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Anomalies and Emergent Symmetries on the Lattice and in the Continuum

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Seiberg-SHS: **2003.10466**, 2004.00015, 2004.06115

Gorantla-Lam-Seiberg-SHS: 2007.04904, 2010.16414, **2103.01257**

Rudelius-Seiberg-SHS: 2012.11592

Introduction

- Common lore: starting with a lattice model with local interactions, there is typically an effective QFT description at low energy/long distances.
- Many exotic lattice systems challenge this piece of lore. They include:
  - Fractons [...Haah 2011...]
  - XY-plaquette lattice model [Paramekanti-Balents-Fisher 2002]
- Do **not** seem to admit a standard continuum field theory limit.

QFT for fractons and other exotic models

[Papers with Gorantla, Lam, Rudelius, Seiberg]

- This motivates us to extend the framework of QFT to find the universal description of these exotic lattice models.
- New features:
  - Space symmetry: discrete rotations (e.g. 90 degree rotations)
  - Exact or emergent subsystem global symmetries
  - Discontinuous field configuration

- Many other interesting works on QFT for fractons [Slagle, Kim, Pretko, Bulmash, Barkeshli, Aasen, Williamson, You, Devakul, Sondhi, Burnell, Fontana, Gomes, Chamon, Karch, Raz, Yamaguchi…]. See Andreas' and Kevin's talks.

This talk

- These continuum field theories exhibit emergent symmetries, new exact dualities, and anomalies.
- Are these phenomena exclusive to the continuum theories? Can we find them on the lattice?
- How should we treat more precisely such a continuum field theory with discontinuous fields?
- We will find a deformation of the original lattice model that exhibits all of the above phenomena. This lattice model makes rigorous our treatment of the discontinuous fields. [Gorantla-Lam-Seiberg-SHS 2021]

Outline

- 1+1d XY model and c=1 free boson
- Modified Villain action for the XY-model
- 2+1d XY-plaquette model

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Warmup: 1+1d XY-model

[…; Jose-Kadanoff-Kirkpatrick-Nelson 1978; …]

- We will use a Euclidean-time, Lagrangian formalism.
- On the lattice, phases $e^{i\phi}$ at the sites with the action ($\mu = x, \tau$):

$$S = -\frac{R^2}{\pi} \sum_{links} \cos(\Delta_\mu \phi)$$

- Natural because of an exact $U(1)$ momentum global symmetry:

$$\phi(x,\tau) \to \phi(x,\tau) + \alpha$$

1+1d c=1 free compact boson

- Continuum limit (assuming $R \geq \sqrt{2}$):

$$S = \frac{R^2}{2\pi} \int d\tau dx \left(\partial_\mu \phi\right)^2 , \phi \sim \phi + 2\pi$$

- This is the c=1 free compact boson CFT with radius R.
- It inherits from the original lattice model the $U(1)$ momentum global symmetry, with Noether currents:

$$j_\mu = -\frac{iR^2}{\pi} \partial_\mu \phi \ , \qquad \partial_\mu j^\mu = 0$$

Emergent symmetries and anomalies

- In addition, there is an **emergent** $\widetilde{U(1)}$ winding global symmetry, which was **not** there in the original lattice model:

$$\tilde{j}_\mu = \frac{\epsilon_{\mu\nu}}{2\pi}\partial_\nu\phi \quad , \qquad \partial_\mu \tilde{j}^\mu = 0$$

- Furthermore, there is a mixed 't Hooft anomaly between the $U(1)$ momentum and the $\widetilde{U(1)}$ winding global symmetries in the continuum field theory.

Mixed anomaly

- Couple the free boson CFT to the background, classical gauge fields $A_\mu, \widetilde{A_\mu}$ for both of the $U(1)\times\widetilde{U(1)}$ symmetries.

$$S = \int d\tau dx [\ \frac{R^2}{2\pi}\left(\partial_\mu\phi - A_\mu\right)^2 + \frac{i}{2\pi}\epsilon_{\mu\nu}\widetilde{A_\mu}\ (\partial_\nu\phi - A_\nu)\ ]$$

- The mixed 't Hooft anomaly implies that the partition function is **not** gauge invariant in the presence of these background gauge fields:

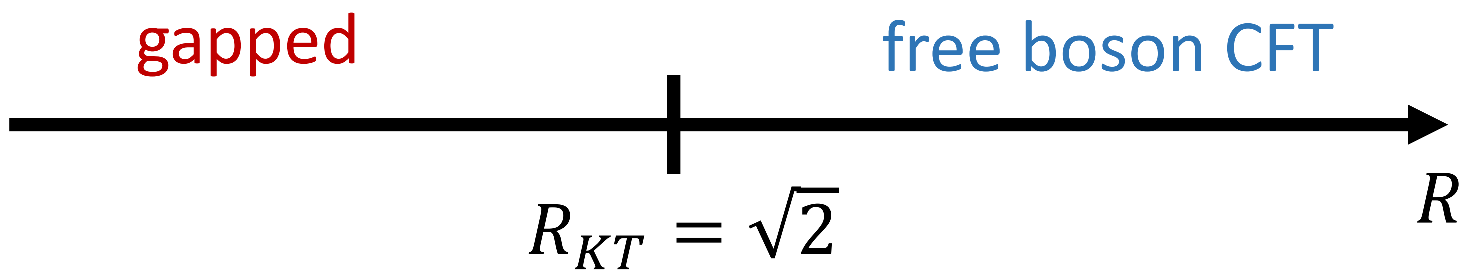
$$\phi \to \phi + \alpha, \qquad A_\mu \to A_\mu + \partial_\mu\alpha, \qquad \widetilde{A_\mu} \to \widetilde{A_\mu} + \partial_\mu\tilde{\alpha}$$

$$S \to S - \frac{i}{2\pi}\int d\tau dx\ \tilde{\alpha}\ (\partial_\tau A_x - \partial_x A_\tau)$$

Kosterlitz-Thouless transition

$$S = \frac{R^2}{2\pi} \int d\tau dx \left(\partial_\mu \phi\right)^2$$

- The continuum limit of the XY model in terms of the c=1 free boson CFT is only correct if $R \geq \sqrt{2}$.
- For $R < \sqrt{2}$, the winding operators are relevant and destabilize the CFT.



Self-duality (T-duality)

$$S = \frac{R^2}{2\pi}\int d\tau dx\left(\partial_\mu \phi\right)^2$$

- In high energy physics, it is common to study the free boson CFT at all values of R.
- The continuum field theory has a self-duality (T-duality) that maps

$$R \leftrightarrow \frac{1}{2R}$$

and exchanges the $U(1)$ momentum and the $\widetilde{U(1)}$ winding global symmetries.

Lattice vs. Continuum

Lattice XY model	Continuum c=1 free boson CFT
Interacting	Free
$U(1)$ momentum symmetry	$U(1)$ momentum symmetry $\widetilde{U(1)}$ winding symmetry
No anomaly	Mixed 't Hooft anomaly
No exact self-duality	Exact self-duality

Winding symmetry on the lattice?

- Prejudice about the lack of winding symmetry on the lattice:
  - No sense of continuity of the fields and hence no winding symmetry.
  - In the continuum there is a mixed anomaly between the momentum and winding symmetries. Not easy to realize it on the lattice.
  - Since there is no winding symmetry, there is no exact self-duality (T-duality).
  - Relatedly, the lattice model has a KT transition.

Outline

- 1+1d XY model and c=1 free boson
- Modified Villain action for the XY model
- 2+1d XY-plaquette model

Villain formulation of the XY model

[Villain 1975]

- Use the Villain formulation

$$S_V = \frac{R^2}{2\pi} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2$$

- Here $\phi \in \mathbb{R}$ and $n_\mu \in \mathbb{Z}$ is an integer gauge field with the $\mathbb{Z}$ gauge symmetry

$$\phi \sim \phi + 2\pi k$$
$$n_\mu \sim n_\mu + \Delta_\mu k$$

- The $\mathbb{Z}$ gauge symmetry effectively makes ϕ a circle-valued field via $U(1) \cong \mathbb{R}/\mathbb{Z}$.

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

- Next, we add the field strength square for the integer gauge field n_μ:

$$\kappa \sum_{plaq} (\Delta_\tau n_x - \Delta_x n_\tau)^2$$

- For $\kappa \to \infty$, n_μ becomes flat and we can replace the action by the modified Villain action

$$S_{mV} = \frac{R^2}{2\pi} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + i \sum_{plaq} \tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

with a Lagrange multiplier field $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

- Both ϕ and $\tilde{\phi}$ are compact . This is not a theory of a noncompact scalar. Indeed, we cannot absorb R in rescaling ϕ.

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

$$S_{mV} = \frac{R^2}{2\pi} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + i \sum_{plaq} \tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- This lattice model has both an **exact** momentum and an **exact** winding symmetry:

$U(1)$ momentum: $\phi \to \phi + \alpha, \quad j_\mu = -i\frac{R^2}{\pi}(\Delta_\mu \phi - 2\pi n_\mu)$

$\widetilde{U(1)}$ winding: $\tilde{\phi} \to \tilde{\phi} + \tilde{\alpha}, \quad \tilde{j}_\mu = \frac{\epsilon_{\mu\nu}}{2\pi}(\Delta_\nu \phi - 2\pi n_\nu)$

- The lattice model realizes the mixed 't Hooft anomaly, in the same way as in the continuum discussion above.

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

$$S_{mV} = \frac{R^2}{2\pi} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + i \sum_{plaq} \tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- It is interesting that the modified Villain lattice model exhibits 't Hooft anomalies between symmetries that act locally:

$$U(1) \quad \text{momentum: } \phi \to \phi + \alpha$$
$$\widetilde{U(1)} \quad \text{winding: } \tilde{\phi} \to \tilde{\phi} + \tilde{\alpha}$$

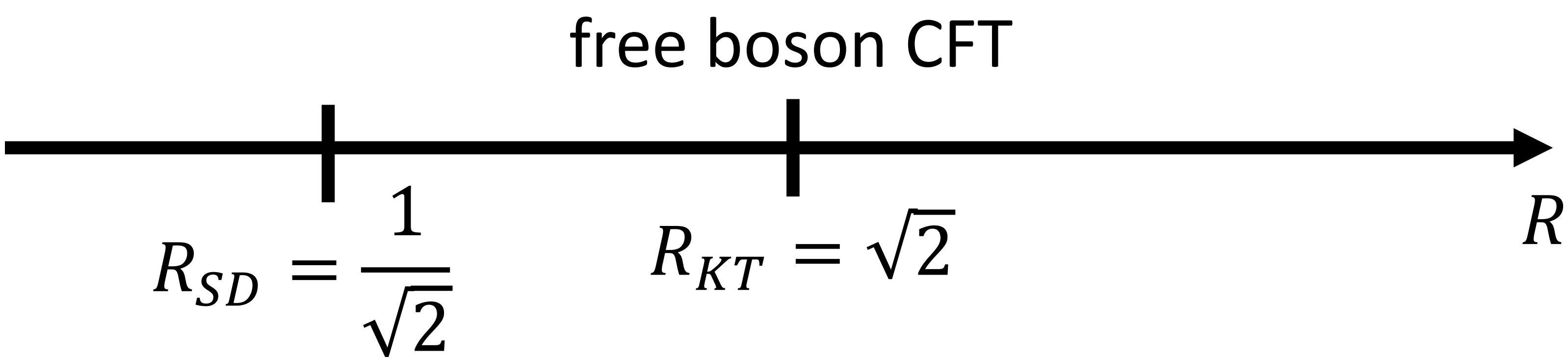
- The anomaly is related to the fact that the local Lagrange density $\mathcal{L}$, or even $\exp(-\mathcal{L})$, are not invariant, but $\exp(-S) = \exp(-\sum \mathcal{L})$ is invariant.

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

$$S_{mV} = \frac{R^2}{2\pi} \sum_{links} \left(\Delta_\mu \phi - 2\pi n_\mu\right)^2 + i \sum_{plaq} \tilde{\phi}(\Delta_\tau n_x - \Delta_x n_\tau)$$

- The mixed ‘t Hooft anomaly implies that the low-energy phase is gapless for **all** values of R. Hence, there is **no** KT transition.
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi}, \;\; R \leftrightarrow \frac{1}{2R}$.



Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

- We can perturb the modified Villain action by the winding operator (i.e., restoring the vortices):

$$\frac{R^2}{2\pi}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}\left(\Delta_\mu n_\nu - \Delta_\nu n_\mu\right) - \lambda\sum_{plaq}\cos\tilde{\phi}$$

$\lambda = \infty\ (\kappa = 0)$

$\lambda = 0\ (\kappa = \infty)$

Villain action

$$S_V = \frac{R^2}{2\pi}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2$$

Modified Villain action

$$S_{mV} = \frac{R^2}{2\pi}\sum_{links}\left(\Delta_\mu\phi - 2\pi n_\mu\right)^2 + i\sum_{plaq}\tilde{\phi}\left(\Delta_\mu n_\nu - \Delta_\nu n_\mu\right)$$

Lattice, continuum, modified Villain

Lattice **XY model**	**c=1 free boson CFT** **and modified Villain lattice model**
Interacting	Free
$U(1)$ momentum symmetry	$U(1)$ momentum symmetry $\widetilde{U(1)}$ winding symmetry
No anomaly	Mixed ‘t Hooft anomaly
No exact self-duality	Exact self-duality

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- 1+1d XY model and c=1 free boson
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2+1d XY-plaquette model

[Paramekanti-Balents-Fisher 2002]

- We will use a Euclidean-time, Lagrangian formulation.
- On the lattice, phases $e^{i\phi}$ at the sites with the action

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos(\Delta_x \Delta_y \phi)$$

- Natural because of global $U(1)$ momentum subsystem symmetry

$$\phi(x, y, \tau) \to \phi(x, y, \tau) + \alpha_x(x) + \alpha_y(y)$$

- While it is not a fracton model, it is related to various gapped and gapless fracton models when lifted to 3+1 dimensions appropriately.

Continuum ϕ-theory

- Continuum field theory:

$$S = \int d\tau dx dy \left(\frac{\mu_0}{2}(\partial_\tau \phi)^2 + \frac{1}{2\mu}\left(\partial_x \partial_y \phi\right)^2\right), \phi \sim \phi + 2\pi$$

- Discontinuous fluctuations of ϕ in, say, x are **not** suppressed as long as they are independent of y.
- It inherits from the original lattice model the $U(1)$ momentum subsystem symmetry, with Noether currents:

$$j_\tau = i\mu_0 \partial_\tau \phi, \qquad j_{xy} = \frac{i}{\mu} \partial_x \partial_y \phi$$

$$\partial_\tau j_\tau = \partial_x \partial_y j_{xy}$$

Continuum ϕ-theory

- In addition, there is an **emergent** $\widetilde{U(1)}$ winding subsystem symmetry, which was **not** there in the original lattice model:

$$\tilde{j}_\tau = \frac{1}{2\pi}\partial_x\partial_y\phi, \qquad \widetilde{j_{xy}} = \frac{1}{2\pi}\partial_\tau\phi$$

$$\partial_\tau\tilde{j}_\tau = \partial_x\partial_y\widetilde{j_{xy}}$$

- Furthermore, there is a mixed 't Hooft anomaly between the $U(1)$ momentum and the $\widetilde{U(1)}$ winding symmetries in the continuum.
- Self-duality [Seiberg-Shao 2020]

$$\mu_0 \leftrightarrow \frac{\mu}{(2\pi)^2} \quad , \quad U(1)\text{ momentum} \leftrightarrow \widetilde{U(1)}\text{ winding}$$

Charged states in the continuum

[Seiberg-Shao 2020]

- All the states charged under the momentum and winding symmetries have energy of order $\frac{1}{a}$ (inverse lattice spacing), which becomes infinite in the continuum. From a conservative viewpoint, they should be ignored in the continuum theory.
- The operators carrying these charges (such as $e^{i\phi}$) create the above states and hence they don't act in the continuum theory – infinitely irrelevant.
- Consequently, these symmetries and the entire continuum theory are robust!
- Can we make the discussion of these discontinuous field configurations and these infinite-energy states rigorous? Can we have the winding subsystem symmetry and exact duality on the lattice?

Villain action of the XY-plaquette model

[Gorantla-Lam-Seiberg-SHS 2021]

- Repeat the discussion of the 1+1d XY model for the 2+1d XY-plaquette model:

$$S = -\beta_0 \sum_{\tau-links} \cos(\Delta_\tau \phi) - \beta \sum_{xy-plaq} \cos(\Delta_x \Delta_y \phi)$$

- Use the Villain form

$$S_V = \frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$

Here $\phi \in \mathbb{R}$ and $n_\tau, n_{xy} \in \mathbb{Z}$ with the $\mathbb{Z}$ tensor gauge symmetry

$$\phi \sim \phi + 2\pi k$$

$$n_\tau \sim n_\tau + \Delta_\tau k, \qquad n_{xy} \sim n_{xy} + \Delta_x \Delta_y k$$

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

- Adding the field strength square for the integer gauge field n_τ, n_{xy}

$$\kappa \sum_{cubes} \left(\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau\right)^2$$

- For $\kappa \to \infty$ we can replace the action by the modified Villain action

$$\frac{\beta_0}{2} \sum_{\tau-links} (\Delta_\tau \phi - 2\pi n_\tau)^2 + \frac{\beta}{2} \sum_{xy-plaq} \left(\Delta_x \Delta_y \phi - 2\pi n_{xy}\right)^2$$

$$+ i \sum_{cubes} \tilde{\phi} (\Delta_\tau n_{xy} - \Delta_x \Delta_y n_\tau)$$

with a Lagrange multiplier $\tilde{\phi} \sim \tilde{\phi} + 2\pi$.

Modified Villain lattice model

[Gorantla-Lam-Seiberg-SHS 2021]

- The modified Villain lattice model is similar to the continuum field theory:
- **Exact** subsystem global symmetry

  $U(1)$ momentum $\phi(x,y,\tau) \to \phi(x,y,\tau) + \alpha_x(x) + \alpha_y(y)$

  $\widetilde{U(1)}$ winding $\tilde{\phi}(x,y,\tau) \to \tilde{\phi}(x,y,\tau) + \tilde{\alpha}_x(x) + \tilde{\alpha}_y(y)$
- It realizes the mixed anomaly between $U(1)$ and $\widetilde{U(1)}$ on the lattice.
- Using Poisson resummation, self-duality: $\phi \leftrightarrow \tilde{\phi},\ \beta_0 \leftrightarrow \frac{1}{(2\pi)^2\beta}$
- In the continuum limit, all the states charged under the momentum and winding symmetries have large energy.

Lattice, continuum, modified Villain

XY-plaquette model	Continuum ϕ-theory and modified Villain lattice model
Interacting	Free
$U(1)$ momentum subsystem sym.	$U(1)$ momentum subsystem sym. $\widetilde{U(1)}$ winding subsystem sym.
No anomaly	Mixed 't Hooft anomaly
No exact self-duality	Exact self-duality

Modified Villain action for other models

[Gorantla-Lam-Seiberg-SHS 2021]

- We have analyzed the modified Villain action for various other standard lattice models, including the XY model, $U(1)$ and $\mathbb{Z}_N$ gauge theory in diverse dimensions.

- We also constructed the modified Villain action for exotic models with subsystem symmetry. These include the XY-plaquette model, various gapless and gapped fracton models such as the X-cube model [Vijay-Haah-Fu 2016].

Summary and further comments

- We considered various standard and exotic lattice systems.
- We deformed them slightly by writing them in the Villain form. This makes them free theories (quadratic, Gaussian).
- A more significant deformation constrains the integer Villain gauge fields to be flat (vanishing field strength).
- The resulting modified Villain lattice action exhibit
  - new exact global symmetries
  - new exact dualities
  - anomalies

Thank you!