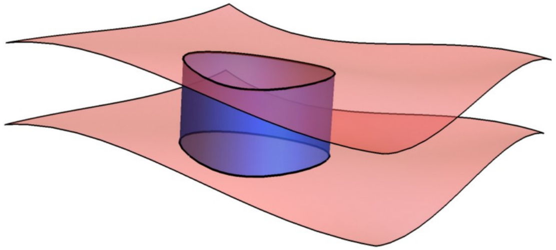
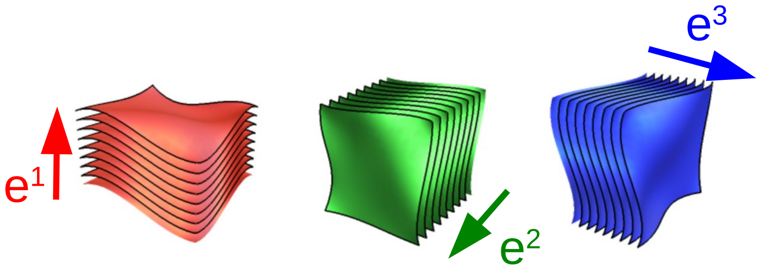
Foliated QFT and Non-Lorenzian Geometry of Fracton Order

Kevin Slagle
Caltech

April 7, 2021

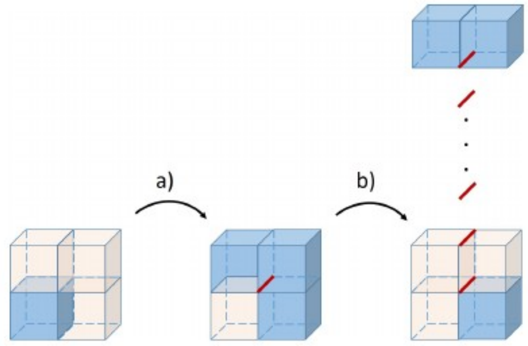
arXiv:1807.00827, 2008.03852
slides: tinyurl.com/fracton





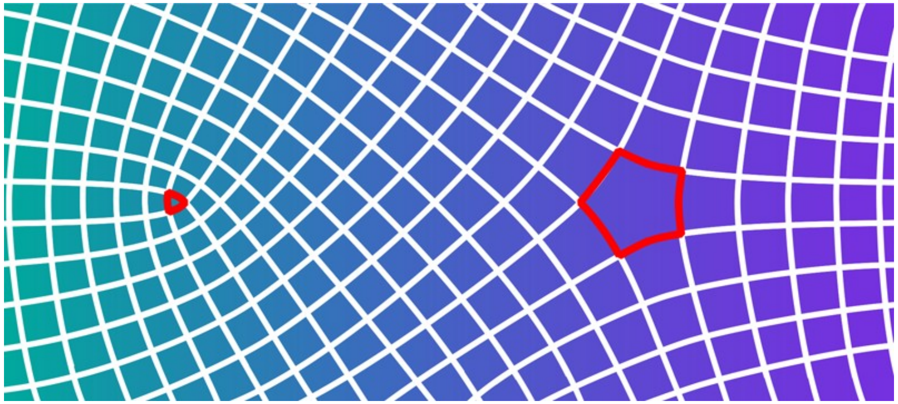
Fracton Motivation

- solvable models of quantum glassiness (Chamon 2005)



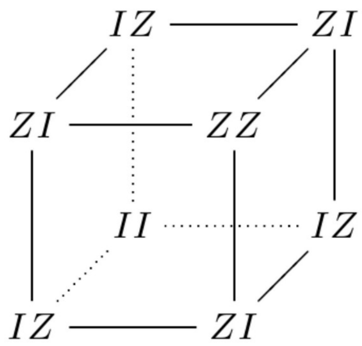
Prem, Haah, Nandkishore 2017

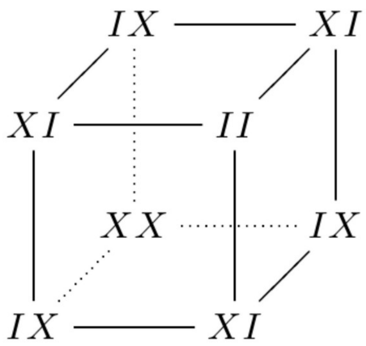
- dualities to elasticity theory of two-dimensional crystals (Pretko, Radzihovsky 2018)

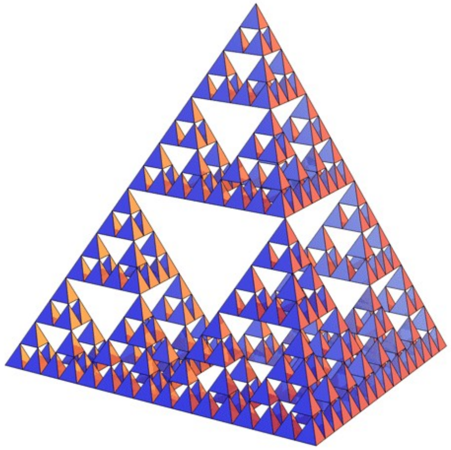


Fracton Motivation

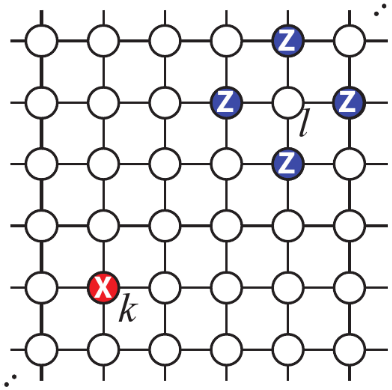
- robust quantum information storage (Haah 2013)







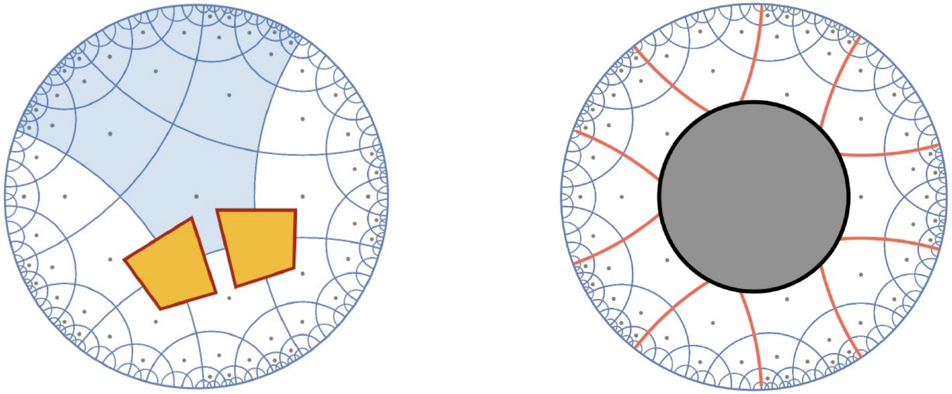
- measurement based quantum computation from ungauged fracton models which have subsystem symmetries (Else, Bartlett, Doherty 2012)



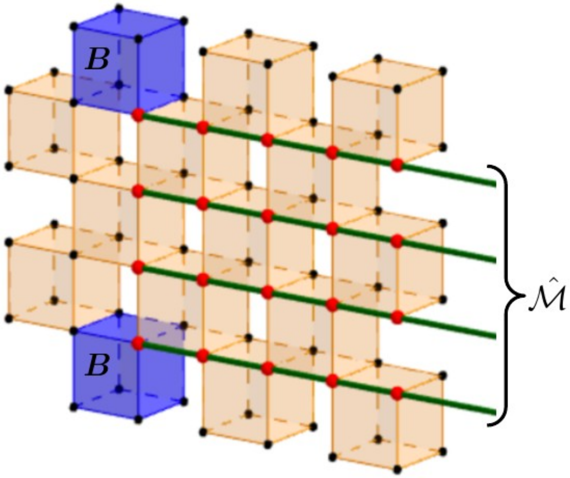
Raussendorf, Okay, Wang, Stephen, Nautrup 2019

Fracton Motivation

- toy model of holography (Yan 2018, ...)



- new phases of matter (Vijay, Haah, Fu 2015)

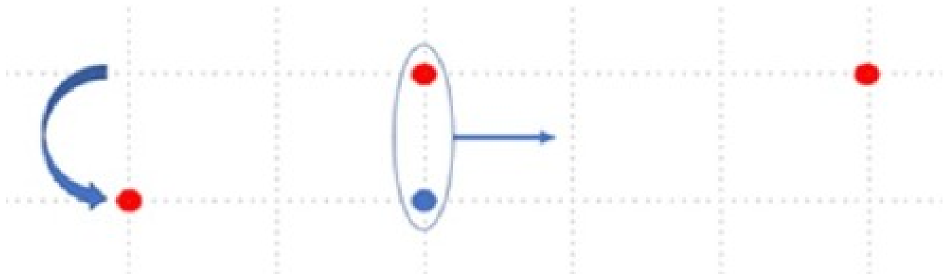


Fracton Motivation

- Linearized gravity → U(1) fracton models
  - Zheng-Cheng Gu, Xiao-Gang Wen (2006, 2009)
  - Cenke Xu (2006, 2006)
  - Cenke Xu, Petr Horava (2010)

- connections to quantum gravity (Pretko 2017)

  fractons move via dipole exchange more easily near other fractons,
  which leads to an effective attractive interaction



Kinds of Fracton Models

(Shao and Karch discussed ungauged versions of the first two kinds)

<table>
<tr><th></th><th>$U(1)$ symmetric tensor gauge theory</th><th>foliated (type-I)</th><th>type-II</th></tr>
<tr><td>example models</td><td>scalar charge
Pretko 2017
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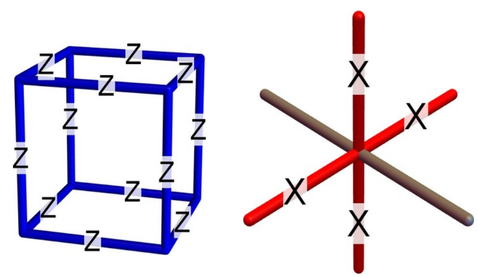
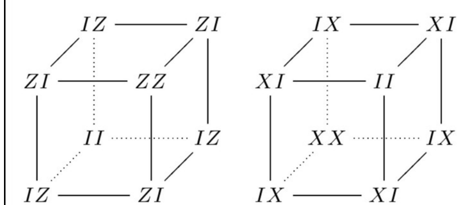

</td><td>Haah's code,
Yoshida's
fractal liquids

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<tr><td>spacetime structure</td><td>Einstein manifolds
$R_{ab} \propto g_{ab}$</td><td>foliated manifolds</td><td>discrete groups?
Tian, Samperton, Wang 2018</td></tr>
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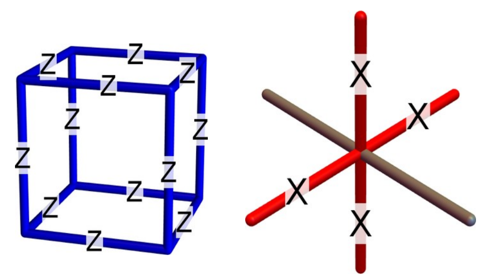
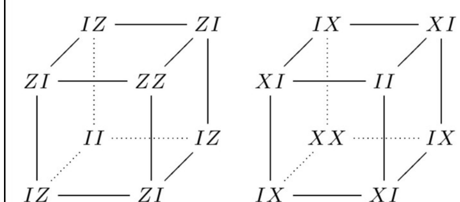
example models	scalar charge Pretko 2017 $H = \int E^2 + B^2$ $B^{ij} = \delta^{jc}\epsilon^{iab}\partial_a A_{bc}$	X-cube Vijay, Haah, Fu 2015 	Haah's code, Yoshida's fractal liquids 
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this talk

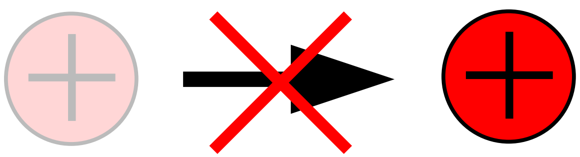
Fractons from U(1) Gauge Theory

Pretko 2017

- Generalize U(1) Maxwell electromagnetism:
- Add dipole moment conservation in addition to charge conservation

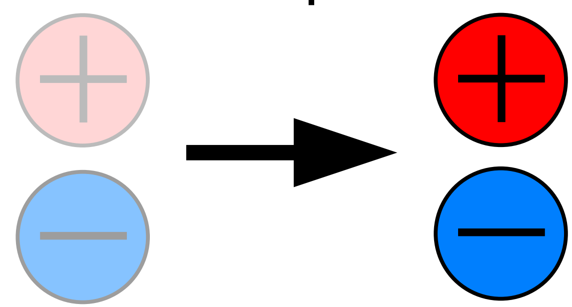
$$P = \int \rho(\vec{r})\, \vec{r}$$

moving change
changes dipole moment



isolated charges
are immobile

moving a dipole
conserves dipole moment



dipoles are mobile

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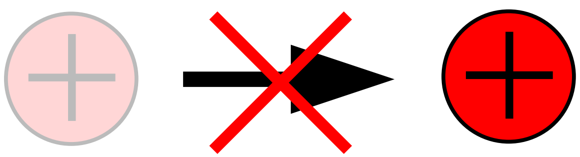

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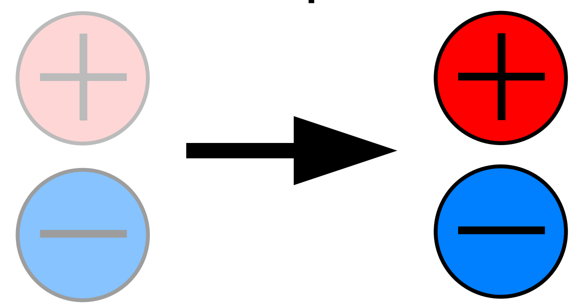
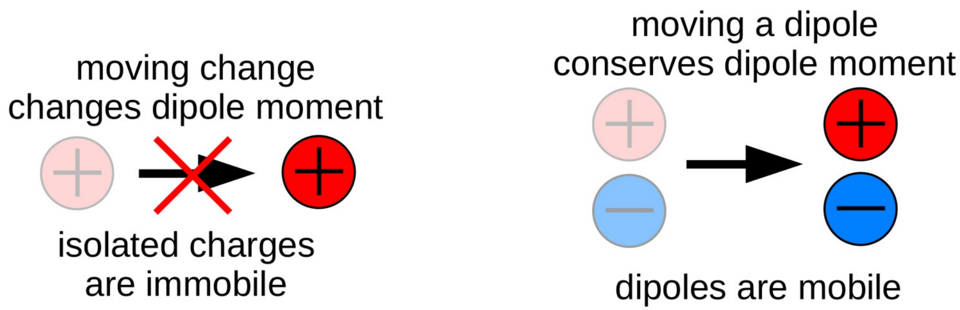


dipoles are mobile

- **fracton**: an immobile particle
  - often a *fraction* of a mobile particle

Fractons from U(1) Gauge Theory

Pretko 2017



previous chalkboard

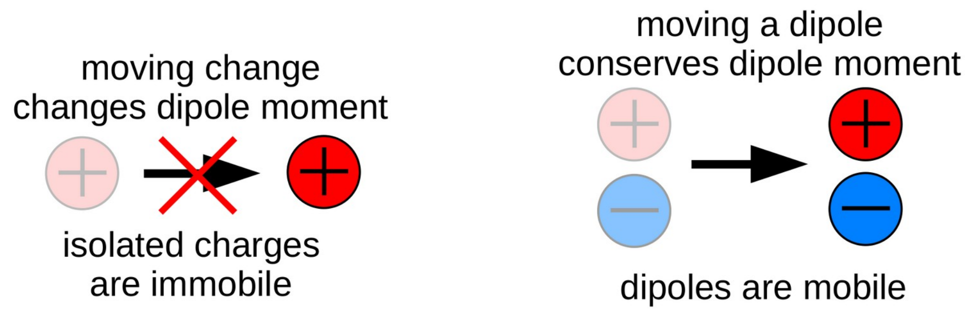
new Gauss law: $\partial_i \partial_j E^{ij} = \rho$

results in charge and dipole moment conservation:

$$\int \mathrm{d}^d x \, \rho = \text{constant} \qquad \int \mathrm{d}^d x \, (\rho \vec{x}) = \text{constant}$$

Fractons from U(1) Gauge Theory

Pretko 2017



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symmetric tensor gauge field and gauge symmetry:

$$A_{ij} \to A_{ij} + \partial_i \partial_j \lambda$$

scalar charge theory:

$$H = \int \mathrm{d}^d x \, \frac{1}{2}(E^2 + B^2) \qquad B^{ij} = \delta^{jc} \epsilon^{iab} \partial_a A_{bc}$$

Fractons from U(1) Gauge Theory

Pretko 2017

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changes dipole moment

isolated charges
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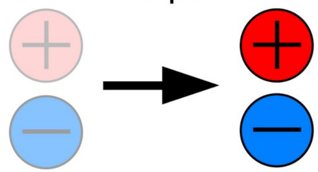

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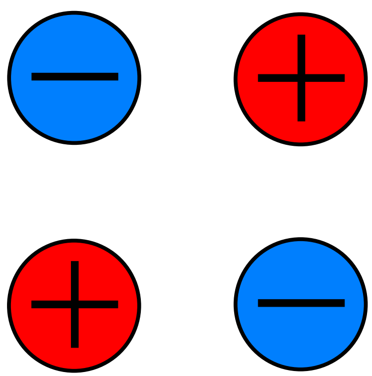
quadrupoles have no
charge or dipole moment



quadrupoles can be
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Fractons from U(1) Gauge Theory

Pretko 2017

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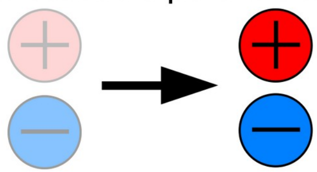

dipoles are mobile

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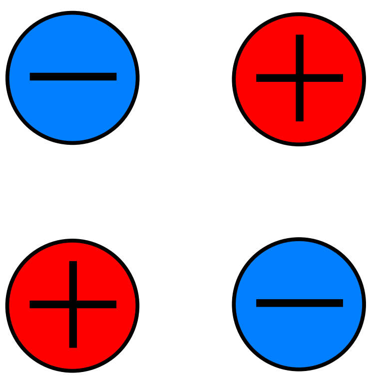
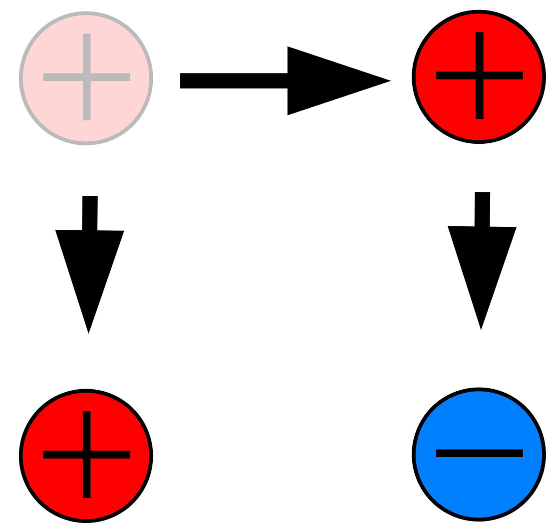


quadrupoles can be
created from the vacuum

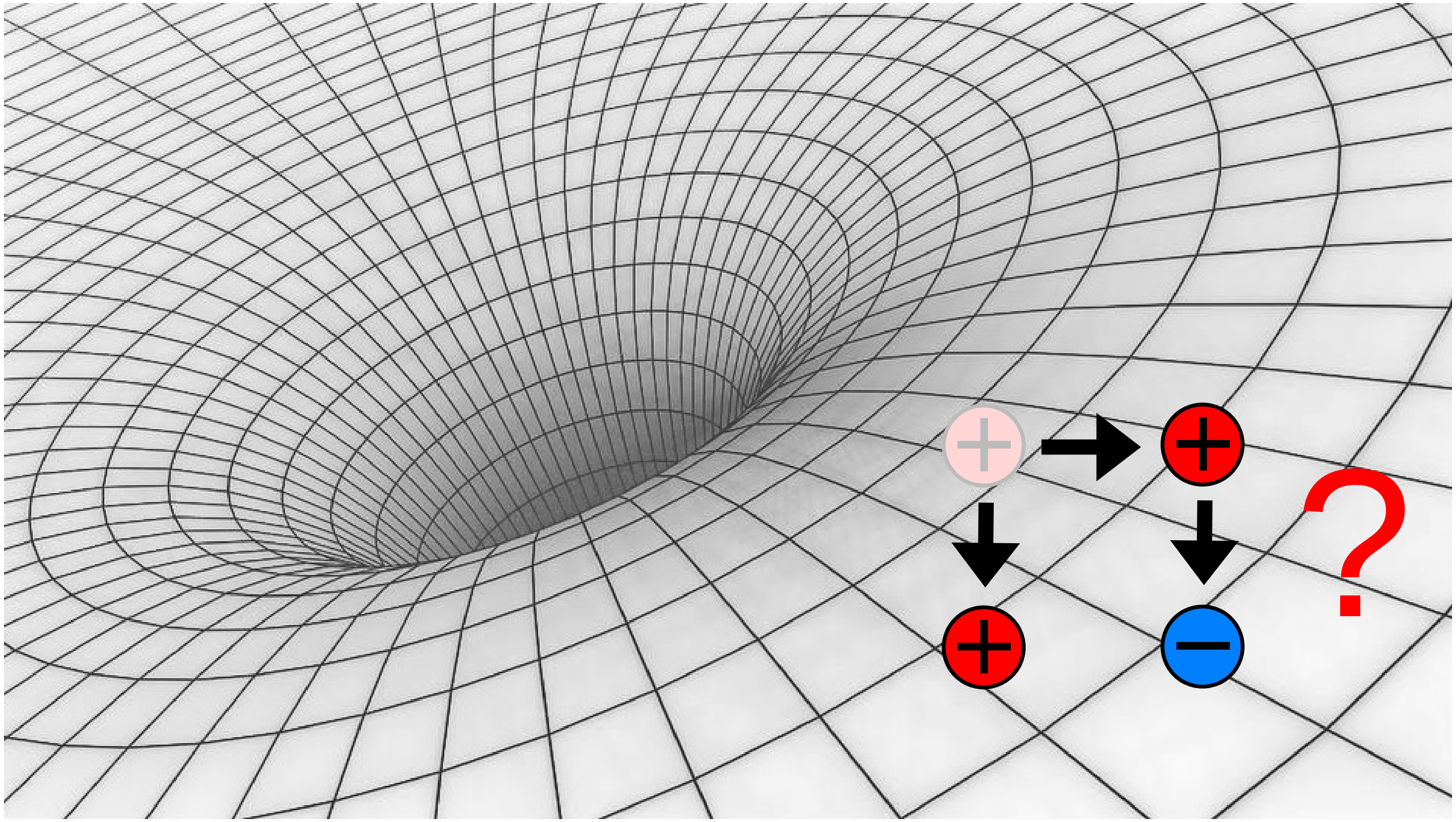
a charge
can move:

if we create
a dipole:



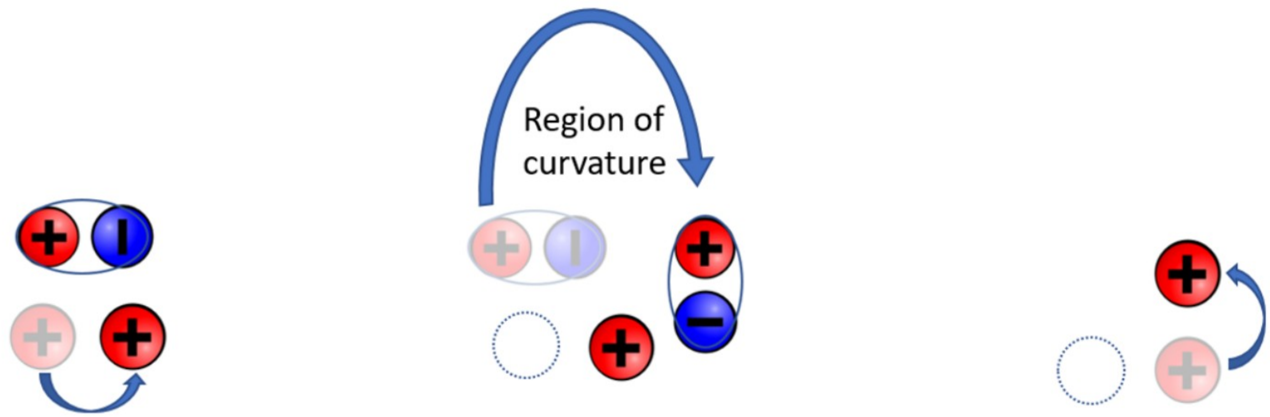
slides: tinyurl.com/fracton

What happens if spacetime is curved?



U(1) Fractons on Curved Space?

- Curvature can grant fractons mobility via dipole mobility
  - dipole rotates via parallel transport



KS, Prem, Pretko 2018

Abhinav Prem

Michael Pretko

U(1) Fractons on Curved Space

- Curvature can grant fractons mobility via dipole mobility
  - also results in a loss of gauge invariance
- Does not occur when on Einstein manifolds for the traceless scalar charge model:

$$R_{ab} \propto g_{ab}$$

  - i.e. manifolds with no torsion or stress-energy tensor

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \underbrace{\frac{8\pi G}{c^4}T_{\mu\nu}}_{\text{must be zero}}$$

Traceless Scalar Charge Model

- Traceless scalar charge model:
  - $g_{ij}E^{ij} = 0$
  - conserved charge, dipole, and a quadrupole $\int \mathrm{d}^d x \, x^2 \rho$
  - fracton dipoles are planons that only move perpendicular to dipole moment

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- On Einstein manifolds:
  - Dipole moment of planons won't change under closed loops

U(1) Fractons on Curved Space

symmetric tensor gauge theory	gauge invariant manifold
3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
3D gapless traceless vector	Einstein with constant curvature
2D gapped traceless scalar	constant curvature
2D gapless traceless scalar	constant curvature
any-D gapless traceful scalar	flat
any-D gapless traceful vector	flat

- We only considered spatial curvature
  - U(1) fractons in curved spacetime is an open problem
    - nontrivial generalization due to no Lorentz symmetry

Questions?

	symmetric tensor gauge theory	gauge invariant manifold
$g_{ij}E^{ij} = 0$	3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
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Part 2

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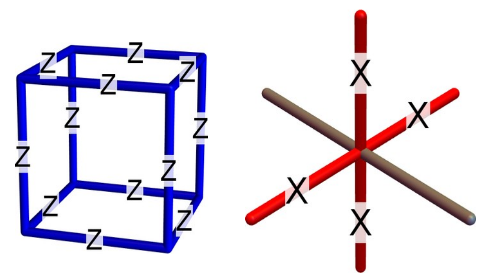
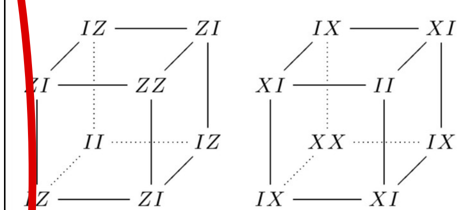
example models	scalar charge Pretko 2017 $H = \int E^2 + B^2$ $B^{ij} = \delta^{jc}\epsilon^{iab}\partial_a A_{bc}$	X-cube Vijay, Haah, Fu 2015 	Haah's code, Yoshida's fractal liquids 
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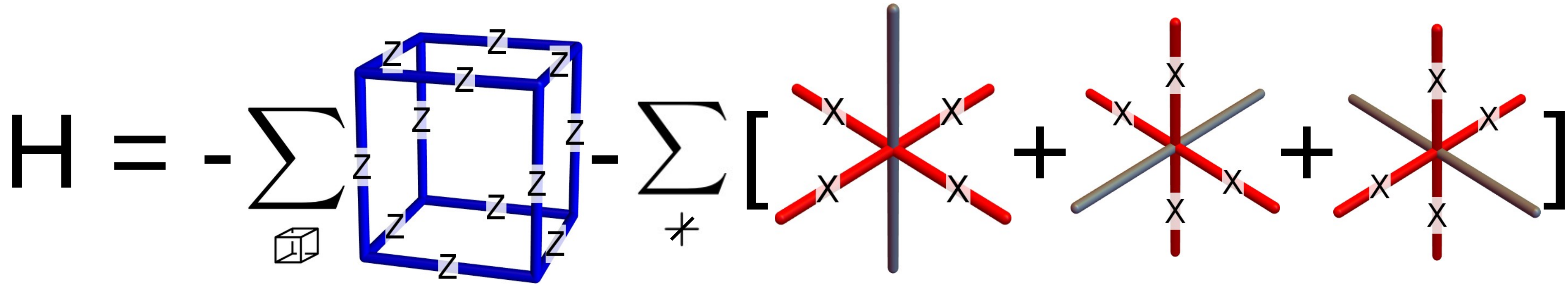
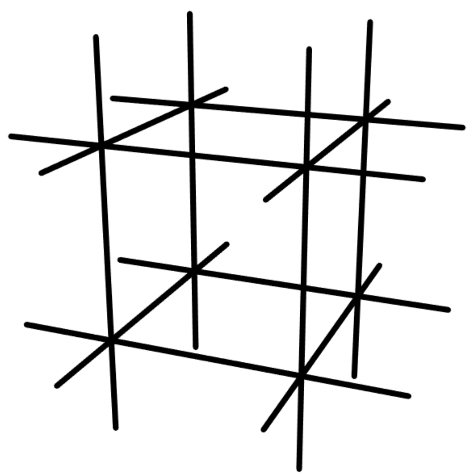
review: X-cube fracton model

Vijay, Haah, Fu 2016



- qubits on links of a cubic lattice
- gapped & exactly solvable
- degen = 2^{6L-3}

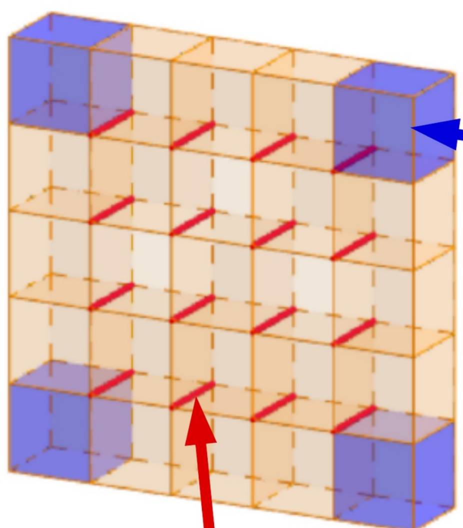
$$Z = \sigma^z$$
$$X = \sigma^x$$



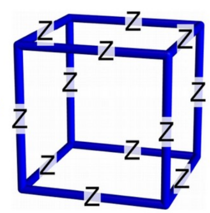
Subdimensional Excitations

Vijay, Haah, Fu 2016

- Create fractons at corners of rectangular operators
- Create lineons at ends of line operators

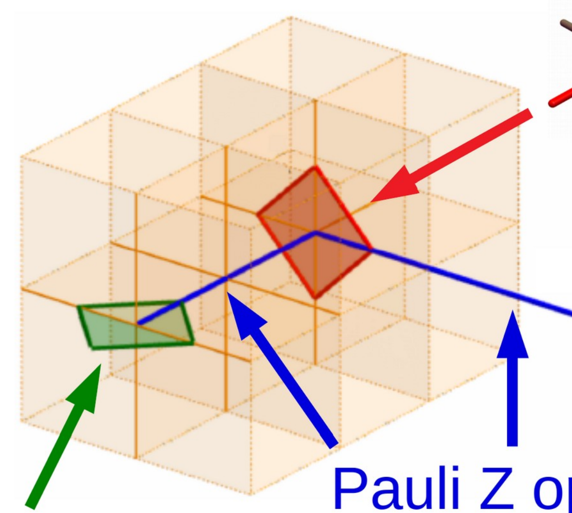


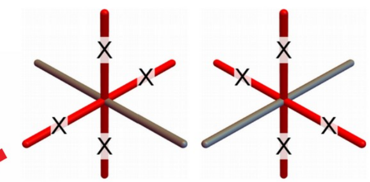
Pauli X operators acting on ground state



= -1

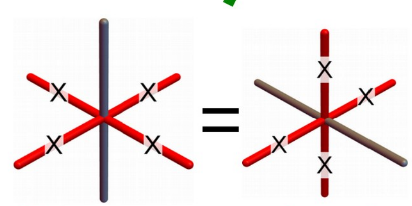
fracton excitation





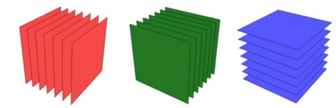
lineon left behind

Pauli Z operators acting on ground state

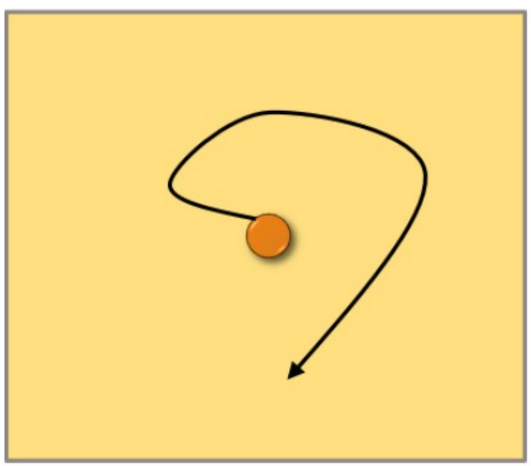


lineon excitation

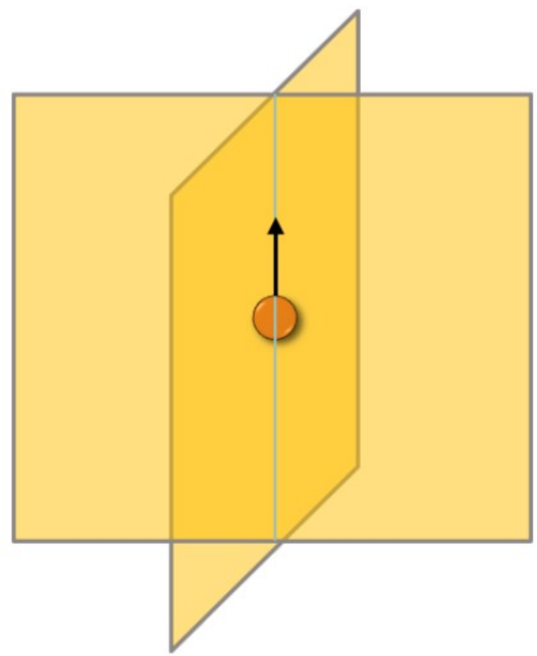
Subdimensional Excitations



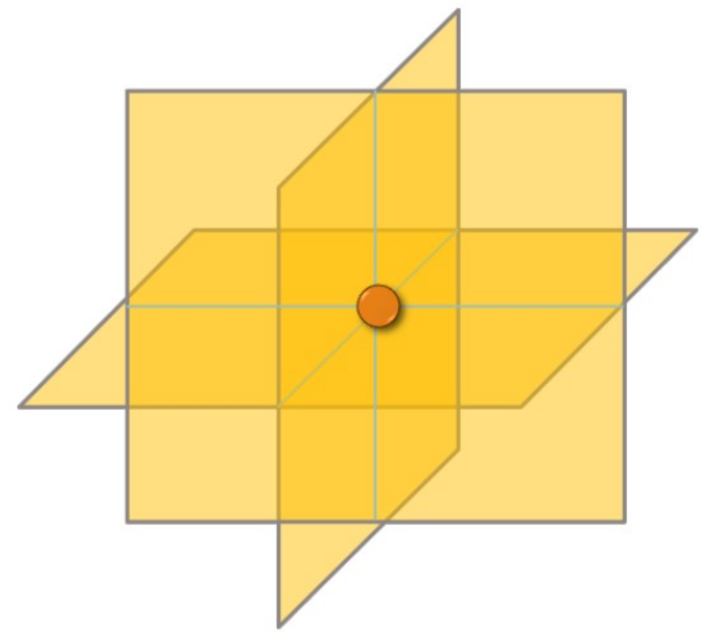
- Mobility restrictions determined by a layering structure



planon
(moves along
2D planes)

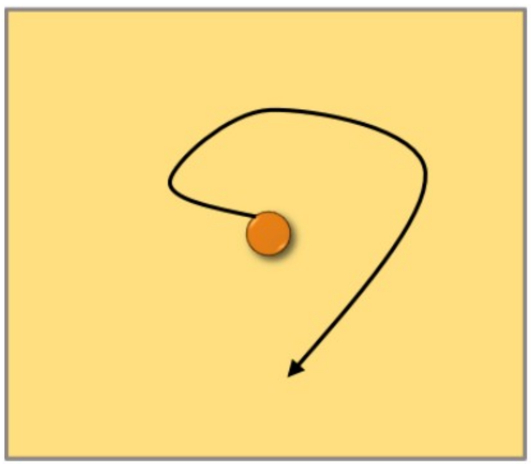


lineon
(moves along
1D lines)



fracton
(immobile)

QFT description?



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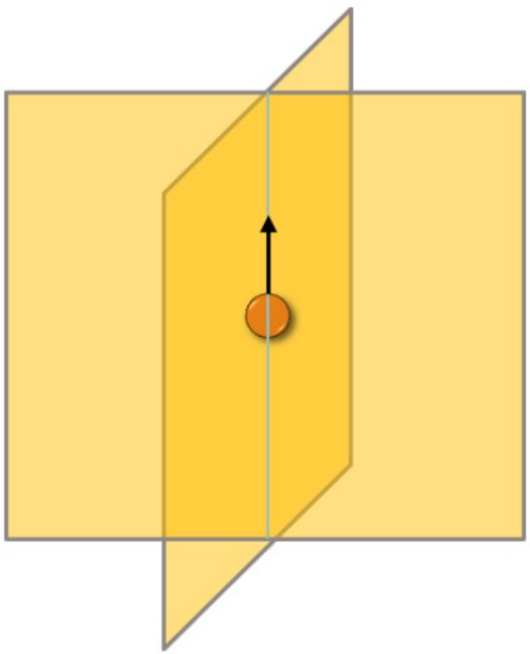

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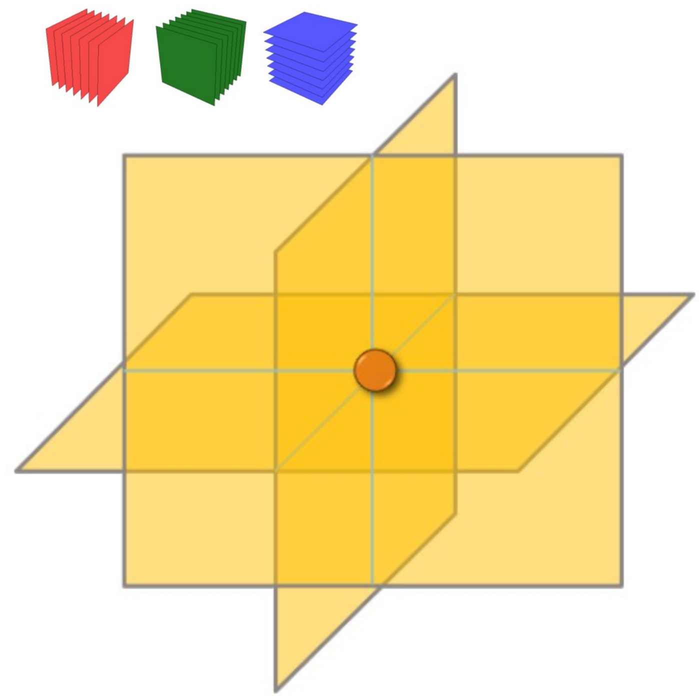
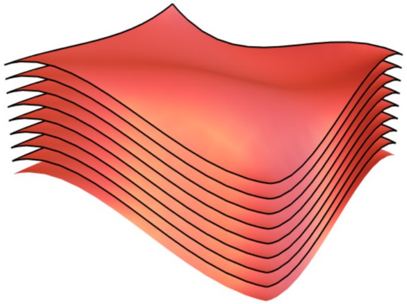
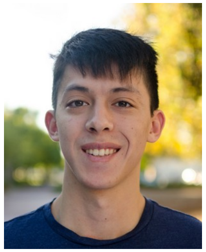


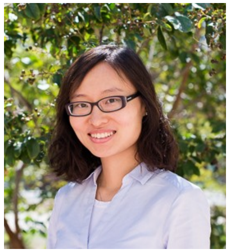
fracton
(immobile)

Shirley, KS, Wang, Chen 2018



Foliation



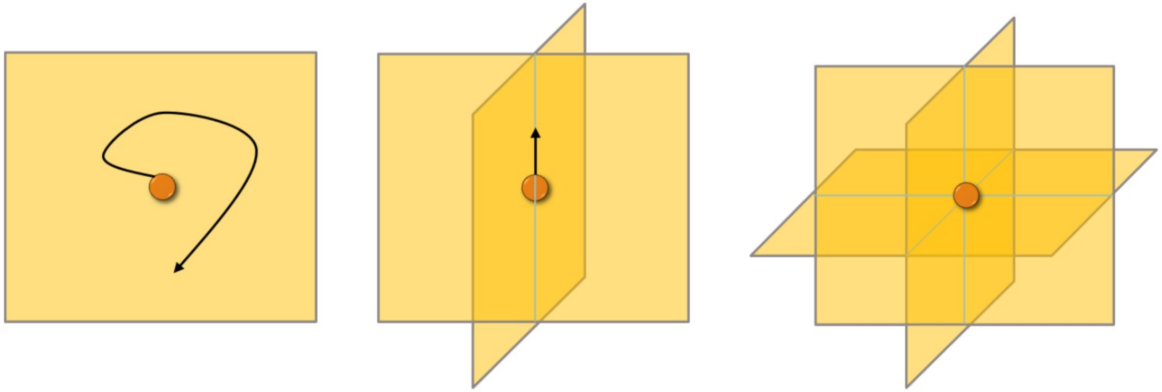


Wilbur Shirley

Zhenghan Wang

Xie Chen

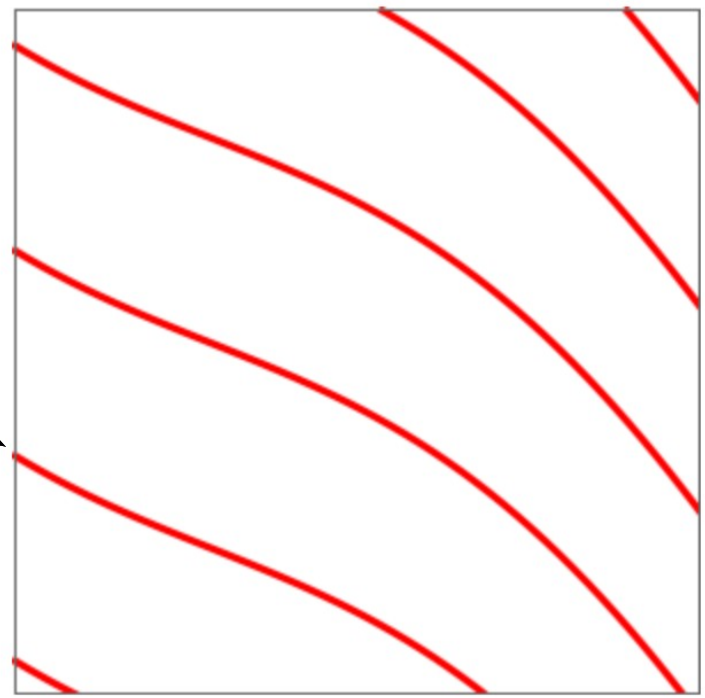
- Foliation (i.e. layered structure) determines:
  - mobility constraints:



  - ground state degeneracy, entanglement, …

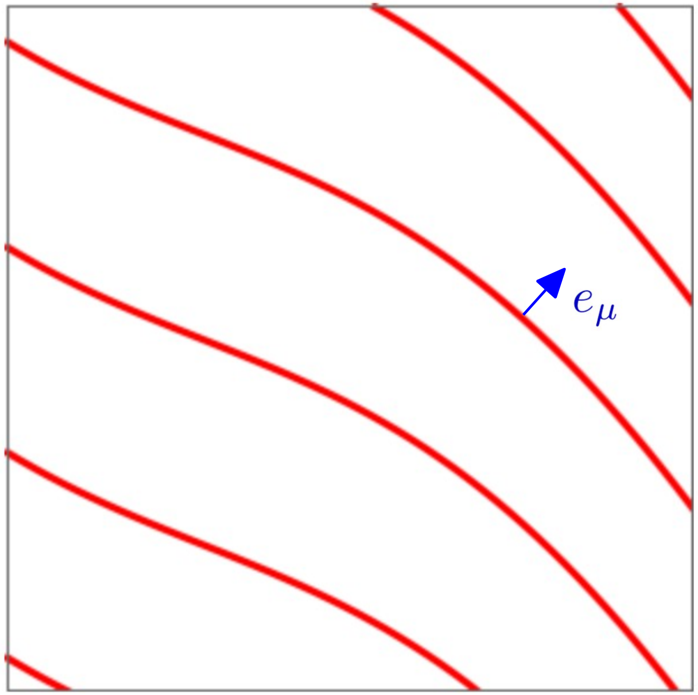
Foliation Intuition

layers of foliation
are locally level
surfaces of $f(\mathbf{x})$



Foliation Field Intuition

layers of foliation are locally level surfaces of $f(\mathbf{x})$



locally define foliation field as

$$e = df$$

e_μ is closed:

$$de = 0$$

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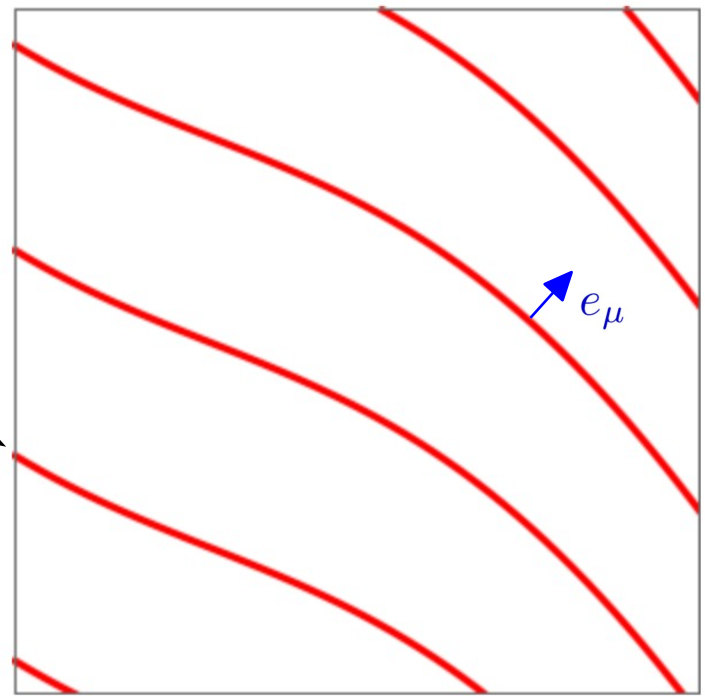


locally define foliation field as

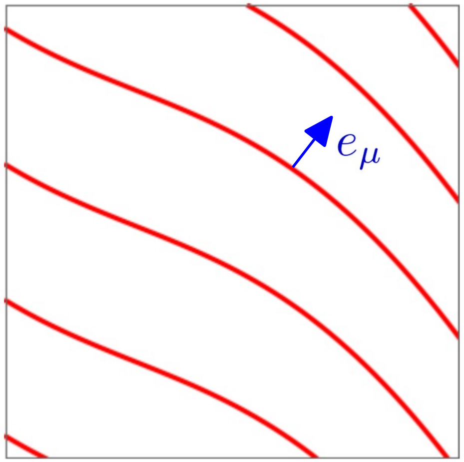
$$e = df$$

e_μ is closed:

$$de = 0$$

- To define foliation field globally, forget about $f(\mathbf{x})$
- Define layers to be orthogonal to e_μ and require $de = 0$
  - i.e. tangent vectors v^μ of layers are null vectors: $v^\mu e_\mu = 0$

Foliation Field



- Define layers to be orthogonal to e_μ and require $de = 0$

previous chalkboard

- More generally: $de = e \wedge \beta$ *[technical note]*
  - where β is a 1-form
  - results from $e \to \gamma e$ transformation
  - cohomology class of $\beta \wedge d\beta$ is the Godbillon-Vey invariant of the foliation
  - often $\beta = 0$

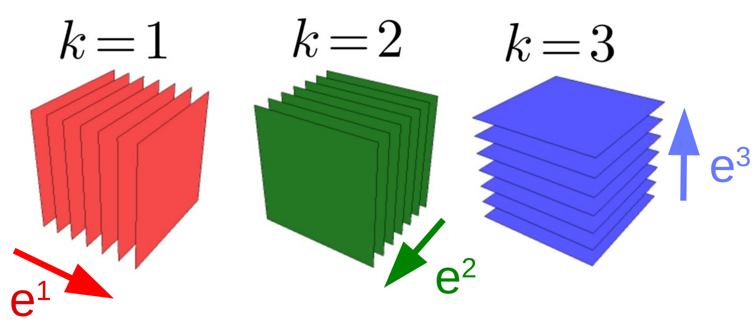
Foliation Field

- Foliating layers are orthogonal to 1-form e_μ

$$de = e \wedge \beta$$

often $\beta = 0$

- Multiple foliation fields e^k
  - $k = 1, 2, .., n_f$ indexes different foliations
  - e.g. $e^k_\mu = \delta^k_\mu$ for flat X-cube foliation



Foliation Field

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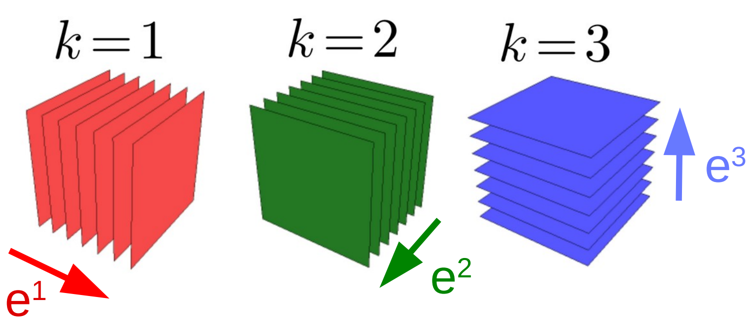


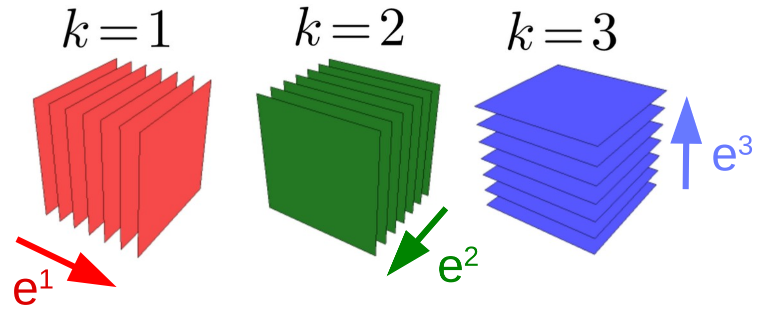
- All compact orientable three-manifolds admit total foliations (Hardorp 1980)

➡ These manifolds admit a foliated QFT with fractons

- Foliating layers are orthogonal to 1-form e_μ^k

$$de^k = e^k \wedge \beta^k$$

often $\beta = 0$



Slagle 2020

Foliated QFT

not magnetic field

1-form gauge fields

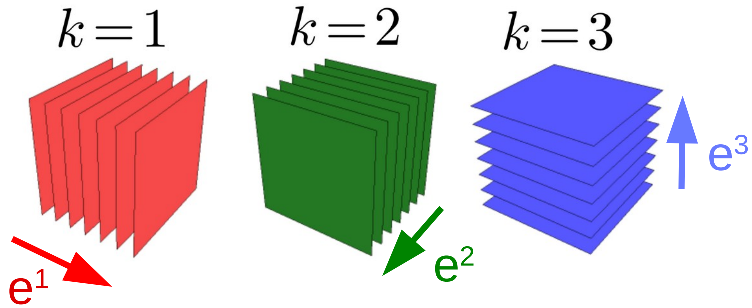
2-form gauge field

$$L = \underbrace{\sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k}_{\text{infinitesimally-spaced stacks of 2+1D BF theory coupled to:}} + \underbrace{\frac{N}{2\pi} b \wedge da}_{\text{3+1D BF theory}}$$

- Foliating layers are orthogonal to 1-form e^k_μ

$$de^k = e^k \wedge \beta^k$$

often $\beta = 0$



Slage 2020

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foliated (1+1)-form gauge field: $A^k \wedge e^k = 0$

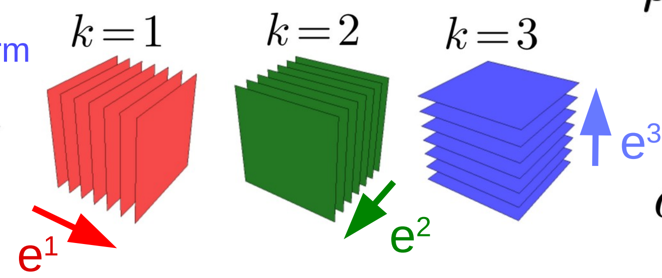
(all gauge fields are compact)

non-dynamical foliation field

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi}(\underbrace{dB^k}_{\text{2-form}} + n_k \underbrace{b}_{\text{1-form}}) \wedge \underbrace{A^k}_{\text{(1+1)-form}} + \frac{N}{2\pi} b \wedge \underbrace{da}_{\text{1-form}}$$

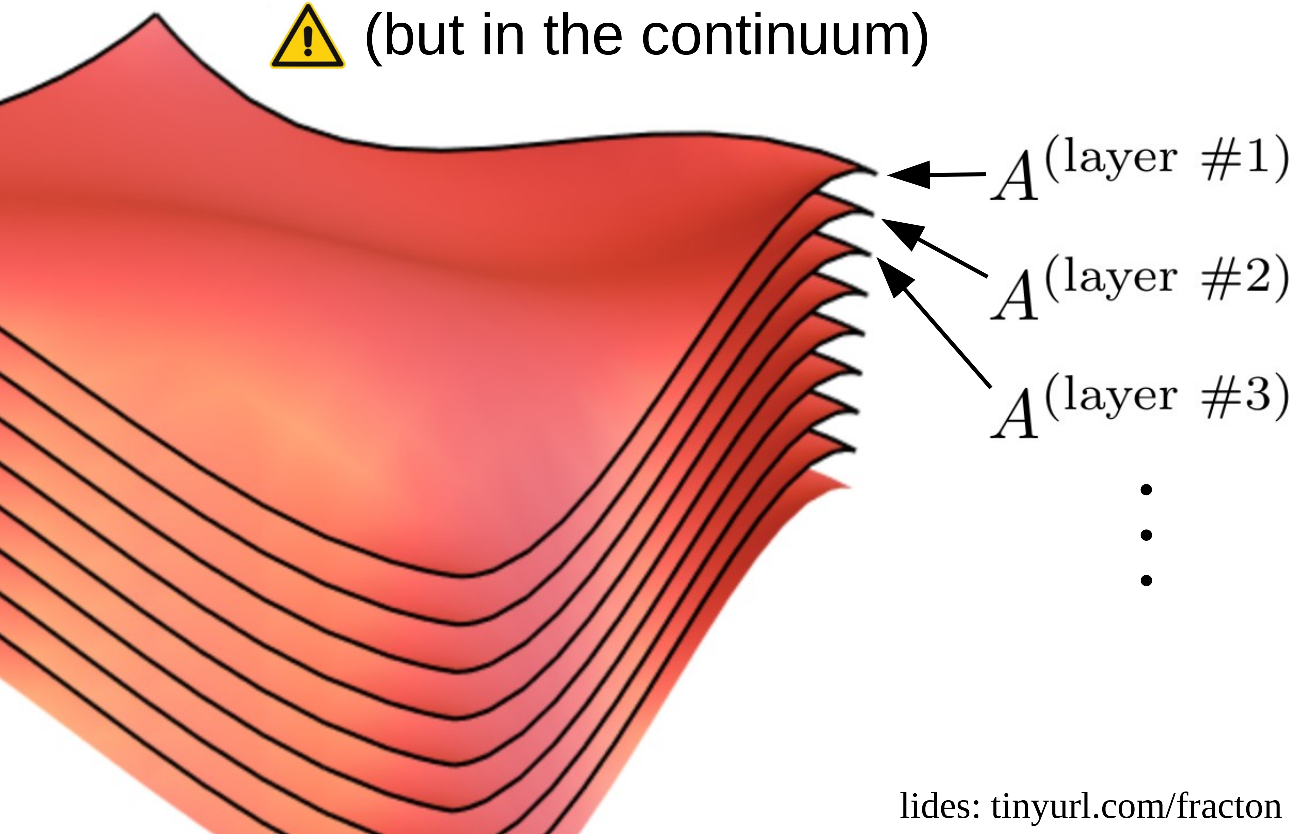
$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Intuition: Foliated Gauge Field

- Similar to an independent gauge field on every layer
  - ⚠ (but in the continuum)



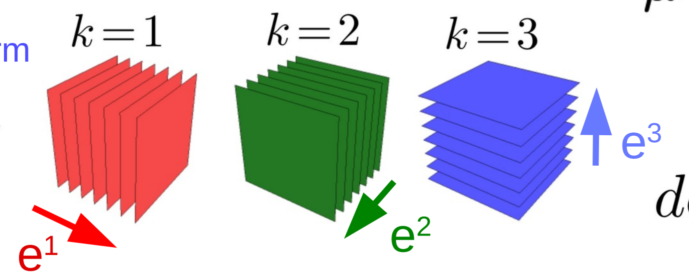
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Intuition: Foliated Gauge Field

- Consider flat foliation example: $e = dz \quad (\text{i.e. } e_\mu = \delta^3_\mu)$

1-form

$$A^{(\text{layer } \#1)}$$

$$= A_0 dt + A_1 dx + A_2 dy$$

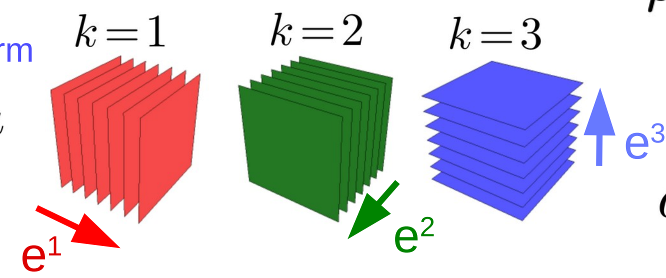
- Foliating layers are orthogonal to 1-form e^k_μ

2-form　　1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form　　(1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Intuition: Foliated Gauge Field

- Consider flat foliation example: $e = dz \quad (\text{i.e. } e_\mu = \delta^3_\mu)$

1-form

$$A^{(\text{layer } \#1)} = A_0 dt + A_1 dx + A_2 dy$$

foliation constraint

(1+1)-form

$$A \wedge e = 0$$

implies

$$A = (A_{03} dt + A_{13} dx + A_{23} dy) dz$$

same degrees of freedom
as 1-form in 2+1D!

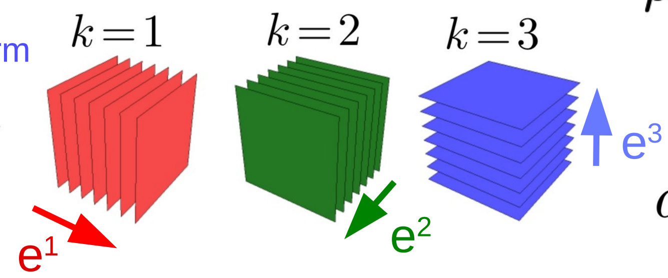
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Continuity Requirements

Gauge Field

- Require dA well-defined

- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

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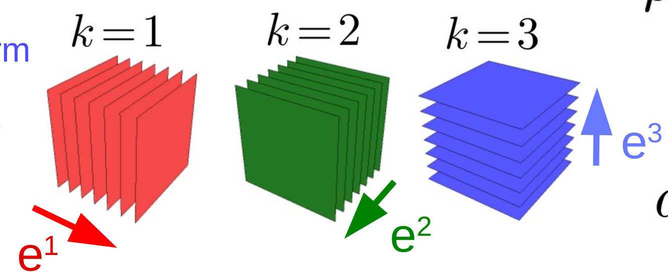

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Continuity Requirements

Gauge Field

- Require dA well-defined

Foliated Gauge Field

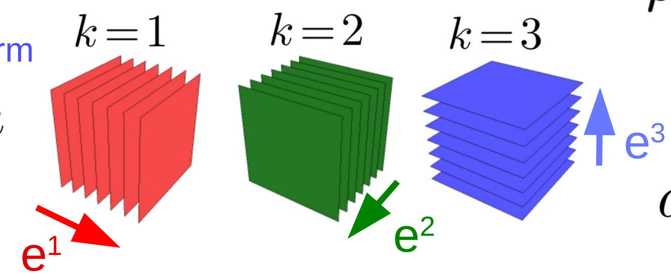
- Require dA well-defined
  - Does not imply continuity between layers (as desired)!

$$e = dz \implies A = (A_{03}dt + A_{13}dx + A_{23}dy)dz$$

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi} \overset{\text{2-form}}{(\underset{\text{1-form}}{dB^k} + n_k b)} \wedge \underset{\text{(1+1)-form}}{A^k} + \frac{N}{2\pi} b \wedge \overset{\text{1-form}}{da}$$

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

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$$e = dz \implies A = (A_{03}dt + A_{13}dx + A_{23}dy)dz$$

- Allow delta-functions
  - e.g. $A = x\,\delta(z)\,dy \wedge dz$ is allowed for $e = dz$

- Foliating layers are orthogonal to 1-form e_μ^k

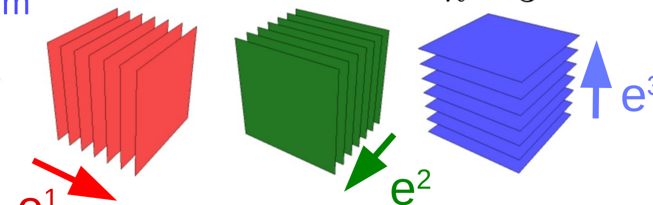
2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$

$k=1$ $k=2$ $k=3$



$$de^k = e^k \wedge \beta^k$$

Continuity Requirements

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Questions?

- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

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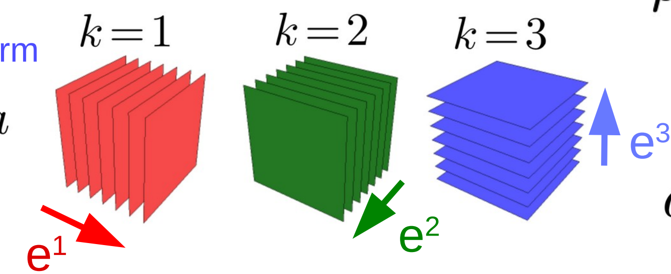
1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Review: Charge Conservation

- Z_N BF theory (Z_N toric code):

$$L = \frac{N}{2\pi} B \wedge dA - A \wedge J - B \wedge I$$

- gauge invariance:

$$A \to A + d\zeta$$

$$B \to B + d\chi$$

- conservation of charge and flux:

$$dJ = dI = 0$$

also constrains gauge invariant operators, eg $e^{\mathrm{i}\int A \wedge J}$

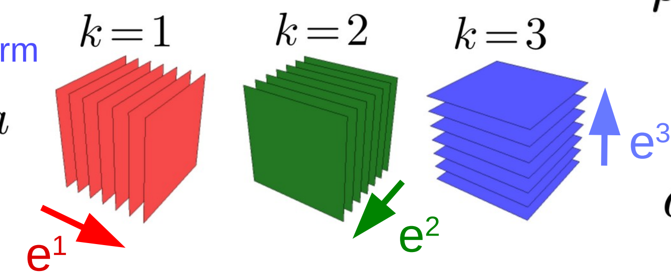
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

- $L' = L - \sum_k A^k \wedge J^k - a \wedge j$ **f** (possible fracton)

planon **p**

- mobility constraints from gauge invariance:

$$a \to a + d\lambda$$

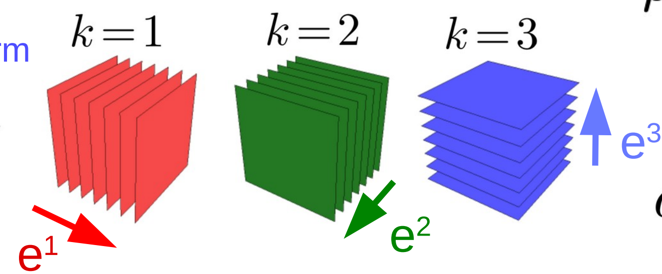
f # conservation:

$$dj = 0$$

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi}(\underset{\text{1-form}}{d\overset{\text{2-form}}{B^k}} + n_k b) \wedge \underset{\text{(1+1)-form}}{A^k} + \frac{N}{2\pi} b \wedge d\overset{\text{1-form}}{a}$$

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

- $L' = L - \sum_k A^k \wedge J^k - a \wedge j$ f (possible fracton)

planon p

- mobility constraints from gauge invariance:

$a \to a + d\lambda$ ⇨ **f** # conservation:

$$dj = 0$$

$A^k \to A^k + d\zeta^k$ (0+1)-form

$\zeta^k \wedge e^k = 0$

current through layer

⇨ $$dJ^k \wedge e^k = -m_k\, j \wedge e^k$$

moving **f** across a
foliation creates planons **p**

$$a \to a - \sum_k m_k \zeta^k$$

$$m_k \equiv \frac{n_k M_k}{N}$$

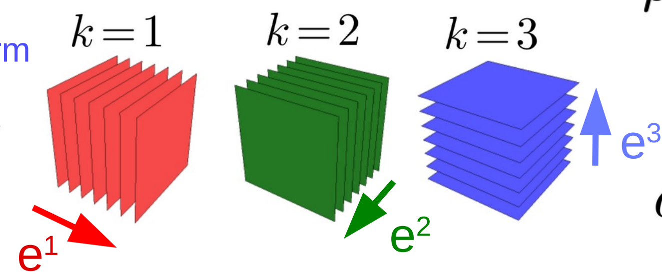
- Foliating layers are orthogonal to 1-form e^k_μ

2-form, 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form, (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

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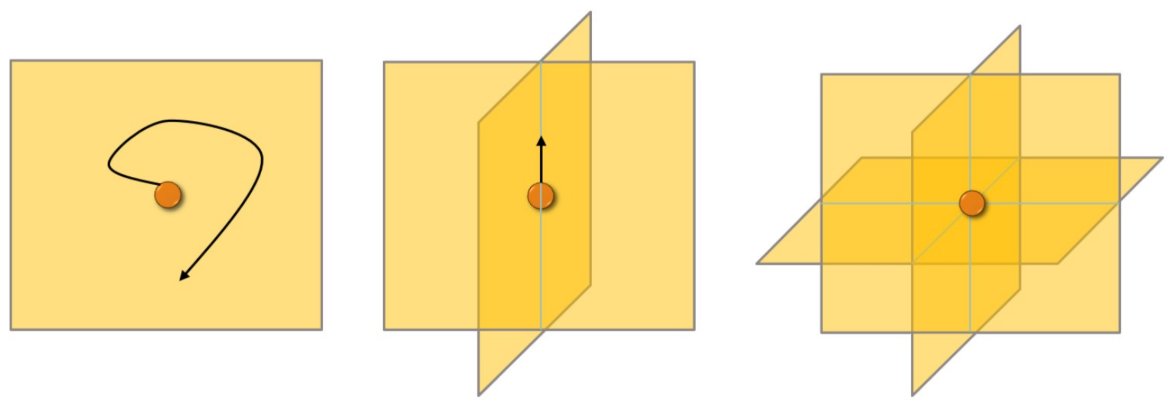
planon **p**

- mobility of **f** depends on # of foliations with $n_k \neq 0$

1 foliation: 2 foliations: 3 foliations:



planon (stuck to 1 layer) | lineon (stuck to 2 layers) | fracton (stuck to 3 layers)

$$dJ^k \wedge e^k = -m_k \, j \wedge e^k$$

$$m_k \equiv \frac{n_k M_k}{N}$$

- Foliating layers are orthogonal to 1-form e^k_μ

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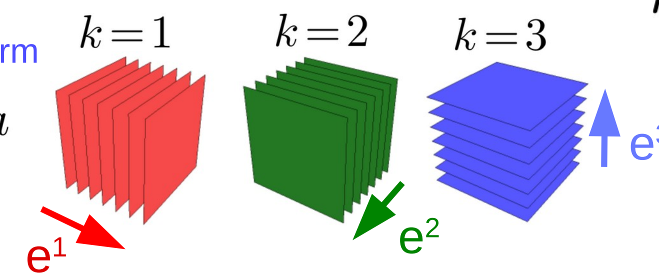
$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

- $$L' = L - \sum_k B^k \wedge I^k - b \wedge i$$

flux string

planon **m**

$B^k \to B^k + d\chi^k$ ⇨ **m** # conserved: $dI^k = 0$

$B^k \to B^k + \alpha^k e^k$ ⇨ **m** restricted to 2D layers: $I^k \wedge e^k = 0$

- Foliating layers are orthogonal to 1-form e^k_μ

2-form

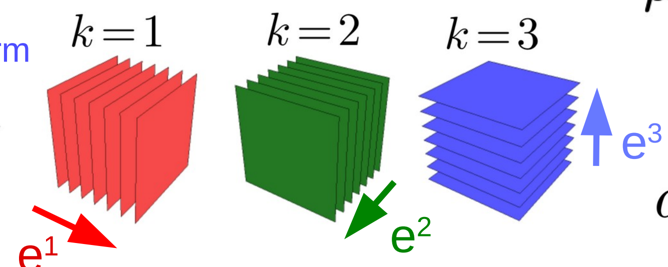
1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form

(1+1)-form

$$A^k \wedge e^k = 0$$



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$b \to b + d\mu$

$B^k \to B^k - n_k \mu$

⇨ **m** bound to flux strings: $di = \sum_k n_k I^k$

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

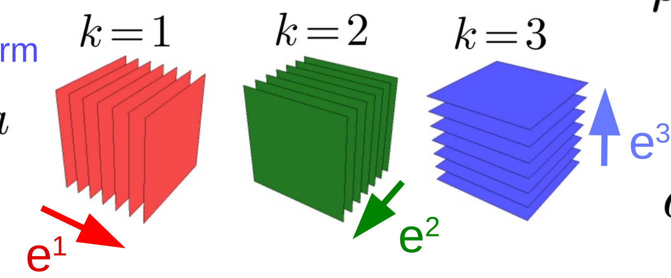
2-form

1-form

1-form

(1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

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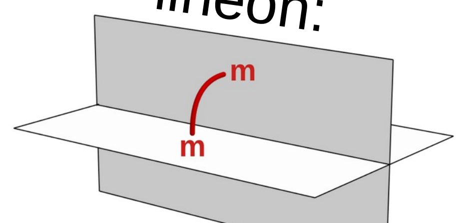
$b \to b + d\mu$

$B^k \to B^k - n_k \mu$ ⇨ **m** bound to flux strings: $di = \sum_k n_k I^k$

lineon:



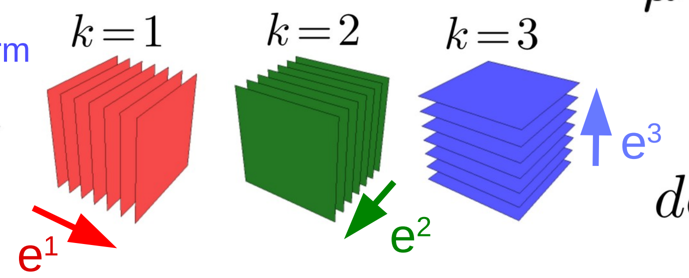
- Foliating layers are orthogonal to 1-form e^k_μ

2-form, 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

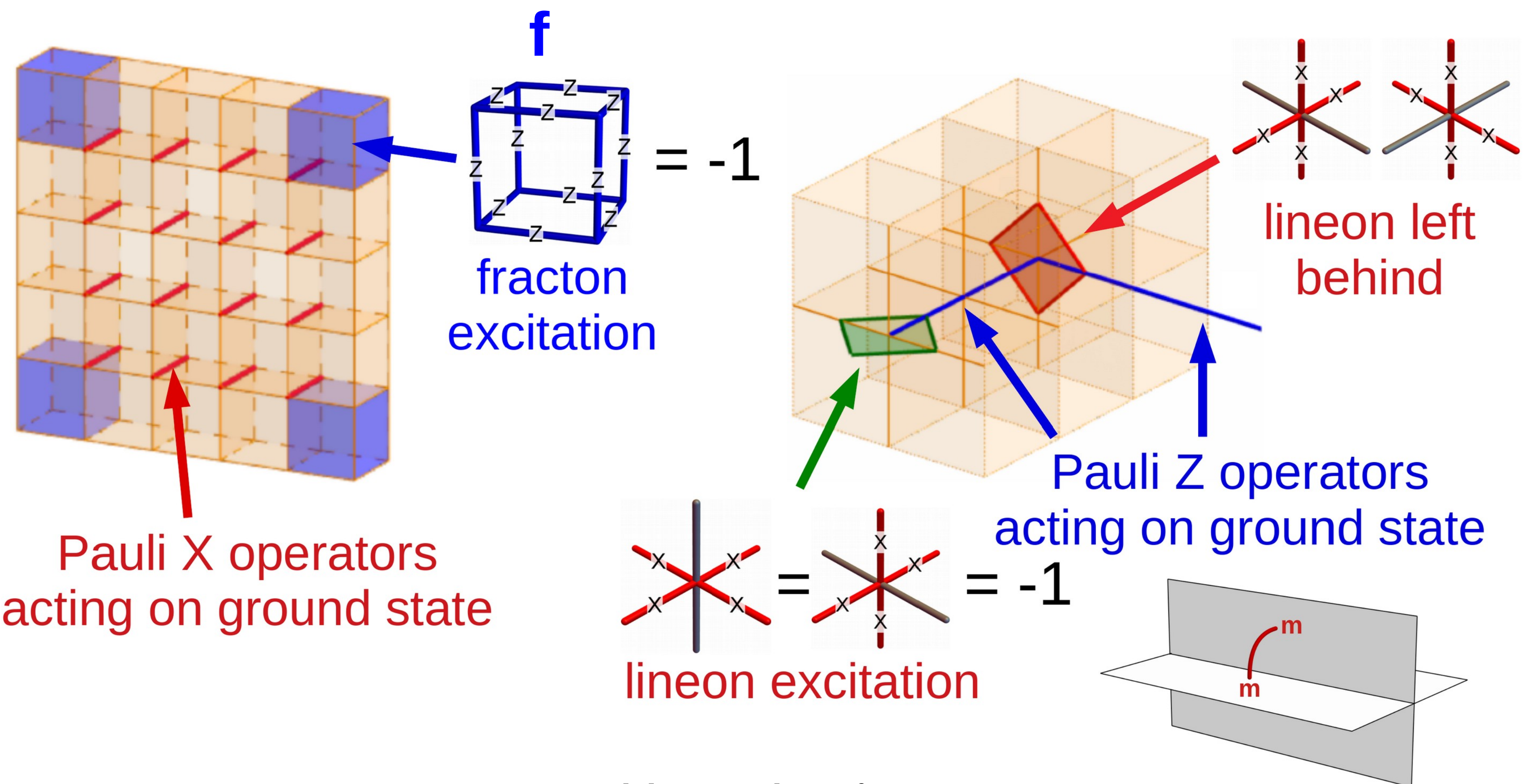
1-form, (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Matches X-cube Model



- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

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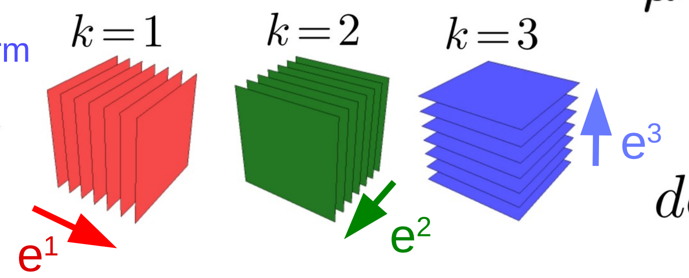
1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

$$L' = L - \sum_k A^k \wedge J^k - a \wedge j - \sum_k B^k \wedge I^k - b \wedge i$$

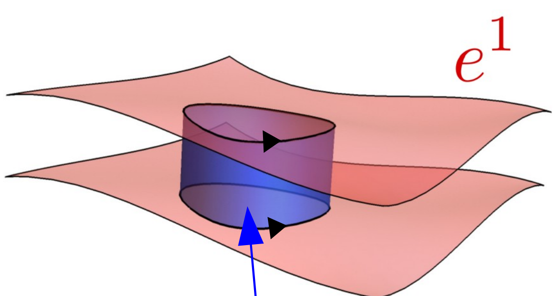
$$dJ^k \wedge e^k = -m_k\, j \wedge e^k \qquad dj = 0$$

$$dI^k = 0 \qquad I^k \wedge e^k = 0$$

$$di = \sum_k n_k I^k$$

$$A^k \to A^k + d\zeta^k \qquad \zeta^k \wedge e^k = 0 \qquad b \to b + d\mu \qquad B^k \to B^k + d\chi^k - n_k \mu + \alpha^k e^k$$

Example Operators

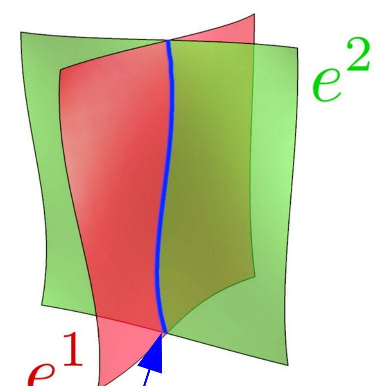


move pair of

fractons $e^{\mathrm{i}\int_{\mathcal{M}} A^1} = e^{\mathrm{i}\int A\wedge J}$

or lineons: $e^{\mathrm{i}\int_{\mathcal{M}}(dB^1 + n_1 b)}$

around a pair of loops



$e^{\mathrm{i}\int_{\mathcal{M}}(B^1 - B^2)}$

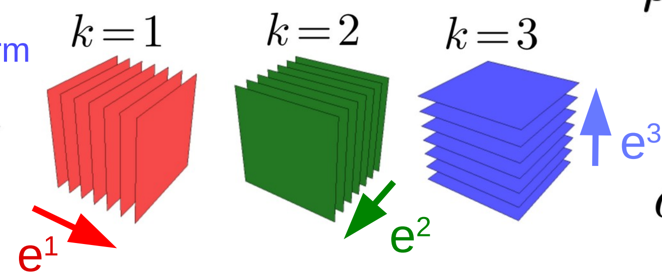
move lineon along intersection of two layers $(n_1 = n_2)$

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b)\wedge A^k + \frac{N}{2\pi} b \wedge da$$

2-form, 1-form, (1+1)-form, 1-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

$$L' = L - \sum_k A^k \wedge J^k - a\wedge j - \sum_k B^k \wedge I^k - b \wedge i$$

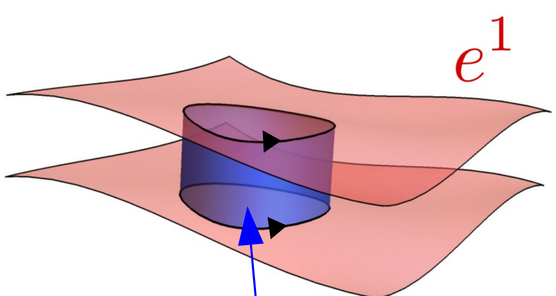
$$dJ^k \wedge e^k = -m_k\, j \wedge e^k \qquad dj = 0$$

$$dI^k = 0 \qquad I^k \wedge e^k = 0$$

$$di = \sum_k n_k I^k$$

$$A^k \to A^k + d\zeta^k \qquad \zeta^k \wedge e^k = 0 \qquad b \to b + d\mu \qquad B^k \to B^k + d\chi^k - n_k\mu + \alpha^k e^k$$

Questions?

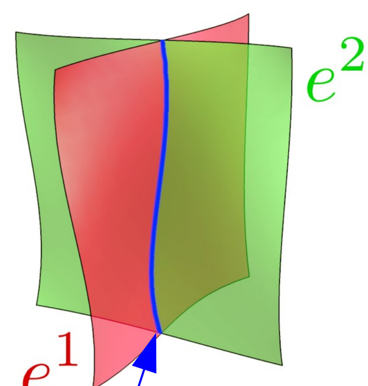


move pair of

fractons $e^{\mathrm{i}\int_{\mathcal{M}} A^1} = e^{\mathrm{i}\int A\wedge J}$

or lineons: $e^{\mathrm{i}\int_{\mathcal{M}}(dB^1 + n_1 b)}$

around a pair of loops



$e^{\mathrm{i}\int_{\mathcal{M}}(B^1 - B^2)}$

move lineon along intersection of two layers $(n_1 = n_2)$

Conclusions

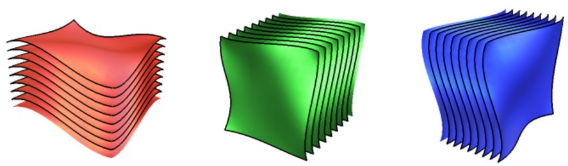
- U(1) fracton models only gauge invariant on certain manifolds

symmetric tensor gauge theory	gauge invariant manifold
3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
3D gapless traceless vector	Einstein with constant curvature
2D gapped traceless scalar	constant curvature
2D gapless traceless scalar	constant curvature
any-D gapless traceful scalar	flat
any-D gapless traceful vector	flat

KS, Prem, Pretko 2018

- Foliated QFT for many type-I fracton models

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$



$A^k \wedge e^k = 0$ $\quad de^k = e^k \wedge \beta^k$ $\quad$ often $\beta = 0$

  - Resembles coupled BF theories with a foliation-dependent constraint
  - Gauge invariance enforces particle mobility constraints

Slagle 2020

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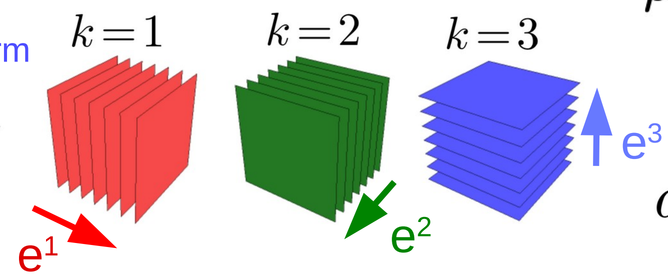
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

$$dJ^k \wedge e^k = -j \wedge e^k$$

$$dI^k = 0, \quad I^k \wedge e^k = 0$$

$$dj = 0$$

$$di = \sum_k I^k$$

$$J^k \to J^k + \phi^k \wedge e^k$$

Equations of Motion

$$\frac{M_k}{2\pi}(dB^k + n_k b) \wedge e^k = J^k \wedge e^k$$

$$\frac{M_k}{2\pi} dA^k = I^k$$

$$\frac{N}{2\pi} db = j$$

$$\frac{N}{2\pi}(da + \sum_k m_k A^k) = i$$

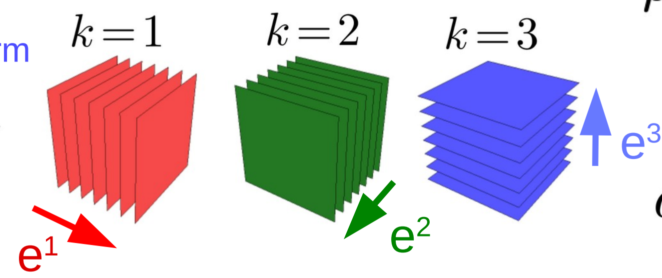
- Foliating layers are orthogonal to 1-form e^k_μ

2-form, 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form, (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Quantized Coefficients

$$A^k \to A^k + d\zeta^k \qquad B^k \to B^k + d\chi^k - \underline{n_k}\mu$$

$$a \to a + d\lambda - \sum_k \underline{m_k}\zeta^k \qquad b \to b + d\mu$$

- compact gauge fields implies: $\underline{m_k} \equiv \frac{n_k M_k}{N} \in \mathbb{Z}$ and $\underline{n_k} \in \mathbb{Z}$

- Foliating layers are orthogonal to 1-form e^k_μ

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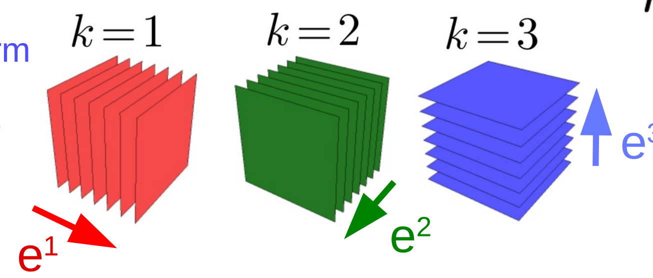

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- compact gauge fields implies: $\underline{m_k} \equiv \frac{n_k M_k}{N} \in \mathbb{Z}$ and $\underline{n_k} \in \mathbb{Z}$

$$\oint L \to \oint L + \sum_k M_k \underbrace{\frac{1}{2\pi}\oint(dB^k \wedge d\zeta^k + d\chi^k \wedge dA^k)}_{\in 2\pi\mathbb{Z}} + N \underbrace{\frac{1}{2\pi}\oint(db \wedge d\lambda + d\mu \wedge da)}_{\in 2\pi\mathbb{Z}}$$

- large gauge transformation invariance implies: $M_k, N \in \mathbb{Z}$

- Foliating layers are orthogonal to 1-form e^k_μ

2-form　　1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

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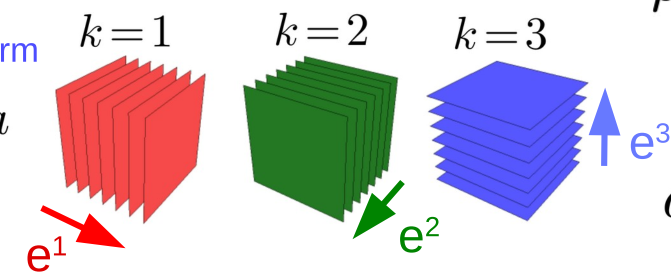
$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Coefficient Identification

Equations of motion:

$$db = dA^k = 0$$

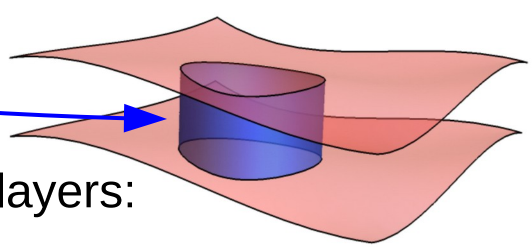
$$\oint b \in \frac{2\pi}{N}\mathbb{Z}$$

$$\oint_F A^k \in \frac{2\pi}{M_k}\mathbb{Z}$$

$$\sum_k \frac{M_k n_k}{2\pi} \int b \wedge A^k \in 2\pi \sum_k \frac{n_k}{N}\mathbb{Z}$$

same action: $\boxed{n_k \sim n_k + N_k}$

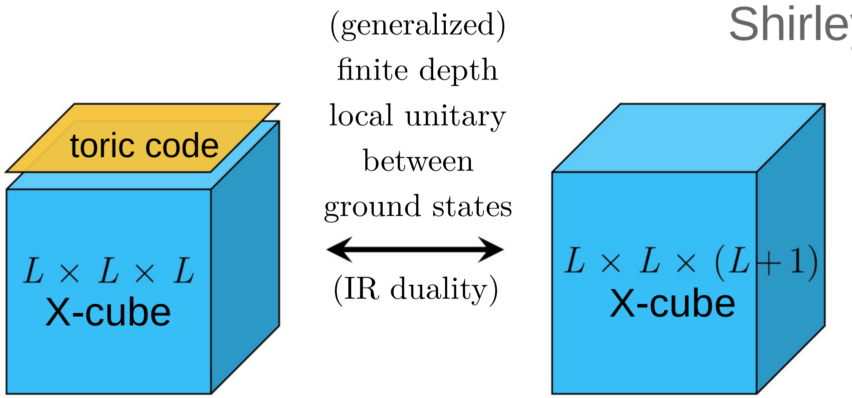
integrate over a
2-manifold between layers:



Foliated Fracton Order (FFO)

- Exfoliation: a local unitary can decouple a layer:

Shirley, Slagle, Chen 2017



- Toric code anyons become X-cube planons
  - Ground state degeneracy on a torus $2^{2L_x+2L_y+2L_z-3}$ is absorbed via $L_z \to L_z + 1$

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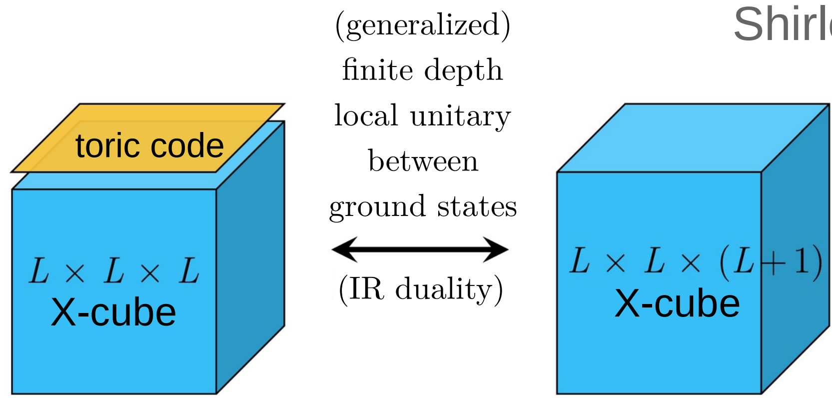


- Toric code anyons become X-cube planons
  - Ground state degeneracy on a torus $2^{2L_x+2L_y+2L_z-3}$ is absorbed via $L_z \to L_z + 1$
- It's *not* possible to decouple all layers
  - On a lattice, at least one layer must remain

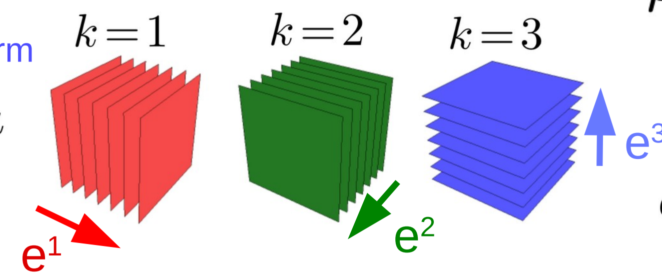
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

Exfoliation in Foliated QFT

- Exfoliate all layers within $z_1 < z < z_2$ $(e_1 = dz)$:
  - i.e. $\tilde{n}_1(z) = \begin{cases} n_1 & z \leq z_1 \text{ or } z \geq z_2 \\ 0 & z_1 < z < z_2 \end{cases}$

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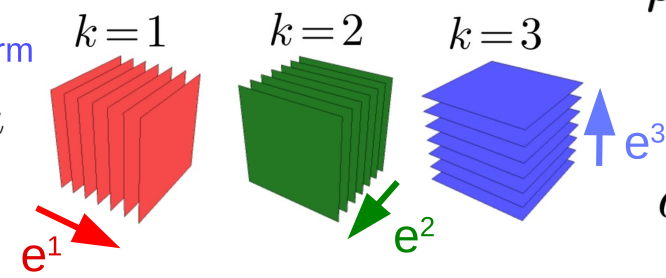

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Exfoliation in Foliated QFT

- Exfoliate all layers within $z_1 < z < z_2$ $(e_1 = dz)$:
  - i.e. $\tilde{n}_1(z) = \begin{cases} n_1 & z \le z_1 \text{ or } z \ge z_2 \\ 0 & z_1 < z < z_2 \end{cases}$
  - claim: this interface can be created and moved by a duality transformation:

$$a \leftrightarrow \tilde{a} = \begin{cases} a & z \le z_1 \text{ or } z_2 \le z \\ a + m_1 \int_{z_1}^{z} A^1 & z_1 < z < z_2 \end{cases}$$

$$A^1 \leftrightarrow \tilde{A}^1 = A^1 + \delta(z - z_2) \int_{z_1}^{z_2} dz\, A^1$$

$$B^1 \leftrightarrow \tilde{B}^1 = \begin{cases} B^1 & z \le z_1 \text{ or } z_2 \le z \\ B^1 - B^1(z_2) + n_1 \int_{z}^{z_2} b & z_1 < z < z_2 \end{cases}$$

- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

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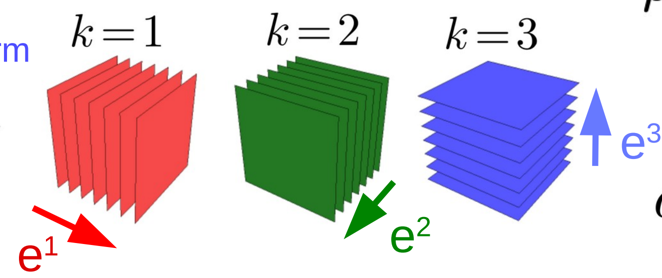
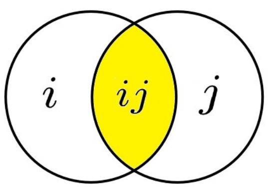


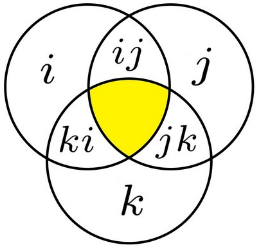
$$de^k = e^k \wedge \beta^k$$

review: Gauge Fields

- Gauge fields on different patches (charts) i, j, k of spacetime are defined up to:
  - 0-form gauge field: $A_i - A_j \in 2\pi\mathbb{Z}$
  - 1-form gauge field: $A_i - A_j = dg_{ij}$
    - Transition functions g_{ij} satisfy cocycle condition



$$g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$$



- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

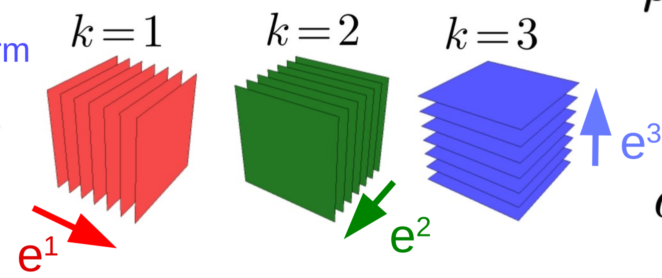
2-form

1-form

1-form

(1+1)-form

$$A^k \wedge e^k = 0$$

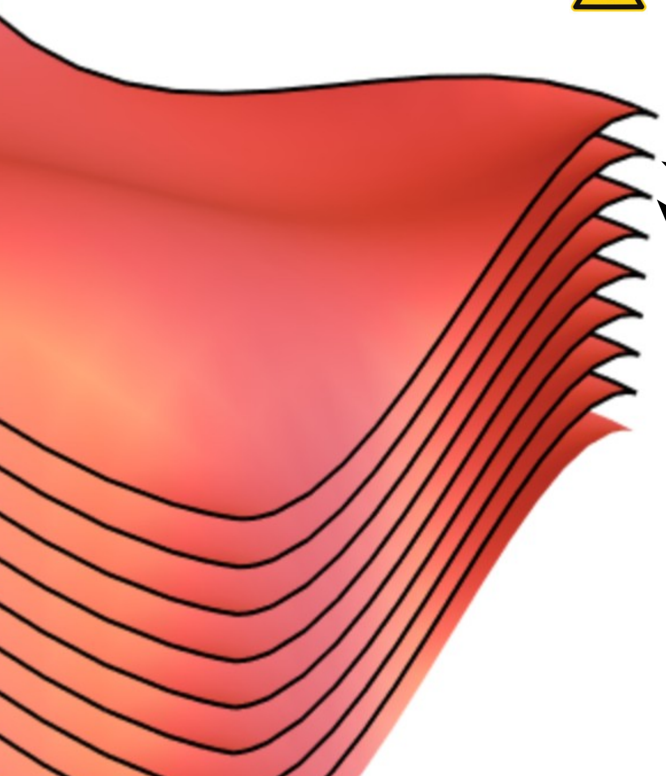


$$de^k = e^k \wedge \beta^k$$

Intuition: Foliated Gauge Field

- Transition function for each layer

⚠ (but in the continuum)



$$A^{(\text{layer } \#1)} \leftarrow A_i^{(\text{layer } \#1)} - A_j^{(\text{layer } \#1)} = dg_{ij}^{(\text{layer } \#1)}$$

$$A^{(\text{layer } \#2)} \leftarrow A_i^{(\text{layer } \#2)} - A_j^{(\text{layer } \#2)} = dg_{ij}^{(\text{layer } \#2)}$$

$$A^{(\text{layer } \#3)}$$

$$\vdots \qquad\qquad \vdots$$

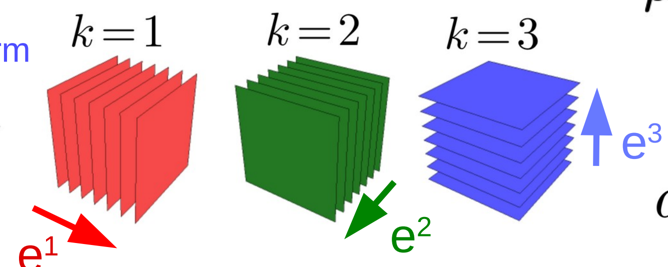
- Foliating layers are orthogonal to 1-form e^k_μ

2-form 1-form

$$L = \sum_k \frac{M_k}{2\pi}(dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

1-form (1+1)-form

$$A^k \wedge e^k = 0$$

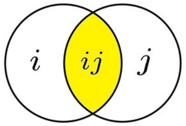


$$de^k = e^k \wedge \beta^k$$

1-form Gauge Field → Foliated (1+1)-form Gauge Field

1-form 0-form

$$A_i - A_j = dg_{ij}$$



(1+1)-form (0+1)-form

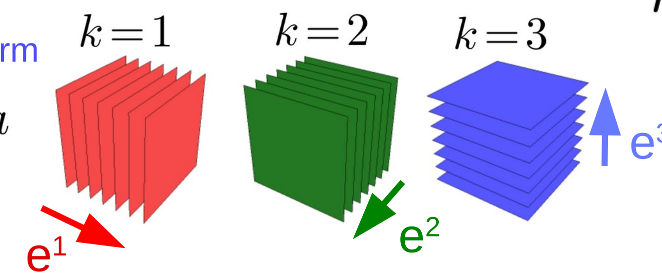
$$A_i - A_j = dg_{ij}$$

g_{ij} is also foliated: $g_{ij} \wedge e = 0$

- Foliating layers are orthogonal to 1-form e^k_μ

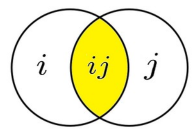
$$L = \sum_k \frac{M_k}{2\pi} (d\overset{\text{2-form}}{B^k} + n_k \underset{\text{1-form}}{b}) \wedge \underset{\text{(1+1)-form}}{A^k} + \frac{N}{2\pi} b \wedge d\overset{\text{1-form}}{a}$$

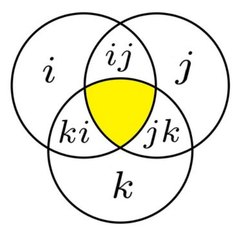
$$A^k \wedge e^k = 0$$



$$de^k = e^k \wedge \beta^k$$

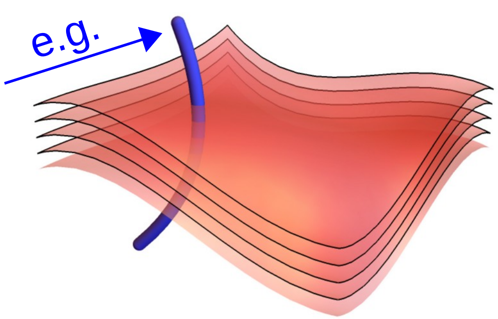
1-form Gauge Field	Foliated (1+1)-form Gauge Field
$A_i - \overset{\text{1-form}}{A_j} = d\overset{\text{0-form}}{g_{ij}}$	$A_i - \overset{\text{(1+1)-form}}{A_j} = d\overset{\text{(0+1)-form}}{g_{ij}}$
	g_{ij} is also foliated: $g_{ij} \wedge e = 0$
$g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$	$\int_s g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$





any line segment

e.g.



Ground State Degeneracy

- solve equations of motion and plug into action:

$$S = \frac{N}{2\pi} \sum_{k \neq b} \frac{L_k}{l_k} \int_0^{l_k} \mathrm{d}x^k \, p_b^k(t, x^k) \, \partial_t q_b^k(t, x^k)$$

$$0 \leq x < l_x, \, 0 \leq y < l_y, \, 0 \leq z < l_z \,, \;\; e_\mu^k = \delta_\mu^k$$

- add a cuttoff for finite degeneracy: $a_k \sim \frac{l_k}{L_k}$.

$$S \sim \frac{N}{2\pi} \sum_{k \neq b} \sum_{x^k = 0, a_k, .., (L_k - 1) a_k} p_b^k(t, x^k) \, \partial_t q_b^k(t, x^k)$$

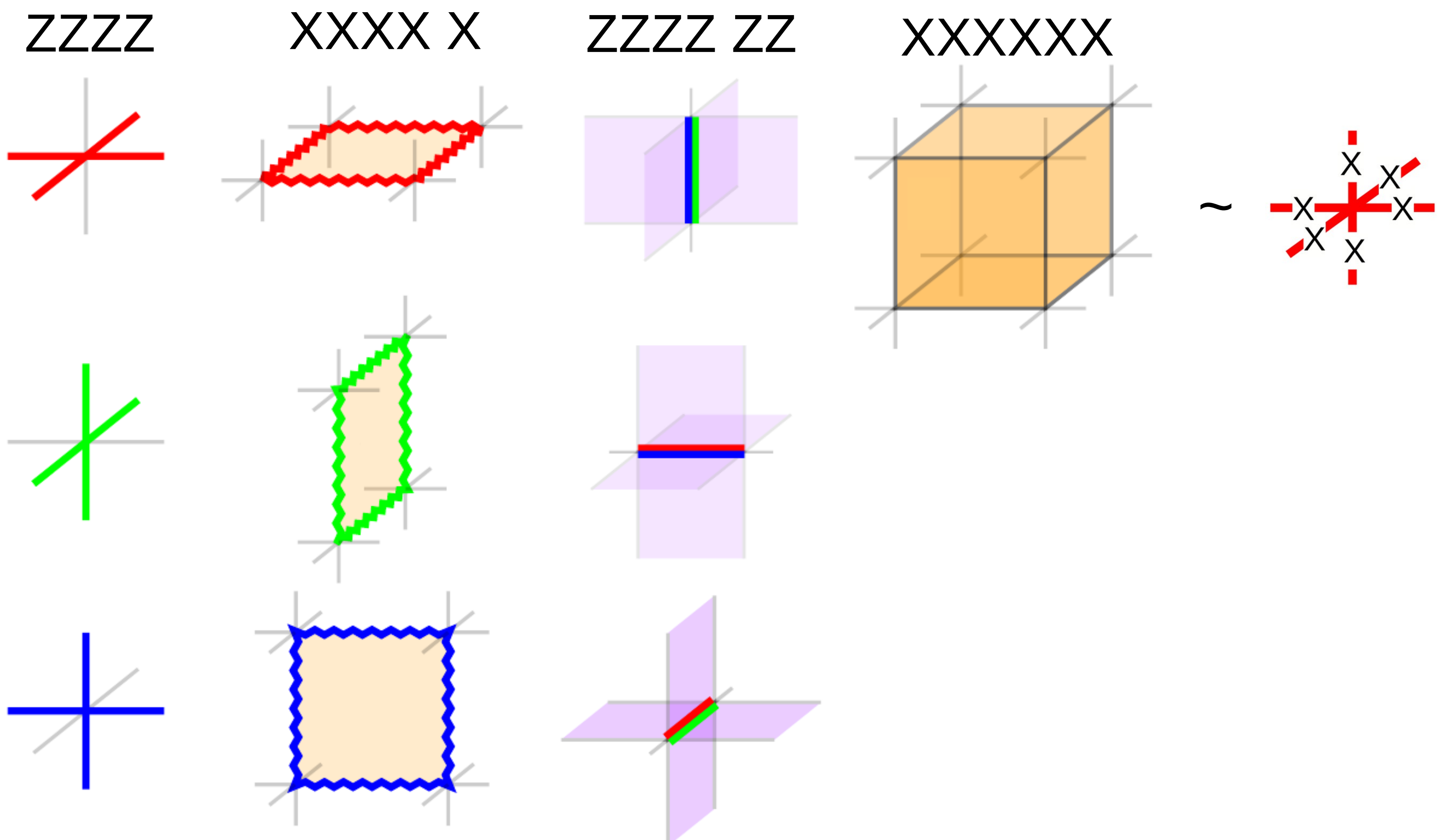
where $q_a^k(t, 0) = p_a^k(t, 0) = 0 \text{ if } \epsilon^{akb} = 1$

- degeneracy $= N^{2L_x + 2L_y + 2L_z - 3}$

Lattice Model

2D toric code
defect layer

coupled to
3D toric code



Example: Discrete Disclination Curvature