

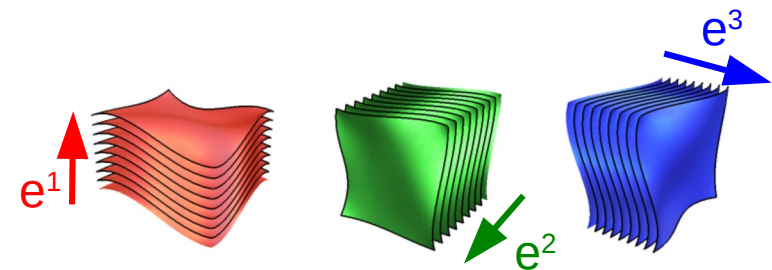
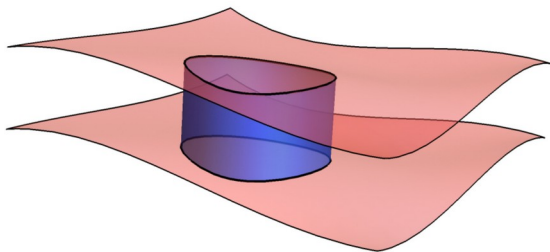
Foliated QFT and Non-Lorenzian Geometry of Fracton Order

Kevin Slagle
Caltech

April 7, 2021

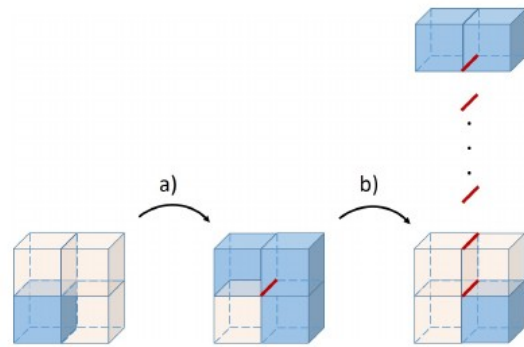
arXiv:1807.00827, 2008.03852

slides: tinyurl.com/fracton



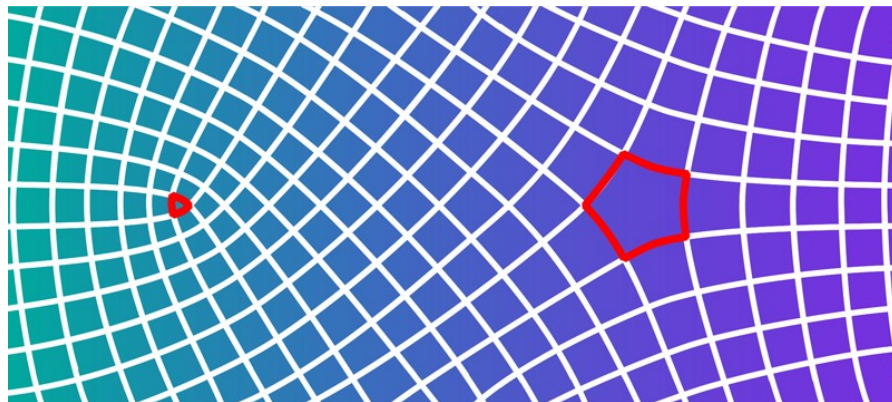
Fracton Motivation

- solvable models of quantum glassiness (Chamon 2005)



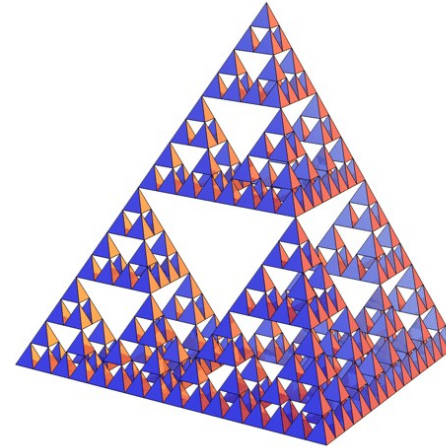
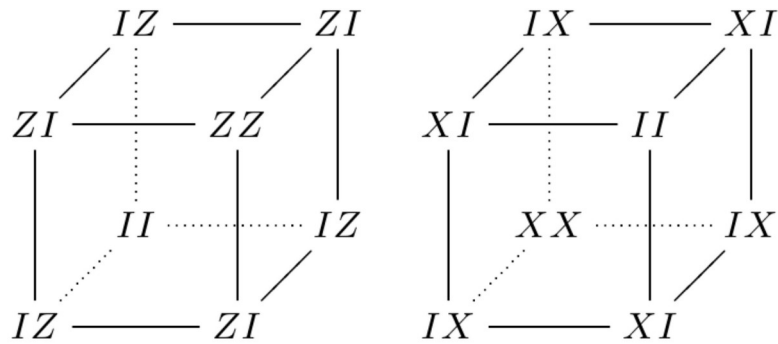
Prem, Haah, Nandkishore 2017

- dualities to elasticity theory of two-dimensional crystals (Pretko, Radzihovsky 2018)

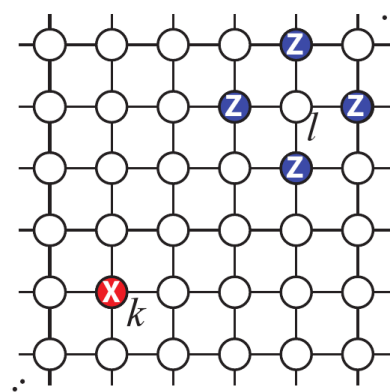


Fracton Motivation

- robust quantum information storage (Haah 2013)



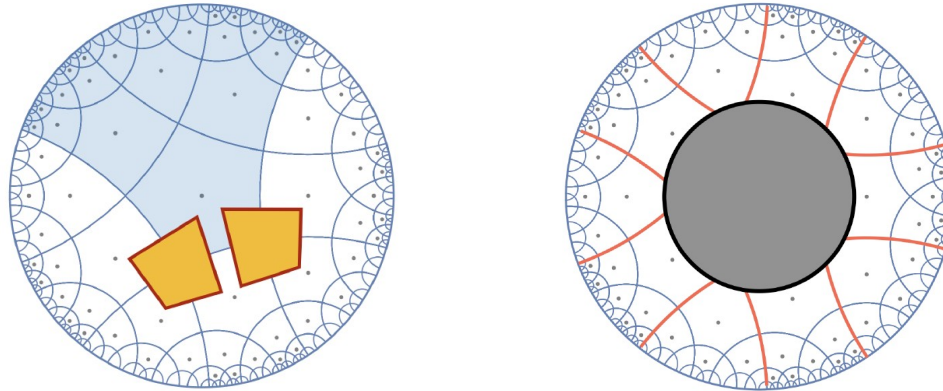
- measurement based quantum computation from ungauged fracton models which have subsystem symmetries (Else, Bartlett, Doherty 2012)



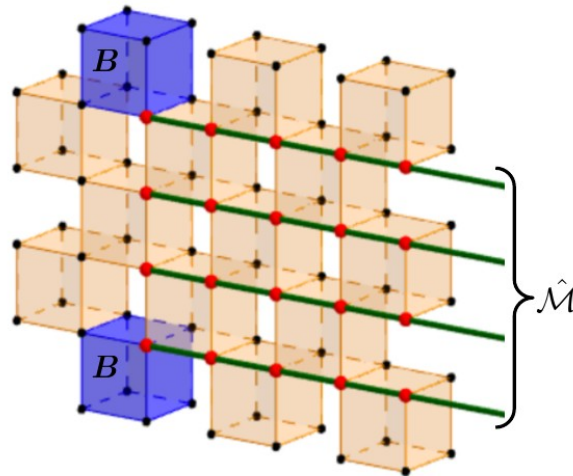
Raussendorf, Okay, Wang, Stephen, Nautrup 2019

Fracton Motivation

- toy model of holography (Yan 2018, ...)



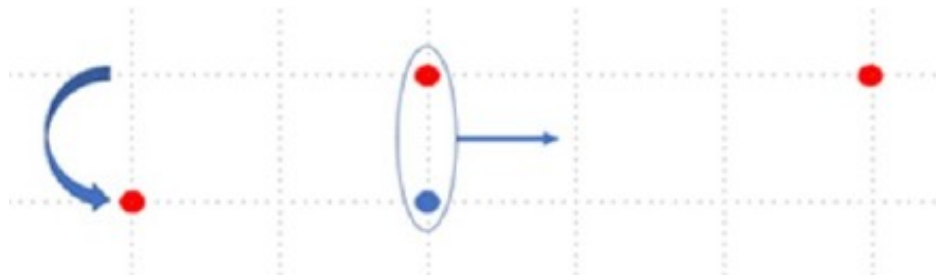
- new phases of matter (Vijay, Haah, Fu 2015)



Fracton Motivation

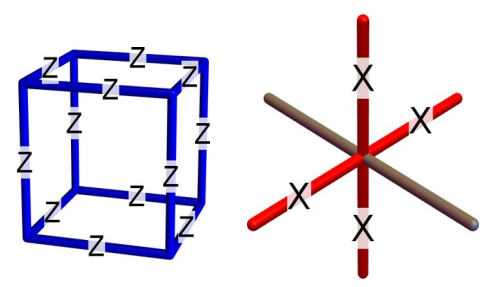
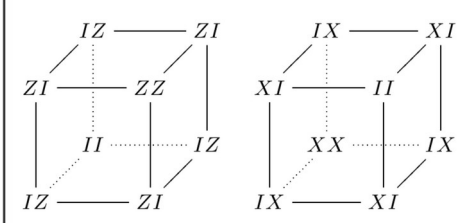
- Linearized gravity \rightarrow U(1) fracton models
 - Zheng-Cheng Gu, Xiao-Gang Wen (2006, 2009)
 - Cenke Xu (2006, 2006)
 - Cenke Xu, Petr Horava (2010)
- connections to quantum gravity (Pretko 2017)

fractons move via dipole exchange more easily near other fractons, which leads to an effective attractive interaction



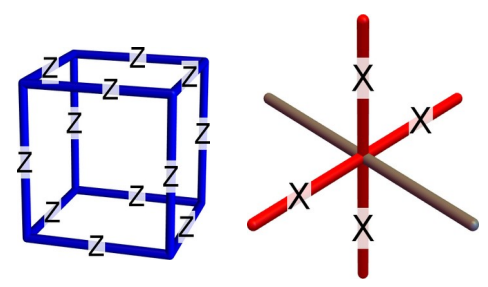
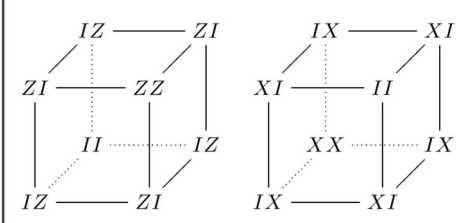
Kinds of Fracton Models

(Shao and Karch discussed ungauged versions of the first two kinds)

	$U(1)$ symmetric tensor gauge theory	foliated (type-I)	type-II
example models	scalar charge Pretko 2017 $H = \int E^2 + B^2$ $B^{ij} = \delta^{jc} \epsilon^{iab} \partial_a A_{bc}$	X-cube Vijay, Haah, Fu 2015 	Haah's code, Yoshida's fractal liquids 
spectrum	gapless	gapped	gapped
charge conservation	conserved dipole moment	conserved on stacks 2D surfaces	conserved on fractal subsets
spacetime structure	Einstein manifolds $R_{ab} \propto g_{ab}$	foliated manifolds	discrete groups? Tian, Samperton, Wang 2018

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this talk

Fractons from U(1) Gauge Theory

Pretko 2017

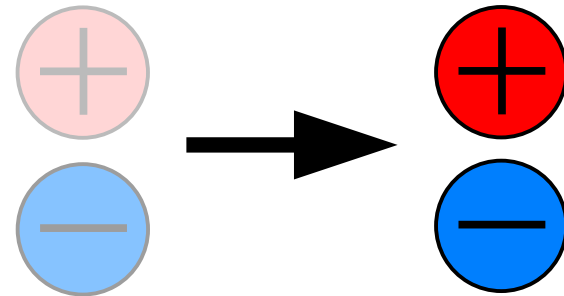
- Generalize U(1) Maxwell electromagnetism:
- Add dipole moment conservation in addition to charge conservation $P = \int \rho(\vec{r}) \vec{r}$

moving charge
changes dipole moment



isolated charges
are immobile

moving a dipole
conserves dipole moment



dipoles are mobile

Fractons from U(1) Gauge Theory

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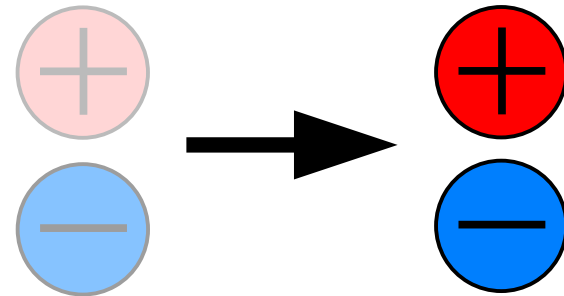
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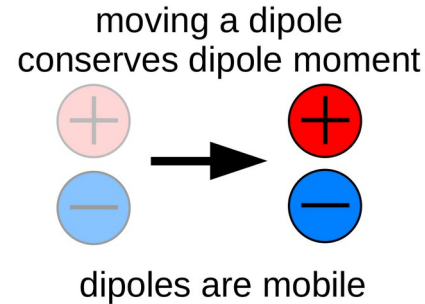
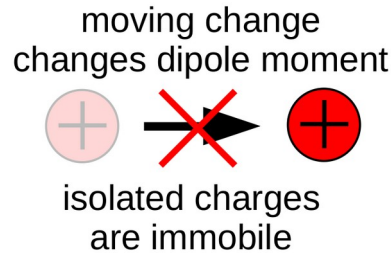


dipoles are mobile

- **fracton**: an immobile particle
 - often a *fraction* of a mobile particle

Fractons from U(1) Gauge Theory

Pretko 2017



↑
previous
chalkboard

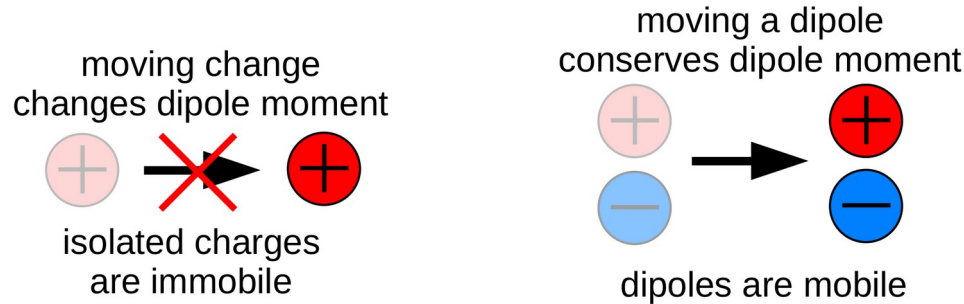
new Gauss law: $\partial_i \partial_j E^{ij} = \rho$

results in charge and dipole moment conservation:

$$\int d^d x \rho = \text{constant} \quad \int d^d x (\rho \vec{x}) = \text{constant}$$

Fractons from U(1) Gauge Theory

Pretko 2017



previous
chalkboard

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symmetric tensor gauge field and gauge symmetry:

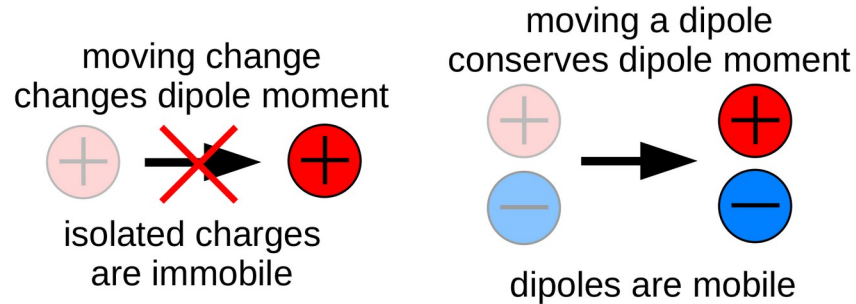
$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \lambda$$

scalar charge theory:

$$H = \int d^d x \frac{1}{2} (E^2 + B^2) \quad B^{ij} = \delta^{jc} \epsilon^{iab} \partial_a A_{bc}$$

Fractons from U(1) Gauge Theory

Pretko 2017



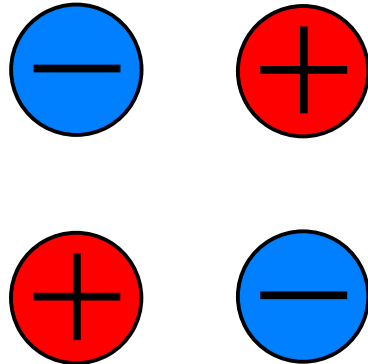
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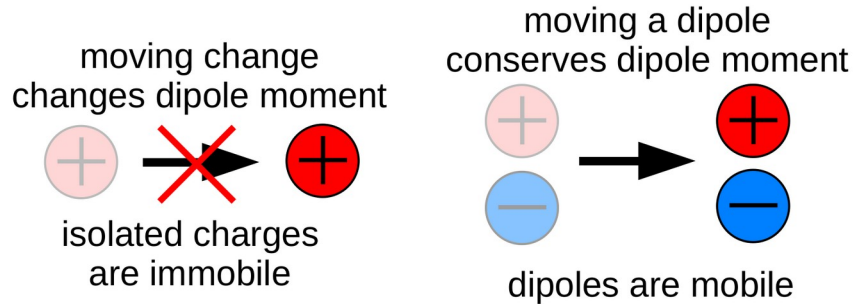
quadrupoles have no
charge or dipole moment



quadrupoles can be
created from the vacuum

Fractons from U(1) Gauge Theory

Pretko 2017



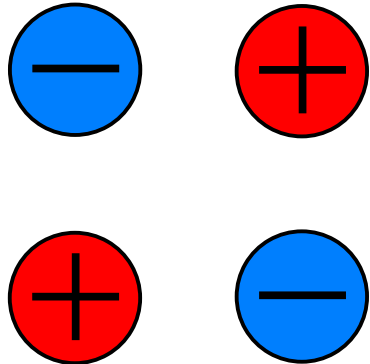
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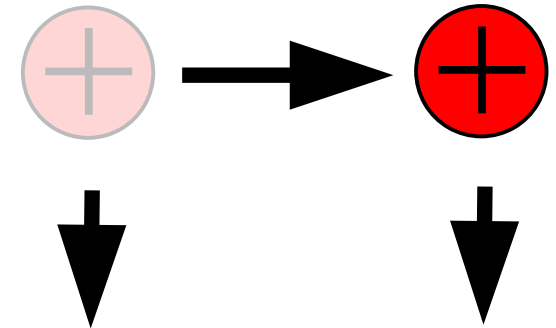
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quadrupoles have no charge or dipole moment



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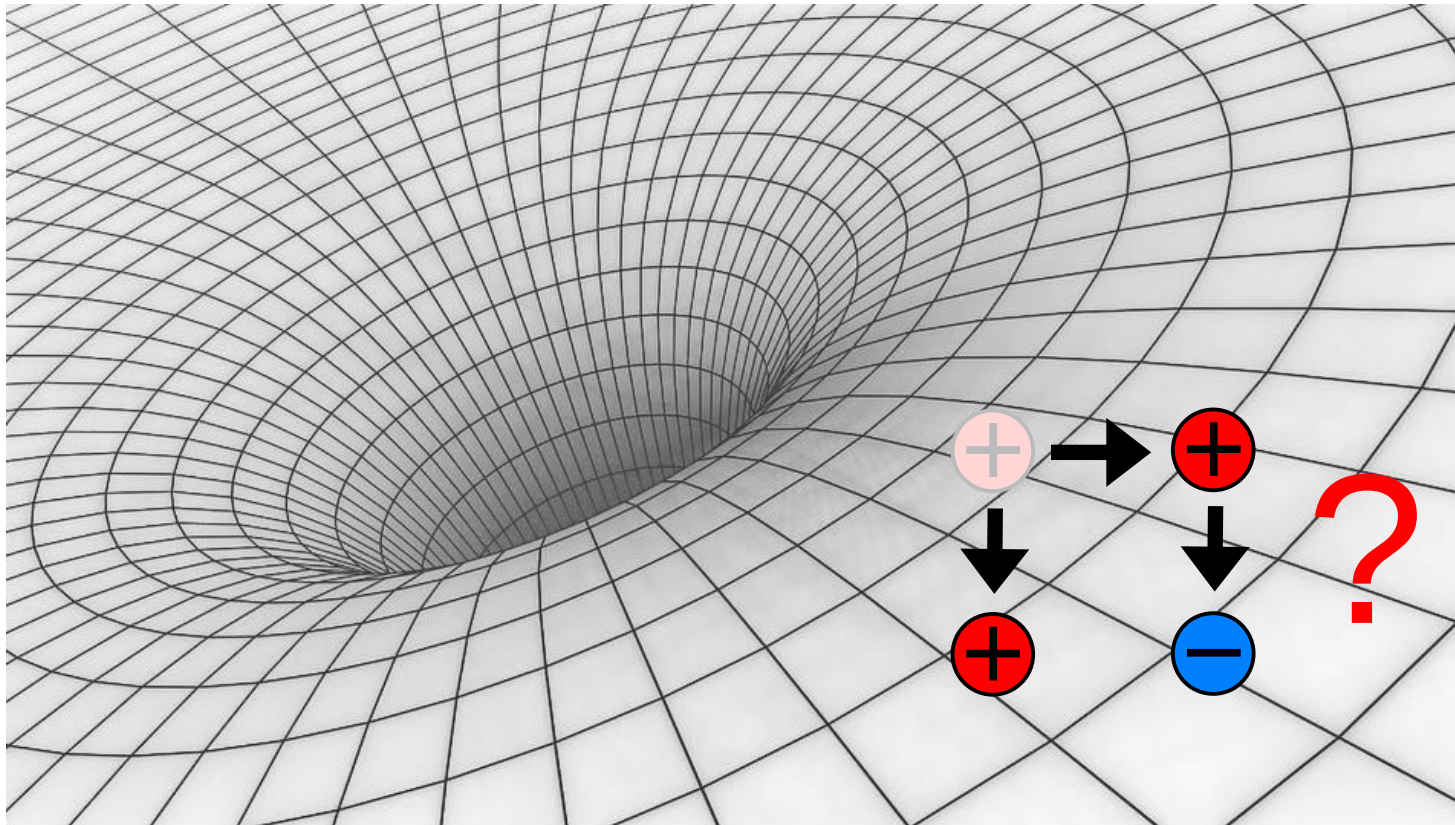
a charge can move:



if we create a dipole:

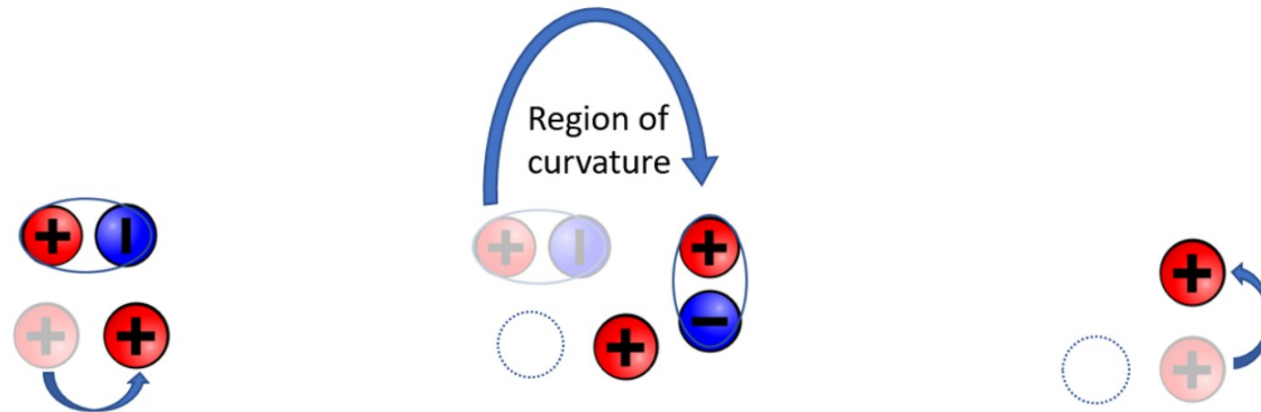


What happens if spacetime is curved?



U(1) Fractons on Curved Space?

- Curvature can grant fractons mobility via dipole mobility
 - dipole rotates via parallel transport



KS, Prem, Pretko 2018



Abhinav Prem



Michael Pretko

U(1) Fractons on Curved Space

- Curvature can grant fractons mobility via dipole mobility
 - also results in a loss of gauge invariance
- Does not occur when on Einstein manifolds for the traceless scalar charge model:

$$R_{ab} \propto g_{ab}$$

- i.e. manifolds with no torsion or stress-energy tensor

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \underbrace{\frac{8\pi G}{c^4} T_{\mu\nu}}_{\text{must be zero}}$$

Traceless Scalar Charge Model

- Traceless scalar charge model:
 - $g_{ij}E^{ij} = 0$
 - conserved charge, dipole, and a quadrupole $\int d^d x x^2 \rho$
 - fracton dipoles are planons that only move perpendicular to dipole moment

Traceless Scalar Charge Model

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- On Einstein manifolds:

- Dipole moment of planons won't change under closed loops



U(1) Fractons on Curved Space

symmetric tensor gauge theory	gauge invariant manifold
3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
3D gapless traceless vector	Einstein with constant curvature
2D gapped traceless scalar	constant curvature
2D gapless traceless scalar	constant curvature
any-D gapless traceful scalar	flat
any-D gapless traceful vector	flat

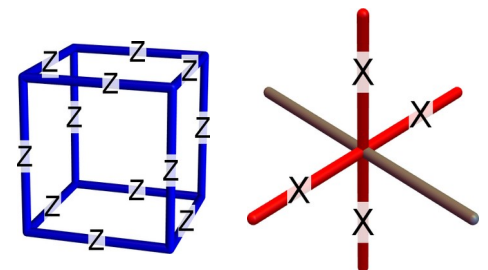
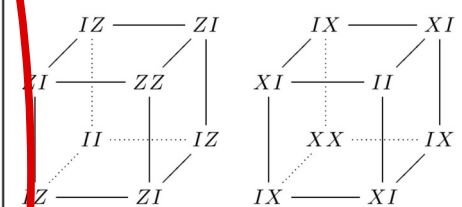
- We only considered spatial curvature
 - U(1) fractons in curved spacetime is an open problem
 - nontrivial generalization due to no Lorentz symmetry

Questions?

symmetric tensor gauge theory	gauge invariant manifold
$g_{ij}E^{ij} = 0$ 3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
3D gapless traceless vector	Einstein with constant curvature
2D gapped traceless scalar	constant curvature
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Part 2

	$U(1)$ symmetric tensor gauge theory	foliated (type-I)	type-II
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spacetime structure	Einstein manifolds $R_{ab} \propto g_{ab}$	foliated manifolds	discrete groups? Tian, Samperton, Wang 2018

review: X-cube fracton model

Vijay, Haah, Fu 2016

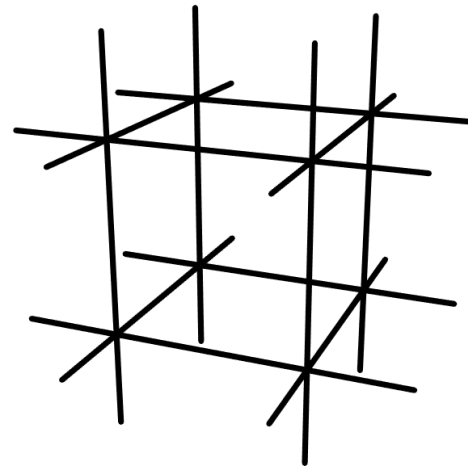
$$H = - \sum_{\text{cube}} Z - \sum_{*} [\text{X-cube} + \text{X-cube} + \text{X-cube}]$$

- qubits on links of a cubic lattice

- gapped & exactly solvable

$$Z = \sigma^z$$

$$X = \sigma^x$$

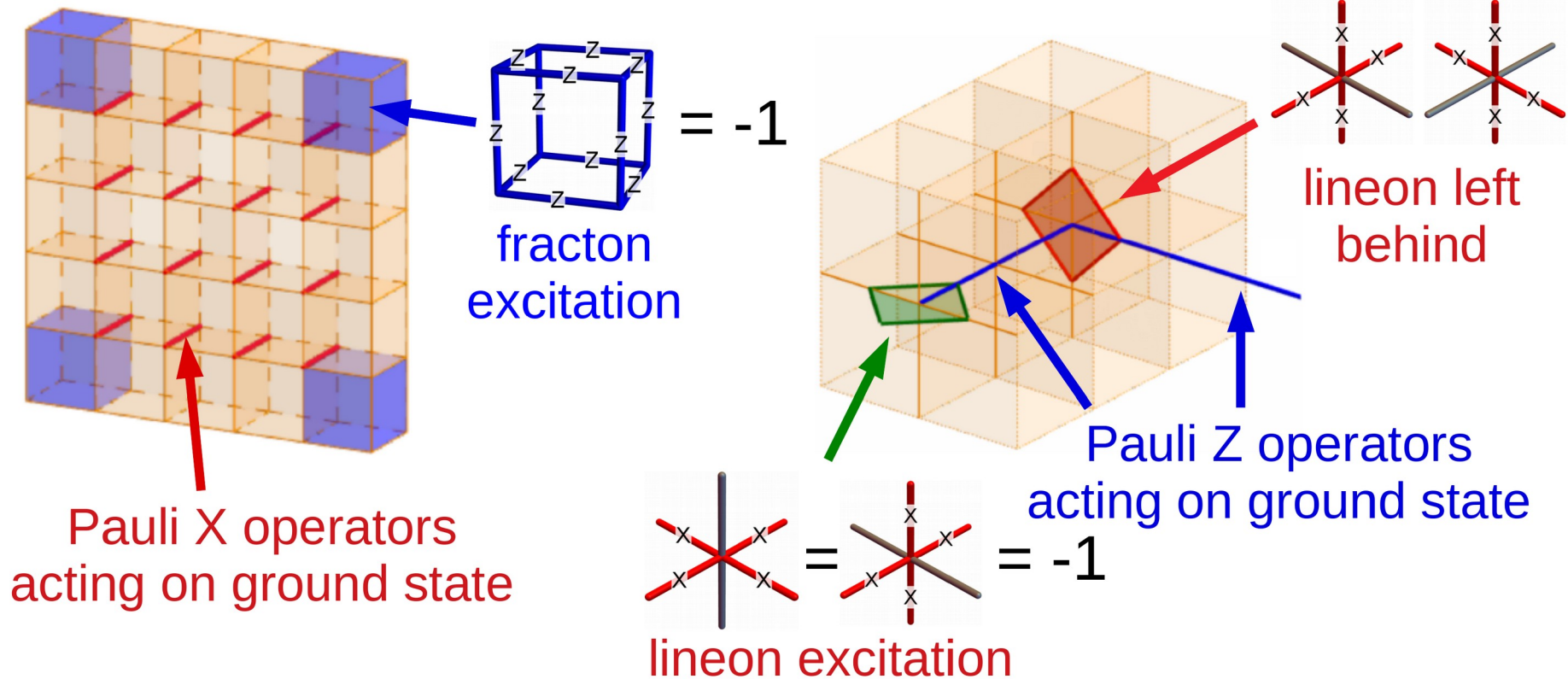


- degen = 2^{6L-3}

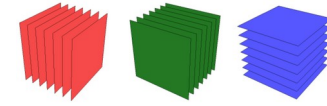
Subdimensional Excitations

Vijay, Haah, Fu 2016

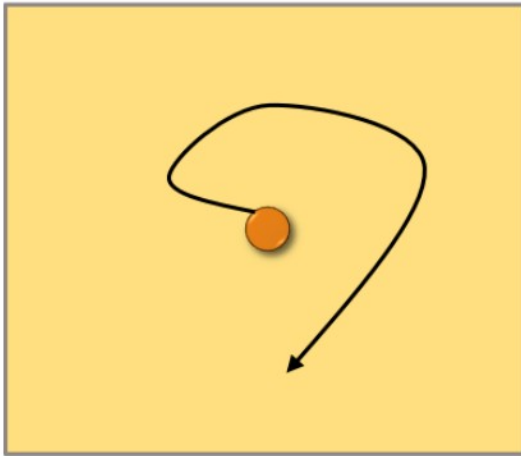
- Create fractons at corners of rectangular operators
- Create lineons at ends of line operators



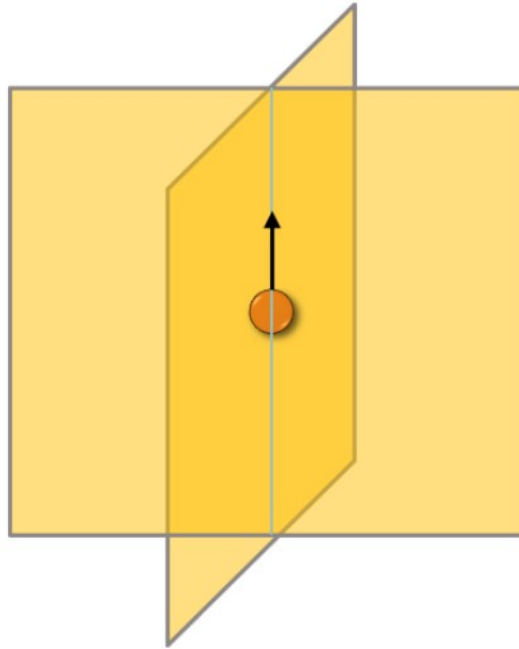
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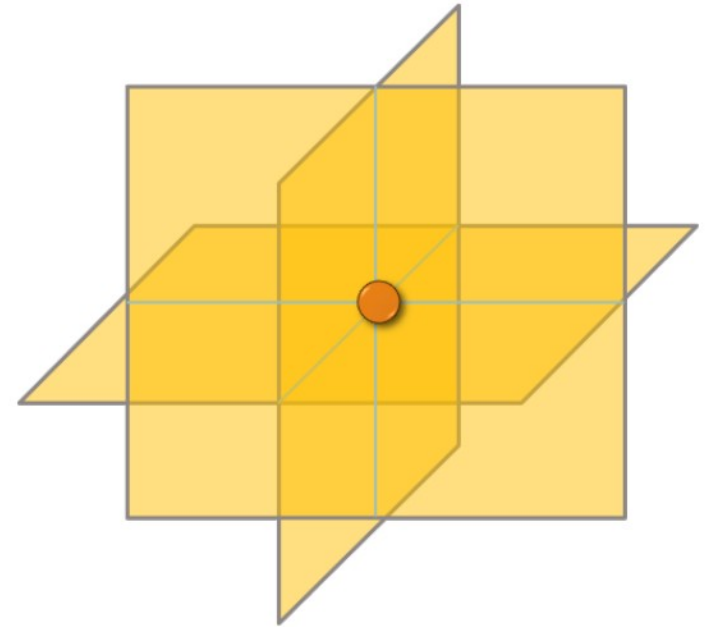
- Mobility restrictions determined by a layering structure



planon
(moves along
2D planes)

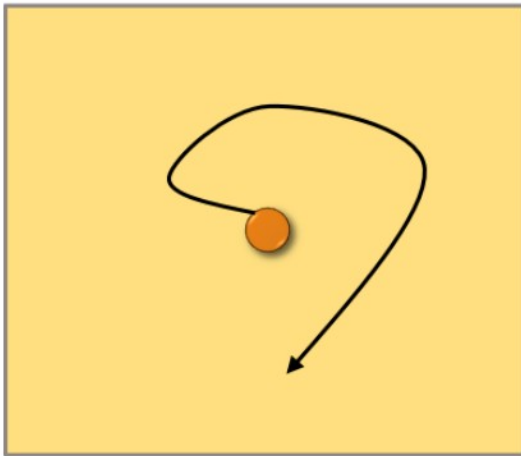


lineon
(moves along
1D lines)

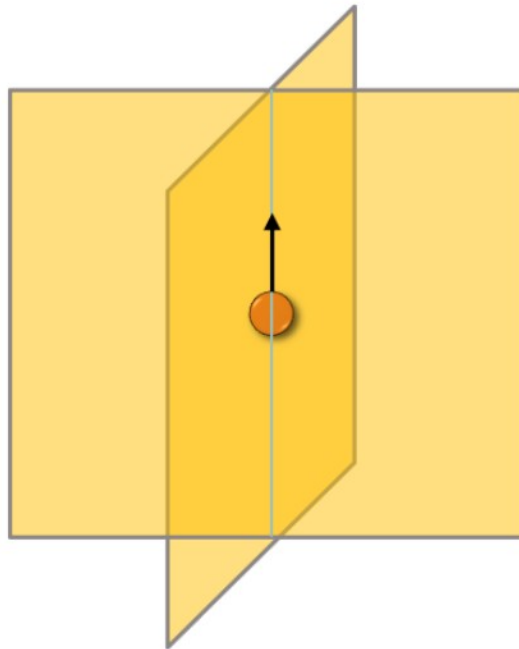


fracton
(immobile)

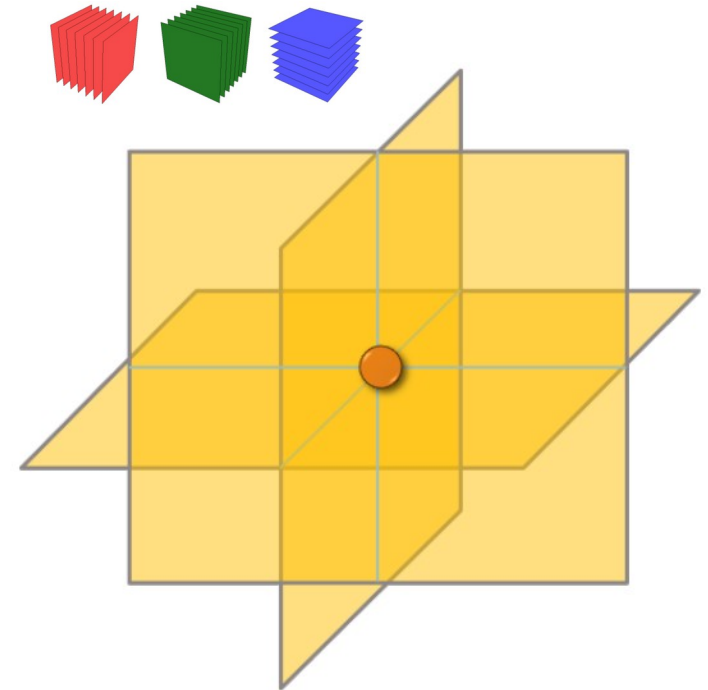
QFT description?



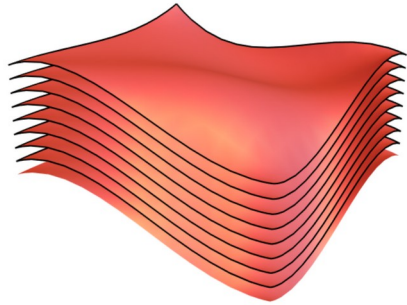
planon
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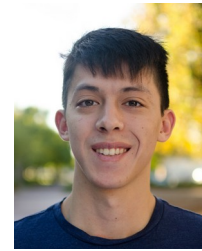
lineon
(moves along
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fracton
(immobile)



Foliation



Wilbur
Shirley

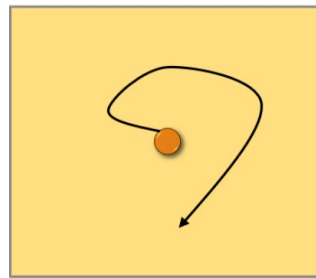


Zhengan
Wang

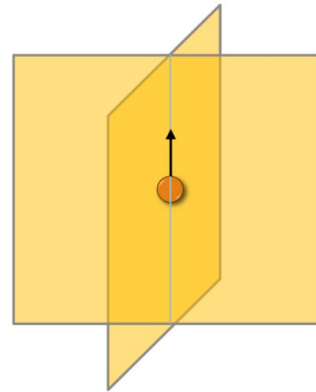


Xie Chen

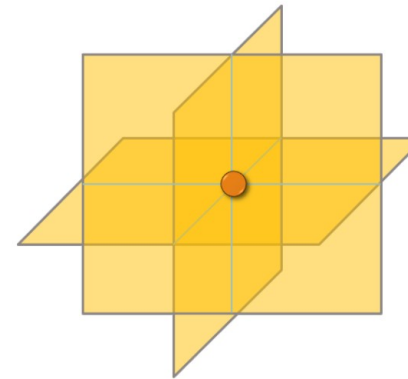
- Foliation (i.e. layered structure) determines:
 - mobility constraints:



planon
(stuck to
1 layer)



lineon
(stuck to
2 layers)

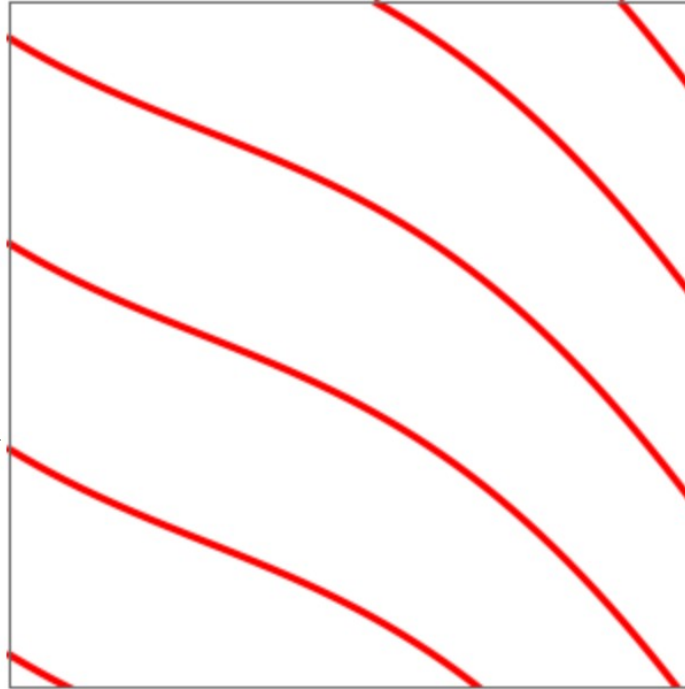
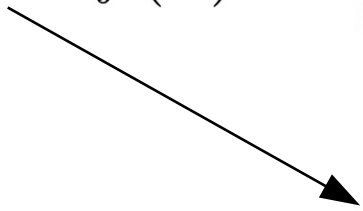


fracton
(stuck to
3 layers)

- ground state degeneracy, entanglement, ...

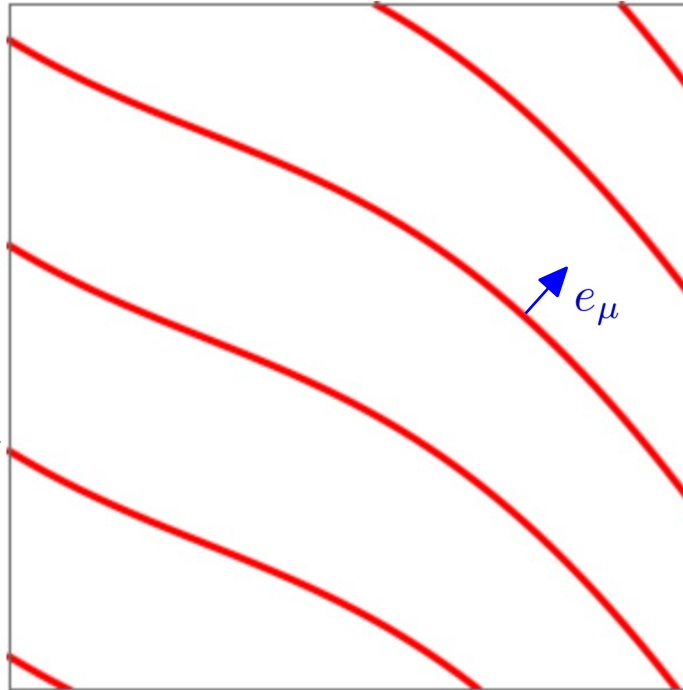
Foliation Intuition

layers of foliation
are locally level
surfaces of $f(\mathbf{x})$



Foliation Field Intuition

layers of foliation
are locally level
surfaces of $f(\mathbf{x})$



locally define
foliation field as

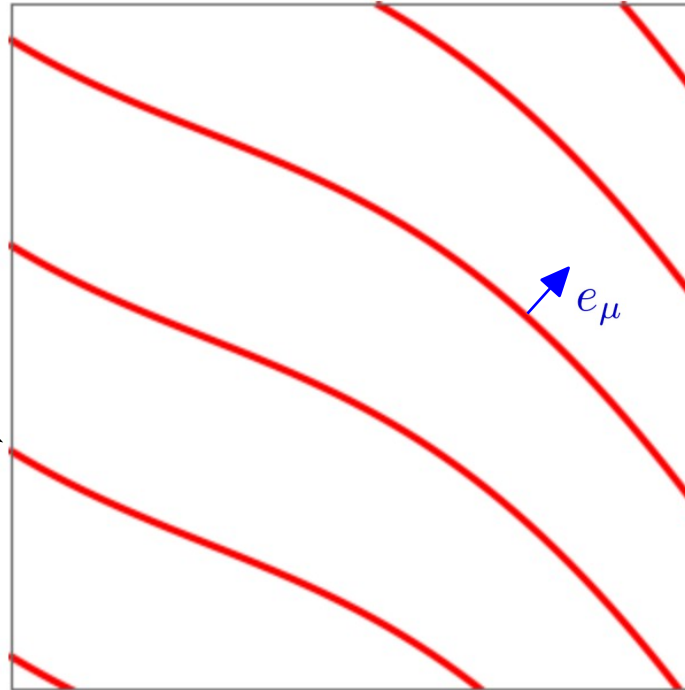
$$e = df$$

e_μ is closed:

$$de = 0$$

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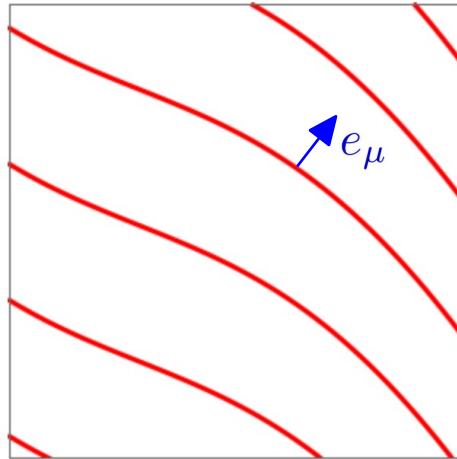
$$e = df$$

e_μ is closed:

$$de = 0$$

- To define foliation field globally, forget about $f(\mathbf{x})$
- Define layers to be orthogonal to e_μ and require $de = 0$
 - i.e. tangent vectors v^μ of layers are null vectors: $v^\mu e_\mu = 0$

Foliation Field



- Define layers to be orthogonal to e_μ and require $de = 0$

-
- More generally: $de = e \wedge \beta$ *[technical note]*
 - where β is a 1-form
 - results from $e \rightarrow \gamma e$ transformation
 - cohomology class of $\beta \wedge d\beta$ is the Godbillon-Vey invariant of the foliation
 - often $\beta = 0$

previous
chalkboard

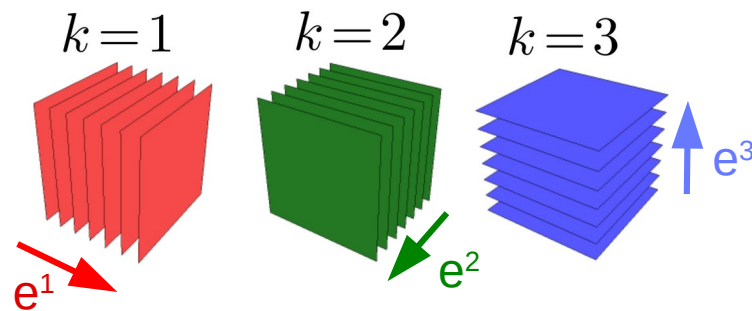
Foliation Field

- Foliating layers are orthogonal to 1-form e_μ

$$de = e \wedge \beta$$

often $\beta = 0$

- Multiple foliation fields e^k
 - $k = 1, 2, \dots, n_f$ indexes different foliations
 - e.g. $e_\mu^k = \delta_\mu^k$ for flat X-cube foliation



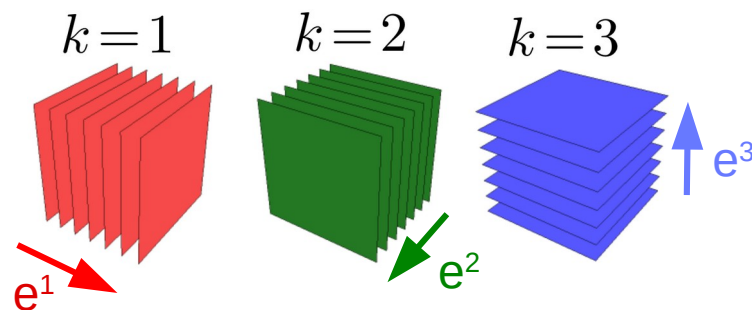
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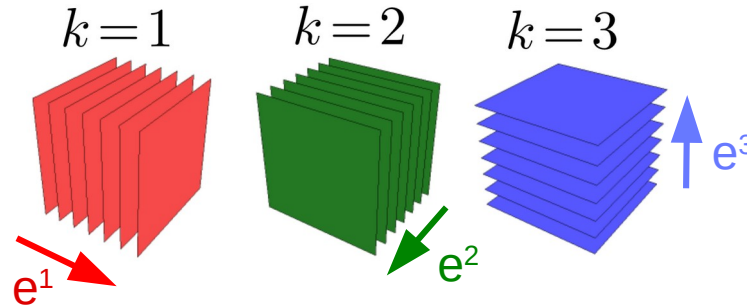


- All compact orientable three-manifolds admit total foliations (Hardorp 1980)
 - ➔ These manifolds admit a foliated QFT with fractons

- Foliating layers are orthogonal to 1-form e_μ^k

$$de^k = e^k \wedge \beta^k$$

often $\beta = 0$



Slagle 2020

Foliated QFT

not magnetic field

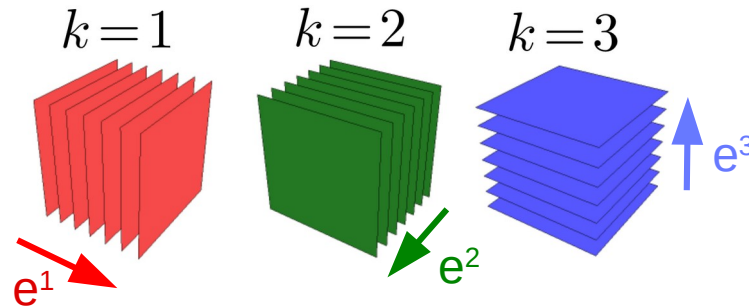
1-form gauge fields
2-form gauge field

$$L = \underbrace{\sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k}_{\text{infinitesimally-spaced stacks of 2+1D BF theory coupled to:}} + \underbrace{\frac{N}{2\pi} b \wedge da}_{\text{3+1D BF theory}}$$

- Foliating layers are orthogonal to 1-form e_μ^k

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Slagle 2020

Foliated QFT

not magnetic field

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foliated (1+1)-form gauge field: $A^k \wedge e^k = 0$

(all gauge fields are compact)

non-dynamical foliation field

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliated structure:

- dB^k is labeled as a 2-form (blue arrow pointing to the term).
- $n_k b$ is labeled as a 1-form (blue arrow pointing to the term).
- A^k is labeled as a (1+1)-form (blue arrow pointing to the term).
- b is labeled as a 1-form (blue arrow pointing to the term).
- da is labeled as a 1-form (blue arrow pointing to the term).

 The diagram shows three stacks of layers:

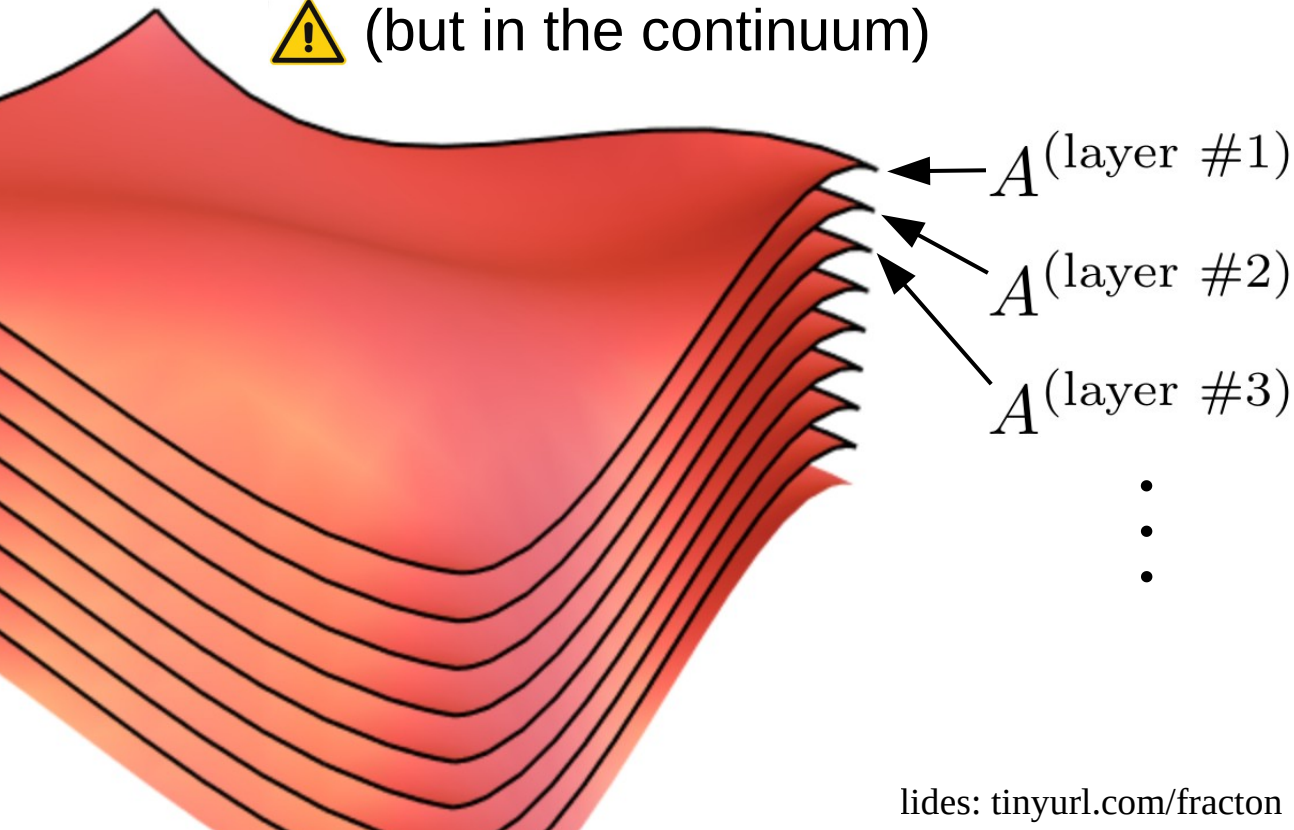
- $k=1$: Red layers, with a red arrow labeled e^1 pointing to the right.
- $k=2$: Green layers, with a green arrow labeled e^2 pointing to the right.
- $k=3$: Blue layers, with a blue arrow labeled e^3 pointing to the right.

 The equation $A^k \wedge e^k = 0$ is shown below the layers. To the right, the expression $de^k = e^k \wedge \beta^k$ is shown.

Intuition: Foliated Gauge Field

- Similar to an independent gauge field on every layer

⚠ (but in the continuum)



- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

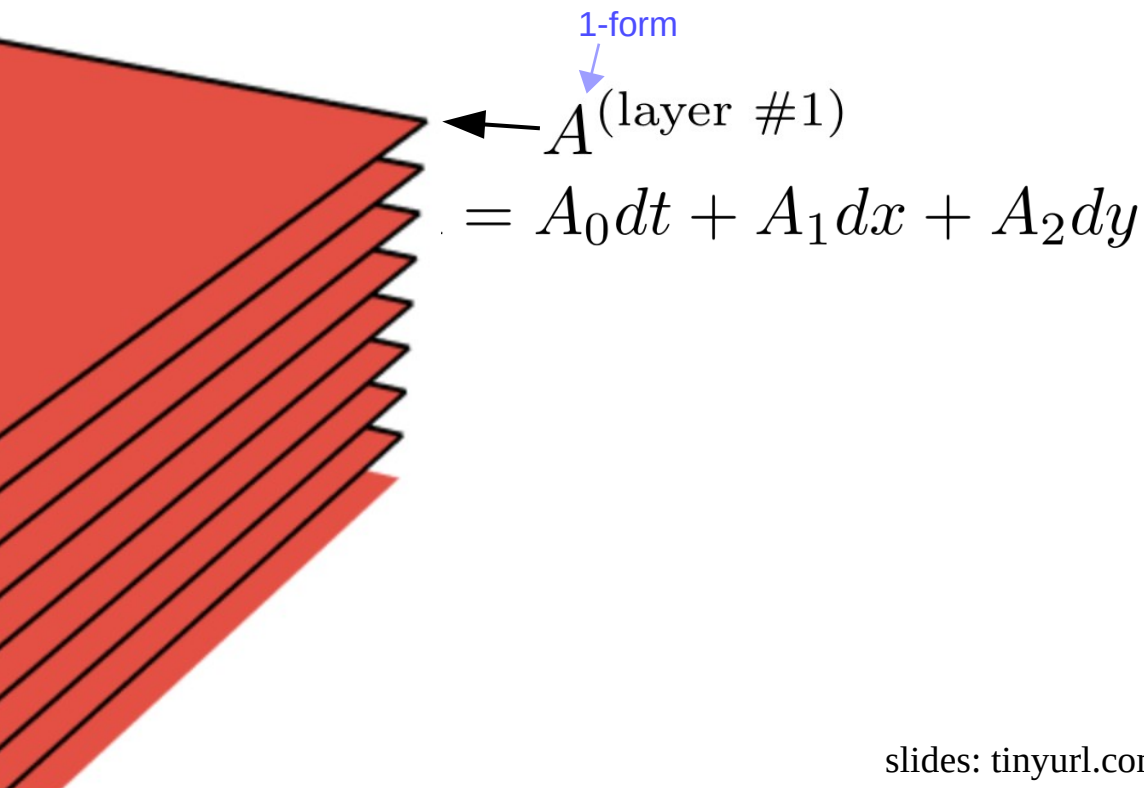
Diagram illustrating the components of the Lagrangian L and the foliation structure:

- $k=1$: Red blocks representing 1-forms A^k and 1-forms e^k (red arrow).
- $k=2$: Green blocks representing 2-forms $n_k b$ and 1-forms e^k (green arrow).
- $k=3$: Blue blocks representing 3-forms B^k and 1-forms e^k (blue arrow).

 The equation $A^k \wedge e^k = 0$ is shown below the blocks. The general relation $de^k = e^k \wedge \beta^k$ is shown to the right.

Intuition: Foliated Gauge Field

- Consider flat foliation example: $e = dz$ (i.e. $e_\mu = \delta_\mu^3$)



- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the foliation constraint $A^k \wedge e^k = 0$. Three stacks of planes are shown for $k=1$ (red), $k=2$ (green), and $k=3$ (blue). The 1-forms e^1 , e^2 , and e^3 are shown as arrows pointing along the planes. The equation $de^k = e^k \wedge \beta^k$ is also shown.

Intuition: Foliated Gauge Field

- Consider flat foliation example: $e = dz$ (i.e. $e_\mu = \delta_\mu^3$)

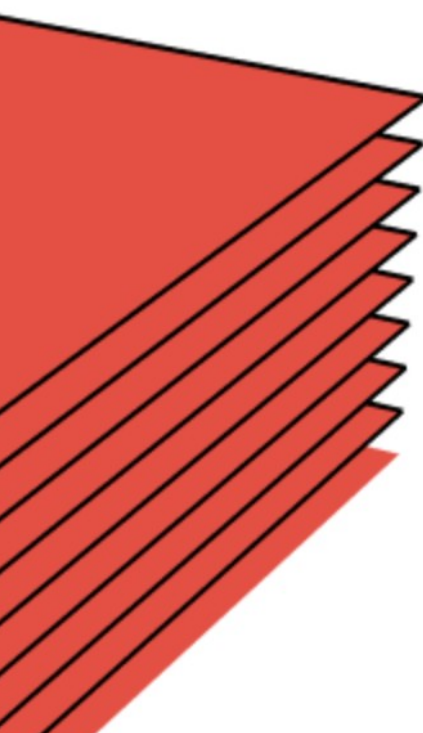


Diagram illustrating the foliation constraint $A \wedge e = 0$. A stack of red planes is shown. The 1-form A (layer #1) is shown as a vector pointing along the planes. The equation $A = A_0 dt + A_1 dx + A_2 dy$ is shown.

foliation constraint

$$A \wedge e = 0$$

implies

$$A = (A_{03} dt + A_{13} dx + A_{23} dy) dz$$

same degrees of freedom as 1-form in 2+1D!

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliating layers:

- dB^k is labeled as a 1-form.
- $n_k b$ is labeled as a 2-form.
- A^k is labeled as a (1+1)-form.
- $b \wedge da$ is labeled as a 1-form.

The foliating layers are represented by stacks of planes:

- $k=1$: Red planes, with a red arrow labeled e^1 pointing to the right.
- $k=2$: Green planes, with a green arrow labeled e^2 pointing down.
- $k=3$: Blue planes, with a blue arrow labeled e^3 pointing up.

The relationship between the layers and the 1-forms is given by:

$$A^k \wedge e^k = 0$$

$$de^k = e^k \wedge \beta^k$$

Continuity Requirements

Gauge Field

- Require dA well-defined

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliation structure:

- dB^k is labeled as a 2-form (blue arrow pointing to the term).
- $n_k b$ is labeled as a 1-form (blue arrow pointing to the term).
- A^k is labeled as a (1+1)-form (blue arrow pointing to the term).
- $b \wedge da$ is labeled as a 1-form (blue arrow pointing to the term).
- The foliation structure is shown with three stacks of layers:
 - $k=1$: Red layers, with a red arrow labeled e^1 pointing to the layers.
 - $k=2$: Green layers, with a green arrow labeled e^2 pointing to the layers.
 - $k=3$: Blue layers, with a blue arrow labeled e^3 pointing to the layers.
- The equation $A^k \wedge e^k = 0$ is shown below the stacks.
- The equation $de^k = e^k \wedge \beta^k$ is shown to the right of the stacks.

Continuity Requirements

Gauge Field

- Require dA well-defined

Foliated Gauge Field

- Require dA well-defined
 - Does not imply continuity between layers (as desired)!

$$e = dz \implies A = (A_{03}dt + A_{13}dx + A_{23}dy)dz$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$de^k = e^k \wedge \beta^k$

Continuity Requirements

Gauge Field

- Require dA well-defined

Foliated Gauge Field

- Require dA well-defined
 - Does not imply continuity between layers (as desired)!
- Allow delta-functions
 - e.g. $A = x \delta(z) dy \wedge dz$ is allowed for $e = dz$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the foliation structure with layers $k=1$ (red), $k=2$ (green), and $k=3$ (blue). The layers are orthogonal to the 1-forms e^1 , e^2 , and e^3 respectively. The equation $A^k \wedge e^k = 0$ is shown below the layers. The general form $de^k = e^k \wedge \beta^k$ is shown to the right.

Continuity Requirements

Gauge Field

- Require dA well-defined

Foliated Gauge Field

- Require dA well-defined
 - Does not imply continuity between layers (as desired)!

$$e = dz \implies A = (A_{03}dt + A_{13}dx + A_{23}dy)dz$$

- Allow delta-functions
 - e.g. $A = x \delta(z) dy \wedge dz$ is allowed for $e = dz$

Questions?

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating foliating layers and their orthogonality to 1-forms e_μ^k . The layers are labeled $k=1$ (red), $k=2$ (green), and $k=3$ (blue). The 1-forms e^1 , e^2 , and e^3 are shown as arrows pointing in the direction of the layers. The equation $A^k \wedge e^k = 0$ is shown below the layers. The equation $de^k = e^k \wedge \beta^k$ is shown to the right.

Review: Charge Conservation

- Z_N BF theory (Z_N toric code):

$$L = \frac{N}{2\pi} B \wedge dA - A \wedge J - B \wedge I$$

- gauge invariance:

$$A \rightarrow A + d\zeta$$

$$B \rightarrow B + d\chi$$

- conservation of charge and flux:

$$dJ = dI = 0$$

also constrains gauge invariant operators, eg $e^{i \int A \wedge J}$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliating layers:

- dB^k is labeled as a 2-form (blue arrow).
- $n_k b$ is labeled as a 1-form (blue arrow).
- A^k is labeled as a (1+1)-form (blue arrow).
- b is labeled as a 1-form (blue arrow).
- da is labeled as a 1-form (blue arrow).
- The foliating layers are shown as stacks of planes:
 - $k=1$ (red planes) with normal vector e^1 (red arrow).
 - $k=2$ (green planes) with normal vector e^2 (green arrow).
 - $k=3$ (blue planes) with normal vector e^3 (blue arrow).
- The relation $A^k \wedge e^k = 0$ is shown below the layers.
- The definition $de^k = e^k \wedge \beta^k$ is shown to the right.

Mobility Constraints

- $L' = L - \sum_k A^k \wedge J^k - a \wedge j$
 - J^k is labeled as **planon p** (blue arrow).
 - $a \wedge j$ is labeled as **f (possible fracton)** (blue arrow).
- mobility constraints from gauge invariance:

$$a \rightarrow a + d\lambda \quad \Rightarrow \quad \mathbf{f} \# \text{ conservation: } dj = 0$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliation structure:

- dB^k is labeled as a 2-form.
- $n_k b$ is labeled as a 1-form.
- A^k is labeled as a (1+1)-form.
- b is labeled as a 1-form.
- da is labeled as a 1-form.
- The foliation structure shows three layers: $k=1$ (red), $k=2$ (green), and $k=3$ (blue).
- Each layer k is orthogonal to the 1-form e^k .
- The condition $A^k \wedge e^k = 0$ is shown.
- The differential of the 1-form is $de^k = e^k \wedge \beta^k$.

Mobility Constraints

- $L' = L - \sum_k A^k \wedge J^k - a \wedge j$
 - f (possible fracton)
 - planon p
- mobility constraints from gauge invariance:

$$a \rightarrow a + d\lambda \quad \Rightarrow \quad f \# \text{ conservation:}$$

$$dj = 0$$

$$A^k \rightarrow A^k + d\zeta^k$$

$\zeta^k \wedge e^k = 0$ (0+1)-form

$$dJ^k \wedge e^k = -m_k j \wedge e^k$$

current through layer \rightarrow

$$a \rightarrow a - \sum_k m_k \zeta^k$$

$$m_k \equiv \frac{n_k M_k}{N}$$

source: tinyurl.com/fracton

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

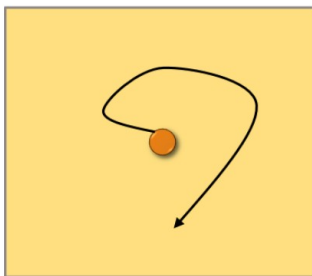
Diagram illustrating the components of the Lagrangian L and the foliating layers:

- dB^k (1-form) and $n_k b$ (2-form) combine to form $(1+1)$ -form $(dB^k + n_k b) \wedge A^k$.
- $b \wedge da$ (1-form) and N (1-form) combine to form $b \wedge da$.
- Foliating layers are shown as stacks of planes for $k=1$ (red), $k=2$ (green), and $k=3$ (blue).
- Orthogonality conditions: $A^k \wedge e^k = 0$ and $de^k = e^k \wedge \beta^k$.
- 1-forms e^1 , e^2 , and e^3 are shown as arrows pointing to the respective layers.

Mobility Constraints

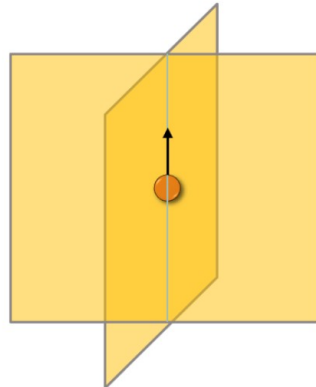
- $L' = L - \sum_k A^k \wedge J^k - a \wedge j$
 - f (possible fracton)
 - planon p
- mobility of f depends on # of foliations with $n_k \neq 0$

1 foliation:



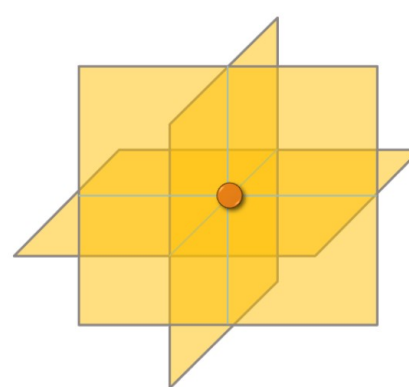
planon
(stuck to
1 layer)

2 foliations:



lineon
(stuck to
2 layers)

3 foliations:



fracton
(stuck to
3 layers)

$$dJ^k \wedge e^k = -m_k j \wedge e^k$$

$$m_k \equiv \frac{n_k M_k}{N}$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliating layers:

- dB^k is labeled as a 2-form.
- $n_k b$ is labeled as a 1-form.
- A^k is labeled as a (1+1)-form.
- b is labeled as a 1-form.
- da is labeled as a 1-form.
- The foliating layers are shown as stacks of planes for $k=1$ (red), $k=2$ (green), and $k=3$ (blue).
- The normal vectors to these layers are e^1 (red arrow), e^2 (green arrow), and e^3 (blue arrow).
- The condition $A^k \wedge e^k = 0$ is shown.
- The definition $de^k = e^k \wedge \beta^k$ is shown.

Mobility Constraints

- $$L' = L - \sum_k B^k \wedge I^k - b \wedge i$$

← flux string
 ← planon m

$$B^k \rightarrow B^k + d\chi^k \quad \Rightarrow \quad m \text{ \# conserved: } dI^k = 0$$

$$B^k \rightarrow B^k + \alpha^k e^k \quad \Rightarrow \quad m \text{ restricted to 2D layers: } I^k \wedge e^k = 0$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the components of the Lagrangian L and the foliating layers:

- dB^k is labeled as a 2-form.
- $n_k b$ is labeled as a 1-form.
- A^k is labeled as a (1+1)-form.
- b is labeled as a 1-form.
- da is labeled as a 1-form.
- The foliating layers are shown as stacks of planes for $k=1$ (red), $k=2$ (green), and $k=3$ (blue).
- The normal vectors to these layers are e^1 (red arrow), e^2 (green arrow), and e^3 (blue arrow).
- The condition $A^k \wedge e^k = 0$ is shown.
- The definition $de^k = e^k \wedge \beta^k$ is shown.

Mobility Constraints

- $L' = L - \sum_k B^k \wedge I^k - b \wedge i$
 - I^k is labeled as a flux string.
 - i is labeled as a planon m .

$$B^k \rightarrow B^k + d\chi^k \quad \Rightarrow \quad m \text{ \# conserved: } dI^k = 0$$

$$B^k \rightarrow B^k + \alpha^k e^k \quad \Rightarrow \quad m \text{ restricted to 2D layers: } I^k \wedge e^k = 0$$

$$b \rightarrow b + d\mu$$

$$B^k \rightarrow B^k - n_k \mu \quad \Rightarrow \quad m \text{ bound to flux strings: } di = \sum_k n_k I^k$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating foliating layers:

- $k=1$: Red layers, 1-form e^1 (red arrow).
- $k=2$: Green layers, 2-form e^2 (green arrow).
- $k=3$: Blue layers, 3-form e^3 (blue arrow).

 The condition $A^k \wedge e^k = 0$ is shown. The general form is $de^k = e^k \wedge \beta^k$.

Mobility Constraints

- $L' = L - \sum_k B^k \wedge I^k - b \wedge i$
 - I^k : planon m (red arrow)
 - i : flux string (red arrow)

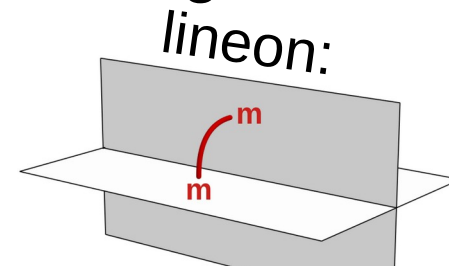
$$B^k \rightarrow B^k + d\chi^k \quad \Rightarrow \quad m \text{ \# conserved: } dI^k = 0$$

$$B^k \rightarrow B^k + \alpha^k e^k \quad \Rightarrow \quad m \text{ restricted to 2D layers: } I^k \wedge e^k = 0$$

$$b \rightarrow b + d\mu$$

$$B^k \rightarrow B^k - n_k \mu$$

m bound to flux strings: $di = \sum_k n_k I^k$



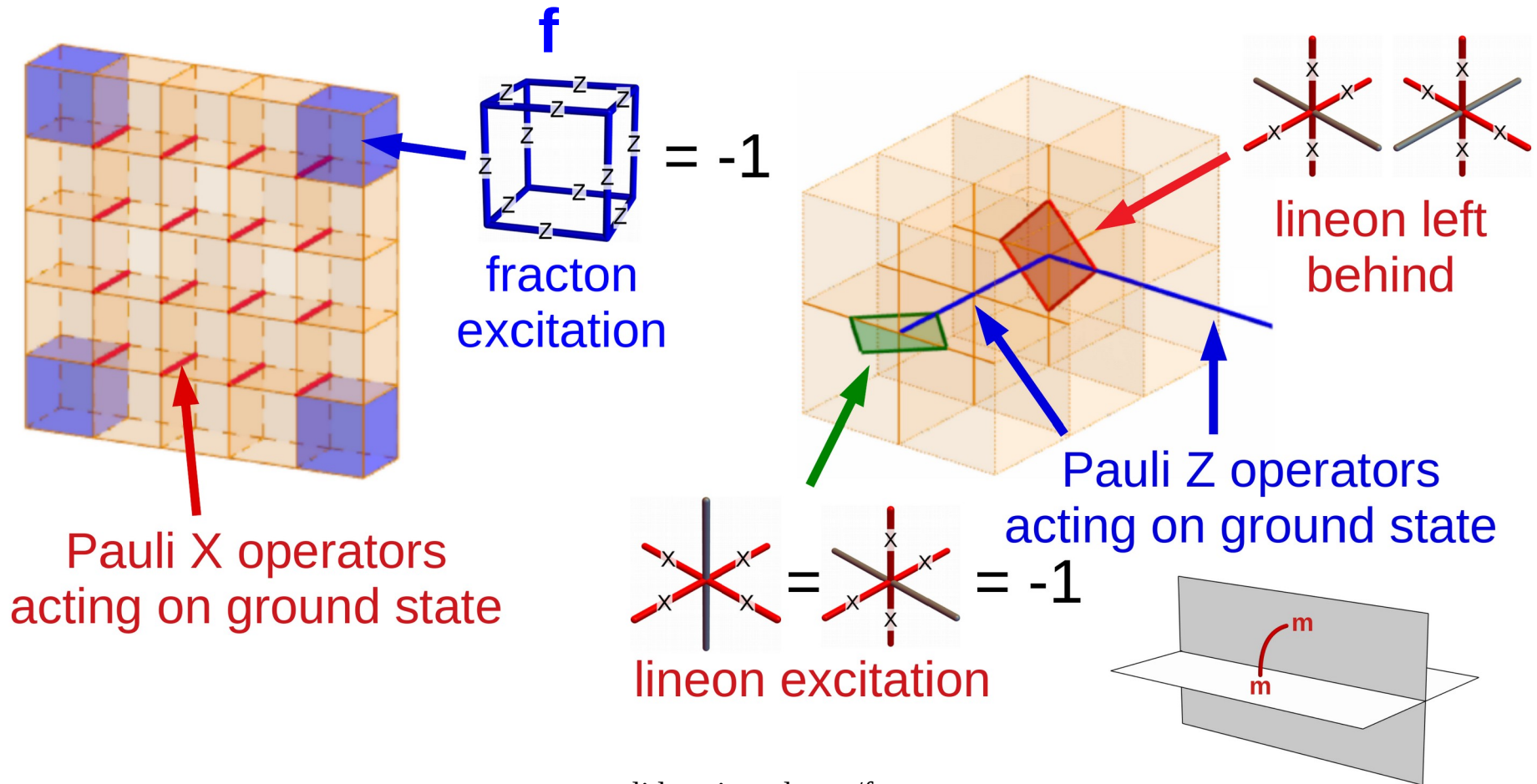
- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$de^k = e^k \wedge \beta^k$

Matches X-cube Model



- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

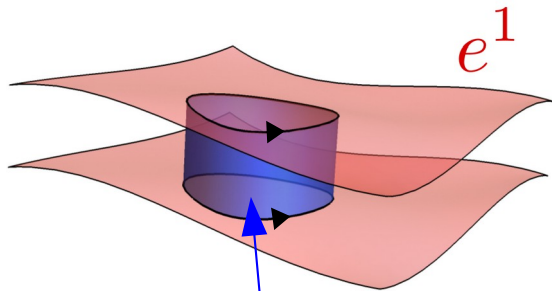
$$L' = L - \sum_k A^k \wedge J^k - a \wedge j - \sum_k B^k \wedge I^k - b \wedge i$$

$$dJ^k \wedge e^k = -m_k j \wedge e^k \quad dI^k = 0 \quad di = \sum_k n_k I^k$$

$$dj = 0 \quad I^k \wedge e^k = 0$$

$$A^k \rightarrow A^k + d\zeta^k \quad \zeta^k \wedge e^k = 0 \quad b \rightarrow b + d\mu \quad B^k \rightarrow B^k + d\chi^k - n_k \mu + \alpha^k e^k$$

Example Operators

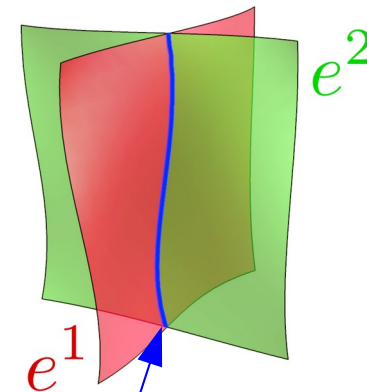


move pair of

fractons $e^i \int_{\mathcal{M}} A^1 = e^i \int A \wedge J$

or lineons: $e^i \int_{\mathcal{M}} (dB^1 + n_1 b)$

around a pair of loops



$$e^i \int_{\mathcal{M}} (B^1 - B^2)$$

move lineon along
intersection of two layers
($n_1 = n_2$)

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

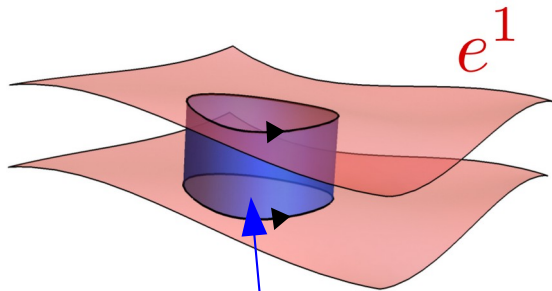
$$L' = L - \sum_k A^k \wedge J^k - a \wedge j - \sum_k B^k \wedge I^k - b \wedge i$$

$$dJ^k \wedge e^k = -m_k j \wedge e^k \quad dI^k = 0 \quad di = \sum_k n_k I^k$$

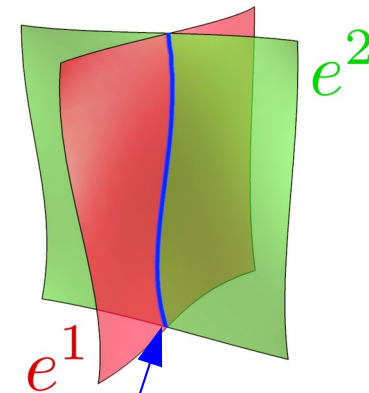
$$dj = 0 \quad I^k \wedge e^k = 0$$

$$A^k \rightarrow A^k + d\zeta^k \quad \zeta^k \wedge e^k = 0 \quad b \rightarrow b + d\mu \quad B^k \rightarrow B^k + d\chi^k - n_k \mu + \alpha^k e^k$$

Questions?



move pair of
fractons $e^i \int_{\mathcal{M}} A^1 = e^i \int A \wedge J$
or lineons: $e^i \int_{\mathcal{M}} (dB^1 + n_1 b)$
around a pair of loops



$e^i \int_{\mathcal{M}} (B^1 - B^2)$
move lineon along
intersection of two layers
($n_1 = n_2$)

Conclusions

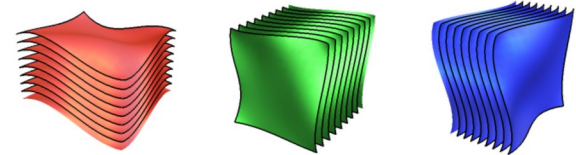
- U(1) fracton models only gauge invariant on certain manifolds

symmetric tensor gauge theory	gauge invariant manifold
3D gapless traceless scalar	Einstein $R_{ab} \propto g_{ab}$
3D gapless traceless vector	Einstein with constant curvature
2D gapped traceless scalar	constant curvature
2D gapless traceless scalar	constant curvature
any-D gapless traceful scalar	flat
any-D gapless traceful vector	flat

KS, Prem, Pretko 2018

- Foliated QFT for many type-I fracton models

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$



$$A^k \wedge e^k = 0$$

$$de^k = e^k \wedge \beta^k$$

often $\beta = 0$

- Resembles coupled BF theories with a foliation-dependent constraint

Slagle 2020

- Gauge invariance enforces particle mobility constraints

kslagle@caltech.edu

slides: tinyurl.com/fracton

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

Mobility Constraints

$$dJ^k \wedge e^k = -j \wedge e^k$$

$$dI^k = 0, \quad I^k \wedge e^k = 0$$

$$dj = 0$$

$$di = \sum_k I^k$$

$$J^k \rightarrow J^k + \phi^k \wedge e^k$$

Equations of Motion

$$\frac{M_k}{2\pi} (dB^k + n_k b) \wedge e^k = J^k \wedge e^k$$

$$\frac{M_k}{2\pi} dA^k = I^k$$

$$\frac{N}{2\pi} db = j$$

$$\frac{N}{2\pi} (da + \sum_k m_k A^k) = i$$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

Quantized Coefficients

$$\begin{aligned}
 A^k &\rightarrow A^k + d\zeta^k & B^k &\rightarrow B^k + d\chi^k - \underline{n_k}\mu \\
 a &\rightarrow a + d\lambda - \sum_k \underline{m_k}\zeta^k & b &\rightarrow b + d\mu
 \end{aligned}$$

- compact gauge fields implies: $\underline{m_k} \equiv \frac{n_k M_k}{N} \in \mathbb{Z}$ and $\underline{n_k} \in \mathbb{Z}$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

Quantized Coefficients

$$\begin{aligned}
 A^k &\rightarrow A^k + d\zeta^k & B^k &\rightarrow B^k + d\chi^k - \underline{n_k}\mu \\
 a &\rightarrow a + d\lambda - \sum_k \underline{m_k}\zeta^k & b &\rightarrow b + d\mu
 \end{aligned}$$

- compact gauge fields implies: $\underline{m_k} \equiv \frac{n_k M_k}{N} \in \mathbb{Z}$ and $\underline{n_k} \in \mathbb{Z}$

$$\oint L \rightarrow \oint L + \underbrace{\sum_k M_k \frac{1}{2\pi} \oint (dB^k \wedge d\zeta^k + d\chi^k \wedge dA^k)}_{\in 2\pi\mathbb{Z}} + N \underbrace{\frac{1}{2\pi} \oint (db \wedge d\lambda + d\mu \wedge da)}_{\in 2\pi\mathbb{Z}}$$

- large gauge transformation invariance implies: $M_k, N \in \mathbb{Z}$

- Foliating layers are orthogonal to 1-form e^k_μ

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the action L with foliating layers. The action is composed of terms involving 1-forms (b , da), 2-forms (dB^k), and (1+1)-forms (A^k). The layers are labeled $k=1$ (red), $k=2$ (green), and $k=3$ (blue). The 1-forms e^1 , e^2 , and e^3 are shown as arrows pointing to the layers. The equation $A^k \wedge e^k = 0$ is shown below the layers. The equation $de^k = e^k \wedge \beta^k$ is shown to the right.

Coefficient Identification

Equations of motion:

$$db = dA^k = 0$$

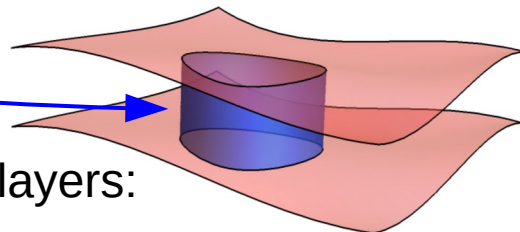
$$\oint b \in \frac{2\pi}{N} \mathbb{Z}$$

$$\int_F A^k \in \frac{2\pi}{M_k} \mathbb{Z}$$

$$\sum_k \frac{M_k n_k}{2\pi} \int b \wedge A^k \in 2\pi \sum_k \frac{n_k}{N} \mathbb{Z}$$

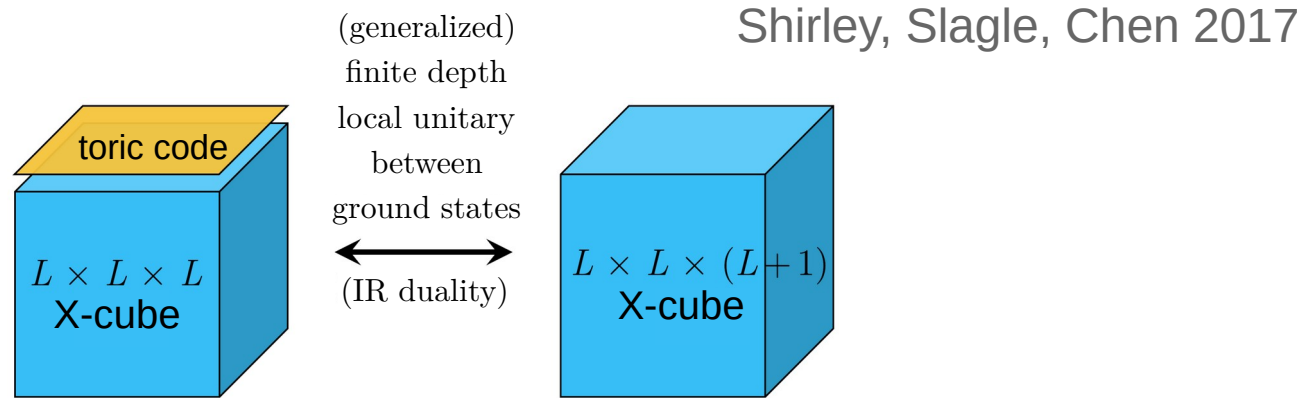
same action: $n_k \sim n_k + N_k$

integrate over a 2-manifold between layers:



Foliated Fracton Order (FFO)

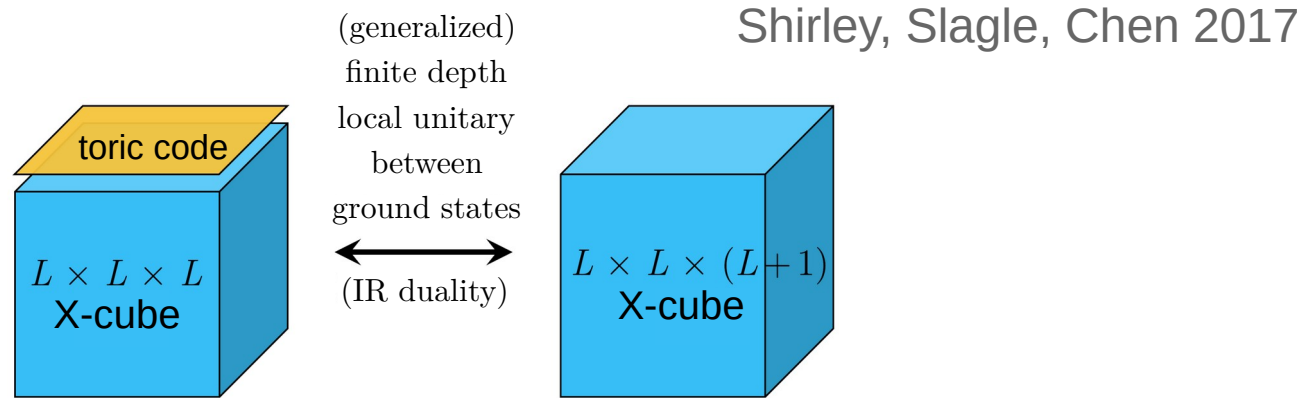
- Exfoliation: a local unitary can decouple a layer:



- Toric code anyons become X-cube planons
 - Ground state degeneracy on a torus $2^{2L_x + 2L_y + 2L_z - 3}$ is absorbed via $L_z \rightarrow L_z + 1$

Foliated Fracton Order (FFO)

- Exfoliation: a local unitary can decouple a layer:



- Toric code anyons become X-cube planons
 - Ground state degeneracy on a torus $2^{2L_x + 2L_y + 2L_z - 3}$ is absorbed via $L_z \rightarrow L_z + 1$
- It's *not* possible to decouple all layers
 - On a lattice, at least one layer must remain

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the foliation structure. Three stacks of layers are shown:

- $k=1$: Red layers, with a red arrow labeled e^1 pointing downwards.
- $k=2$: Green layers, with a green arrow labeled e^2 pointing downwards.
- $k=3$: Blue layers, with a blue arrow labeled e^3 pointing upwards.

 Labels for the forms in the equation:

- dB^k : 1-form (blue arrow pointing to B^k)
- $n_k b$: 2-form (blue arrow pointing to b)
- A^k : (1+1)-form (blue arrow pointing to A^k)
- b : 1-form (blue arrow pointing to b)
- da : 1-form (blue arrow pointing to da)

 The condition $A^k \wedge e^k = 0$ is shown below the stacks. To the right, the definition $de^k = e^k \wedge \beta^k$ is given.

Exfoliation in Foliated QFT

- Exfoliate all layers within $z_1 < z < z_2$ ($e_1 = dz$):

- i.e. $\tilde{n}_1(z) = \begin{cases} n_1 & z \leq z_1 \text{ or } z \geq z_2 \\ 0 & z_1 < z < z_2 \end{cases}$

- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

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- claim: this interface can be created and moved by a duality transformation:

$$a \leftrightarrow \tilde{a} = \begin{cases} a & z \leq z_1 \text{ or } z_2 \leq z \\ a + m_1 \int_{z_1}^z A^1 & z_1 < z < z_2 \end{cases}$$

$$A^1 \leftrightarrow \tilde{A}^1 = A^1 + \delta(z - z_2) \int_{z_1}^{z_2} dz A^1$$

$$B^1 \leftrightarrow \tilde{B}^1 = \begin{cases} B^1 & z \leq z_1 \text{ or } z_2 \leq z \\ B^1 - B^1(z_2) + n_1 \int_z^{z_2} b & z_1 < z < z_2 \end{cases}$$

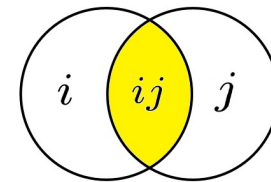
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$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

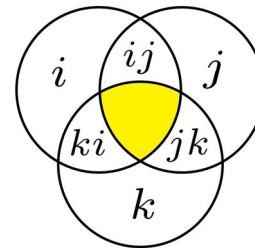
$A^k \wedge e^k = 0$

review: Gauge Fields

- Gauge fields on different patches (charts) i, j, k of spacetime are defined up to:
 - 0-form gauge field: $A_i - A_j \in 2\pi\mathbb{Z}$
 - 1-form gauge field: $A_i - A_j = dg_{ij}$
 - Transition functions g_{ij} satisfy cocycle condition



$$g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$$



- Foliating layers are orthogonal to 1-form e_μ^k

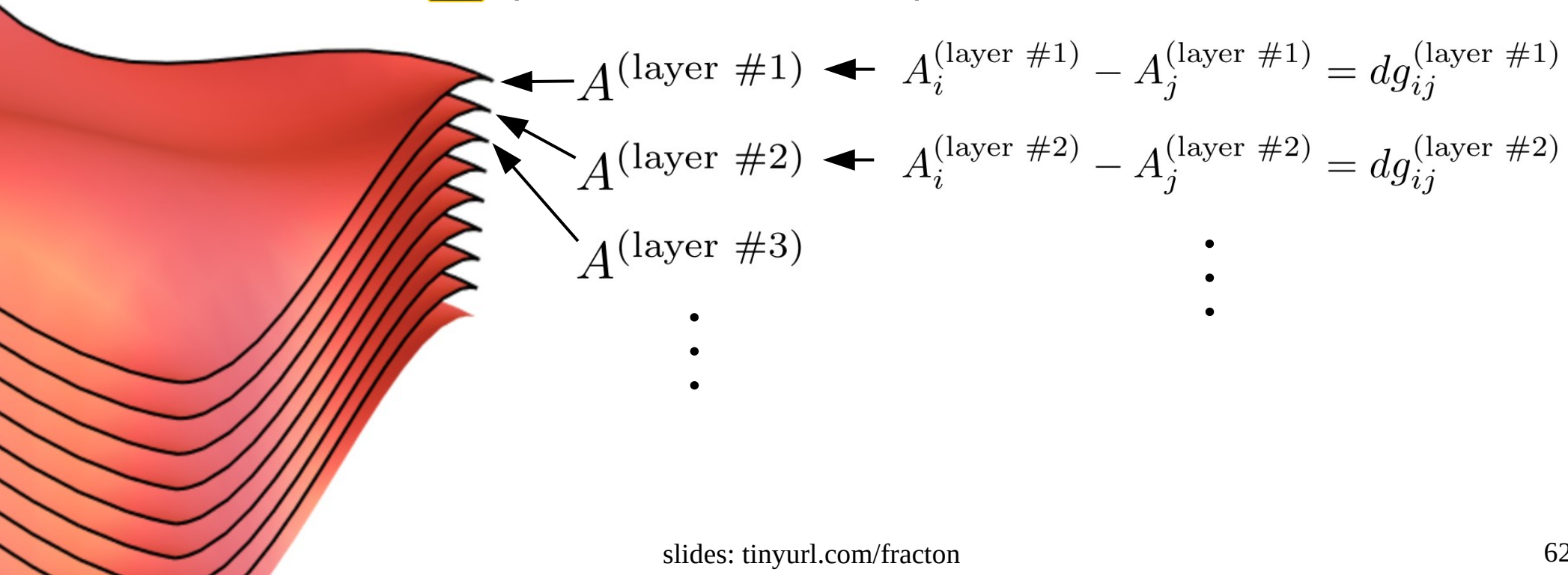
$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

Intuition: Foliated Gauge Field

- Transition function for each layer

⚠ (but in the continuum)

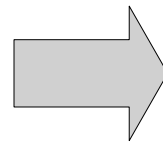


- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

Diagram illustrating the foliation of layers $k=1, 2, 3$. The layers are represented by red, green, and blue stacks of planes. The 1-forms e^1, e^2, e^3 are shown as arrows pointing along the layers. The equation $A^k \wedge e^k = 0$ is shown below the layers. The equation $de^k = e^k \wedge \beta^k$ is shown to the right.

1-form
Gauge Field



Foliated (1+1)-form
Gauge Field

$$A_i - A_j = dg_{ij}$$

Diagram illustrating the relationship between the 1-form A_i and the 0-form A_j in the gauge field. The equation is shown with a Venn diagram where two overlapping circles labeled i and j have a yellow intersection labeled ij .

$$A_i - A_j = dg_{ij}$$

g_{ij} is also foliated: $g_{ij} \wedge e = 0$

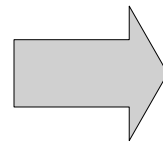
- Foliating layers are orthogonal to 1-form e_μ^k

$$L = \sum_k \frac{M_k}{2\pi} (dB^k + n_k b) \wedge A^k + \frac{N}{2\pi} b \wedge da$$

$A^k \wedge e^k = 0$

$$de^k = e^k \wedge \beta^k$$

1-form
Gauge Field



Foliated (1+1)-form
Gauge Field

$$A_i - A_j = dg_{ij}$$

$$A_i - A_j = dg_{ij}$$

g_{ij} is also foliated: $g_{ij} \wedge e = 0$

$$g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$$

$$\int_s g_{ij} + g_{jk} + g_{ki} \in 2\pi\mathbb{Z}$$

Ground State Degeneracy

- solve equations of motion and plug into action:

$$S = \frac{N}{2\pi} \sum_{k \neq b} \frac{L_k}{l_k} \int_0^{l_k} dx^k p_b^k(t, x^k) \partial_t q_b^k(t, x^k)$$

$$0 \leq x < l_x, 0 \leq y < l_y, 0 \leq z < l_z, e_\mu^k = \delta_\mu^k$$

- add a cutoff for finite degeneracy: $a_k \sim \frac{l_k}{L_k}$

$$S \sim \frac{N}{2\pi} \sum_{k \neq b} \sum_{x^k=0, a_k, \dots, (L_k-1)a_k} p_b^k(t, x^k) \partial_t q_b^k(t, x^k)$$

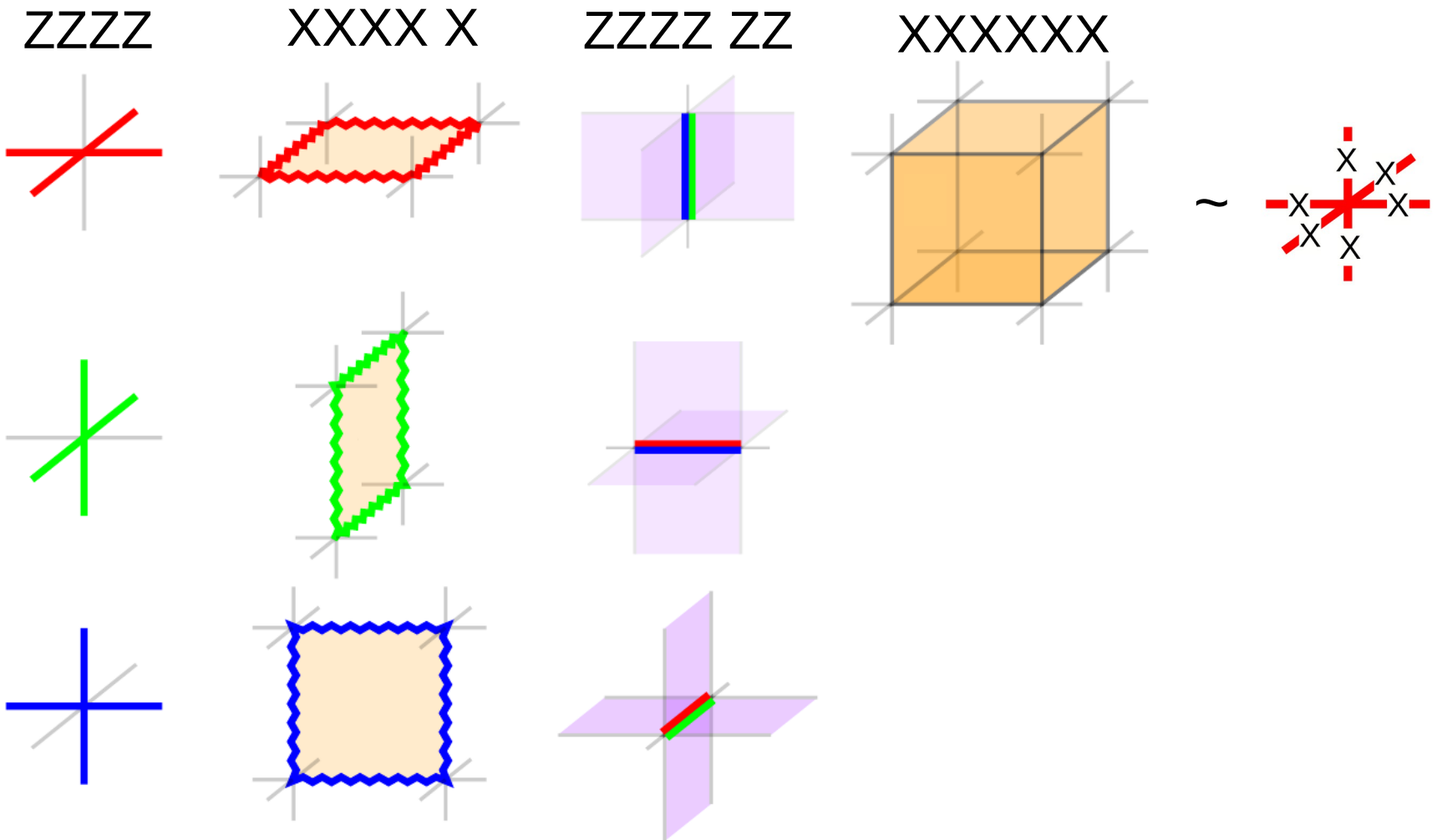
where $q_a^k(t, 0) = p_a^k(t, 0) = 0$ if $\epsilon^{akb} = 1$

- degeneracy = $N^2 L_x + 2L_y + 2L_z - 3$

Lattice Model

2D toric code
defect layer

coupled to
3D toric code



Example: Discrete Disclination Curvature

