\geq\; \frac{\hbar}{2}, \qquad \Delta \tilde{x} \, \Delta \tilde{p} \;\geq\; \frac{\hbar}{2}. \tag{0.57}$$

However, it is worth noticing that the momentum and its dual can be measured at the same time:

Relation with the symplectic structure of the doubled space.

Let us now focus on the subalgebra generated by the operators $\hat{x}^\mu$ and $\hat{\tilde{x}}_\mu$. This will be given by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{x}^\mu, \hat{\tilde{x}}_\nu] = \pi i \hbar \alpha' \delta^\mu_\nu, \quad [\hat{\tilde{x}}_\mu, \hat{\tilde{x}}_\nu] = 0 \tag{0.59}$$

Notice that this can be seen as an ordinary $2n$-dimensional Heisenberg algebra $\mathfrak{h}(2n)$. This means that such an algebra is immediately given by a symplectic manifold $(\mathcal{M}, \varpi)$ with $\mathcal{M} \cong \mathbb{R}^{2n}$ and symplectic form $\varpi := \pi\hbar\alpha' \mathrm{d}x^\mu \wedge \mathrm{d}\tilde{x}_\mu$. This symplectic structure on the doubled space as introduced by Vaisman.

Back to T-duality

We will obtain the following generating function on the phase space of the zero-mode doubled string:

$$\begin{aligned} F(\mathbb{x},\mathbb{p}) &= \tilde{p}^{\mu}\tilde{x}_{\mu} - p_{\mu}x^{\mu} + \pi\alpha' p_{\mu}\tilde{p}^{\mu} \\ &= \omega_{MN}\mathbb{p}^{M}\mathbb{x}^{N} + \frac{\pi\alpha'}{2}\eta_{MN}\mathbb{p}^{M}\mathbb{p}^{N}. \end{aligned} \tag{0.60}$$

Such a symplectomorphism is simply the $O(n,n)$ transformation of the doubled coordinates and momenta by $(\mathbb{x}^{M},\mathbb{p}^{M}) \mapsto (\eta_{MN}\mathbb{x}^{N},\, \eta_{MN}\mathbb{p}^{N})$.
Now apply the machinary of geometric quatisation, to get the map between Hilbert spaces gives the following, **stringy Fourier transformation**:

$$\widetilde{\psi}_{\tilde{w}}(\tilde{x}) = \int_{L} \mathrm{d}^{n}x\, \mathrm{d}^{n}w \exp\frac{i}{\hbar}\left(\omega_{MN}\mathbb{p}^{M}\mathbb{x}^{N} + \frac{\pi\alpha'}{2}\eta_{MN}\mathbb{p}^{M}\mathbb{p}^{N}\right)\psi_{w}(x)\,. \tag{0.61}$$

This is the transformation between the wavefunctions in different duality frames. Mathematically it is the isomorphism $\mathrm{L}^2(L,\mathbb{C}) \cong \mathrm{L}^2(\widetilde{L},\mathbb{C})$. In undoubled coordinates we can explicitly rewrite such a stringy Fourier transformation as it follows:

$$\widetilde{\psi}_{\tilde{w}}(\tilde{x}) \;=\; \int_L \mathrm{d}^n x\, \mathrm{d}^n w \exp\frac{i}{\hbar}\big(\tilde{w}_\mu x^\mu - w^\mu \tilde{x}_\mu + \pi\alpha' \tilde{w}_\mu w^\mu\big)\, \psi_w(x), \tag{0.62}$$

A phase term in the change of polarisation.

Finally, notice that, if we restrict our generalised winding to ordinary integer winding $w, \tilde{w} \in \mathbb{Z}^n$, we will obtain a change of polarisation of the form

$$\widetilde{\psi}_{\tilde{w}}(\tilde{x}) = \sum_{w\in\mathbb{Z}^n} e^{\frac{i}{\hbar}\pi\alpha'\tilde{w}_\mu w^\mu} \int_M \mathrm{d}^n x\, e^{\frac{i}{\hbar}(\tilde{w}_\mu x^\mu - w^\mu \tilde{x}_\mu)}\psi_w(x). \quad (0.63)$$

In this context, as already observed using different arguments, T-duality does not simply act as a "double" Fourier transformation of the wave-function of a string, because there will be an extra phase contribution given by $\exp\big(i\pi\frac{\alpha'}{\hbar}\tilde{w}_\mu w^\mu\big)$ for any term with $w, \tilde{w} \neq 0$. Since we are restricting now to the case where $w, \tilde{w}$ are integers and $\sqrt{\hbar/\alpha'}$ is just the unit of momentum, we immediately conclude that the only possible phase contributions are $\exp\big(i\pi\frac{\alpha'}{\hbar}\tilde{w}_\mu w^\mu\big) \in \{+1, -1\}$, depending on the product $\tilde{w}_\mu w^\mu \equiv p_\mu w^\mu$ being even or odd.

Minimal scale of the doubled space

In analogy with usual quantum mechanics we can form a double coherent state in doubled space.

$$|\mathbb{z}\rangle \;=\; \int \frac{\mathrm{d}^{2n}\mathbb{p}}{(2\pi)^{2n}} \exp\!\left(\frac{i}{\hbar}\mathbb{p}_M \mathbb{x}^M - \frac{\pi\alpha'}{4\hbar}\delta^{MN}\mathbb{p}_M\mathbb{p}_N\right)|\mathbb{p}\rangle \qquad (0.64)$$

where $\mathbb{x}^M$ is the mean doubled position of the coherent state $|\mathbb{z}\rangle$. Now we can transform wave-functions $\psi(\mathbb{p}) := \langle \mathbb{p} \,|\, \psi\rangle$ on the doubled momentum space to wavefunctions $\psi(\mathbb{x}) := \langle \mathbb{z} \,|\, \psi\rangle$ expressed in the basis of the coherent states. (Here $\mathbb{x}^M$ denotes the mean position of $|\mathbb{z}\rangle$ and is not a coordinate.)

Let us now choose the free particle state $|\psi\rangle = |\mathbb{p}\rangle$, on the doubled momentum space which will have wave-function $\psi(\mathbb{p}) = 1$.

$$\psi(\mathbb{x}) \;=\; \exp\left(-\frac{\left|\mathbb{x}^M\right|^2}{\pi\hbar\alpha'}\right) \tag{0.65}$$

which is a Gaussian distribution on the doubled space and not a delta function. This means that, even if the doubled momentum is maximally spread, the uncertainty on the doubled coordinates cannot be zero. This is because each couple of T-dual coordinates can shrink only to a minimal area proportional to $\ell_s^2 = \hbar\alpha'$. Thus α' is the parameter which controls the fuzziness of doubled space between physical and T-dual coordinates.

Conclusions

Geometric Quantisation is a natural formalism for the doubled string.

The doubled space is naturally a phase space itself with a deformation controlled by the string length.