Non-Lorentzian Geometries and their Applications to Theoretical Physics

The Post-Newtonian Description of Inspiralling Compact Binaries

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Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

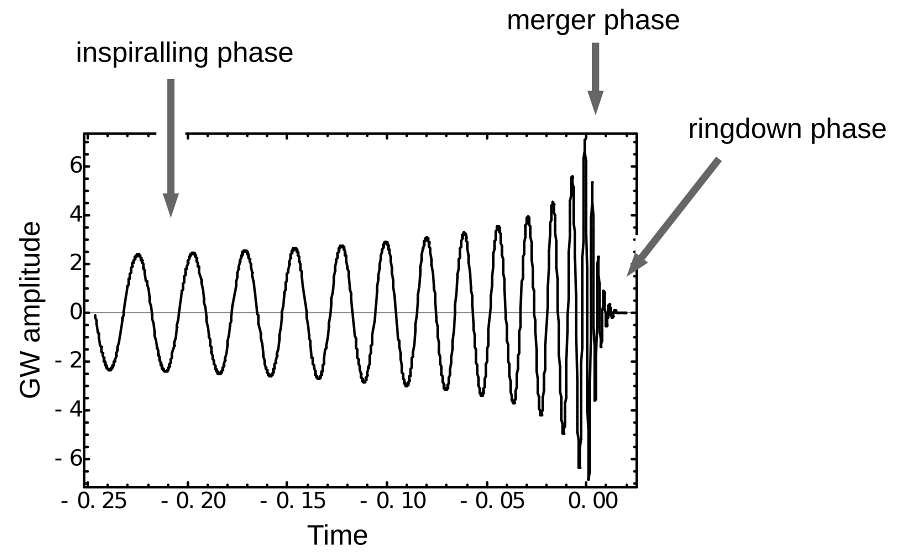
10 Juin 2021

Plan of the talk

1. Post-Newtonian parameters describing the compact binary inspiral
2. Gravitational wave generation formalism for slow-moving isolated systems
3. Toward 4.5PN parameters in orbital phase and 4PN in amplitude

POST-NEWTONIAN PARAMETERS

The gravitational chirp of binary black holes



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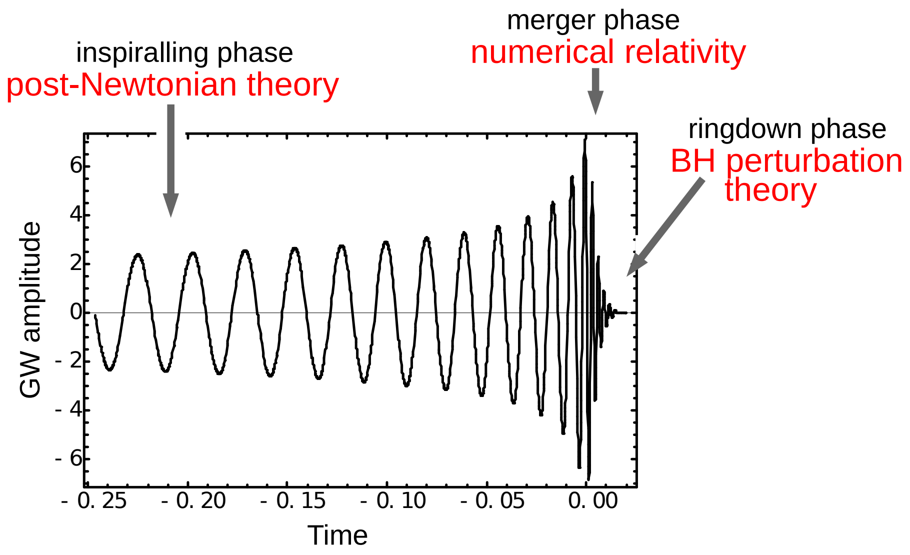
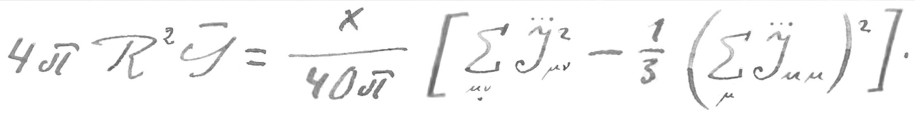


Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1945]



1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4R}\left\{\frac{\mathrm{d}^2Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

$$F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Radiation reaction and balance equations

1. Conserved Newtonian energy in the source

$$E = \int \mathrm{d}^3\mathbf{x}\,\rho\left[\frac{\mathbf{v}^2}{2} + \Pi - \frac{U}{2}\right]$$

2. Eulerian equations of motion in the source

$$\rho\frac{\mathrm{d}v^i}{\mathrm{d}t} = -\partial_i P + \rho\partial_i U - \overbrace{\frac{2G}{5c^5}\rho\, x^j\frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5}}^{\boldsymbol{F}^{\text{reac}}}$$

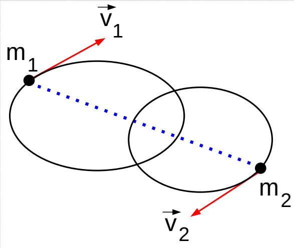
3. Energy loss is due to the work of the radiation reaction force

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\boldsymbol{v}\cdot\boldsymbol{F}^{\text{reac}} = -\frac{G}{5c^5}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \text{total time derivative}}$$

4. Obtain the balance equation after averaging over one period

$$\langle\frac{\mathrm{d}E}{\mathrm{d}t}\rangle = -\langle\mathcal{F}^{\text{GW}}\rangle \quad\Longrightarrow\quad \phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot\omega}\,\mathrm{d}\omega$$

Application to compact binaries [Peters & Mathews 1963; Peters 1964]



$$\begin{cases} a \text{ semi-major axis of relative orbit} \\ e \text{ eccentricity of relative orbit} \\ \omega = \frac{2\pi}{P} \text{ orbital frequency} \end{cases}$$

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{M}$$

$$\boxed{\nu = \frac{\mu}{M} \qquad 0 < \nu \leqslant \frac{1}{4}}$$

Averaged energy and angular momentum balance equations

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = -\langle \mathcal{F}^{\mathsf{GW}} \rangle \qquad \left\langle \frac{\mathrm{d}J_i}{\mathrm{d}t} \right\rangle = -\langle \mathcal{G}_i^{\mathsf{GW}} \rangle$$

are applied to a Keplerian orbit (using Kepler's law $GM = \omega^2 a^3$)

$$\boxed{\begin{aligned} \left\langle \frac{\mathrm{d}P}{\mathrm{d}t} \right\rangle &= -\frac{192\pi}{5c^5}\nu \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{73}{24}e^2 + \frac{37}{96}e^4}{(1-e^2)^{7/2}} \\ \left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle &= -\frac{608\pi}{15c^5}\nu \frac{e}{P} \left(\frac{2\pi GM}{P}\right)^{5/3} \frac{1 + \frac{121}{304}e^2}{(1-e^2)^{5/2}} \end{aligned}}$$

Orbital phase evolution of compact binaries

[Dyson 1969; Esposito & Harrison 1975; Wagoner 1975]

1. Compact binaries are circularized when they enter the detector's bandwidth

$$\boxed{E = -\frac{Mc^2}{2}\nu\, x \qquad \mathcal{F}^{\mathsf{GW}} = \frac{32}{5}\frac{c^5}{G}\nu^2 x^5}$$

where $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ denotes a small PN parameter defined with ω

2. Equating $\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F}^{\mathsf{GW}}$ gives a differential equation for x

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{64}{5}\frac{c^3\nu}{GM}x^5 \qquad \Longleftrightarrow \qquad \frac{\dot{\omega}}{\omega^2} = \frac{96\nu}{5}\nu\left(\frac{GM\omega}{c^3}\right)^{5/3}}$$

3. This permits to solve for the orbital phase

$$\phi = \int \omega\,\mathrm{d}t = \int \frac{\omega}{\dot{\omega}}\,\mathrm{d}\omega$$

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1. The amplitude and phase evolution follow an adiabatic chirp in time

$$a(t) = \left(\frac{256}{5}\frac{G^3M^3\nu}{c^5}(t_c - t)\right)^{1/4}$$

$$\phi(t) = \phi_c - \frac{1}{32\nu}\left(\frac{256}{5}\frac{c^3\nu}{GM}(t_c - t)\right)^{5/8}$$

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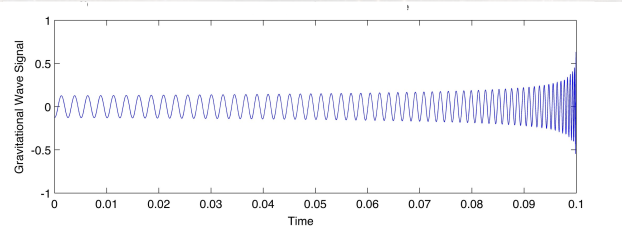

The quadrupole formula works for GW150914

- The GW frequency is given in terms of the chirp mass $\mathcal{M} = \mu^{3/5} M^{2/5}$ by

$$f = \frac{1}{\pi}\left[\frac{256}{5}\frac{G\mathcal{M}^{5/3}}{c^5}(t_c - t)\right]^{-3/8}$$

- Therefore the chirp mass is directly measured as

$$\boxed{\mathcal{M} = \left[\frac{5}{96}\frac{c^5}{G\pi^{8/3}} f^{-11/3}\dot{f}\right]^{3/5}}$$

which gives $\mathcal{M} = 30 M_\odot$ thus $M \geqslant 70 M_\odot$

- The GW amplitude is predicted to be[1]

$$h_{\text{eff}} \sim 4.1 \times 10^{-22} \left(\frac{\mathcal{M}}{M_\odot}\right)^{5/6} \left(\frac{100\,\text{Mpc}}{R}\right) \left(\frac{100\,\text{Hz}}{f_{\text{merger}}}\right)^{-1/6} \sim 1.6 \times 10^{-21}$$

- The distance $R = 400\,\text{Mpc}$ is measured from the signal itself [Schutz 1986]

[1] $h_{\text{eff}} \sim h\sqrt{N}$ where $N \sim \omega^2/\dot{\omega}$ is the number of cycles around frequency ω

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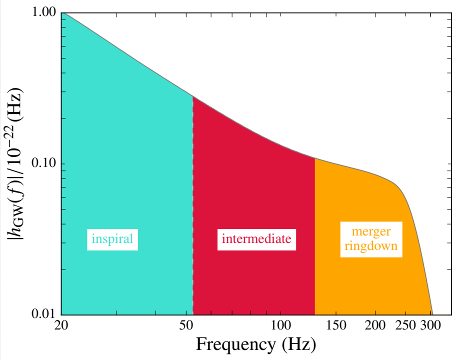
The inspiral-merger-ringdown models

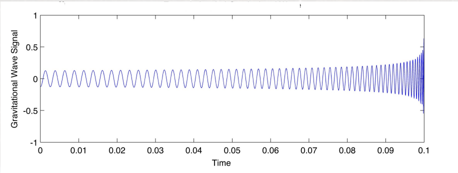


These models interpolate between the different phases play a crucial role

- The effective-one-body (EOB) approach [Buonanno & Damour 1999]
- The inspiral-merger-ringdown (IMR) [Ajith *et al.* 2008]

$$\{\underbrace{\text{PN parameters;}}_{\text{inspiral}}\ \underbrace{\beta_2,\beta_3}_{\text{intermediate}}\ ;\ \underbrace{\alpha_2,\alpha_3,\alpha_4}_{\text{merger-ringdown}}\ \}$$

PN parameters in the orbital phase evolution



- The PN parameters come from a mixture of conservative and dissipative effects through the energy balance equation

$$\frac{\mathrm{d}\overbrace{E}^{\text{conservative energy}}}{\mathrm{d}t} = -\underbrace{\mathcal{F}^{\mathsf{GW}}}_{\text{dissipative energy flux}}$$

- The orbital phase $\phi = \int \omega\,\mathrm{d}t$ is obtained as a function of $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ and the symmetric mass ratio $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$

$$\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \Big(\varphi_{p\mathsf{PN}}(\nu) + \varphi^{(l)}_{p\mathsf{PN}}(\nu) \, \log x \Big) \, x^p + \mathcal{O}[(\log x)^2]$$

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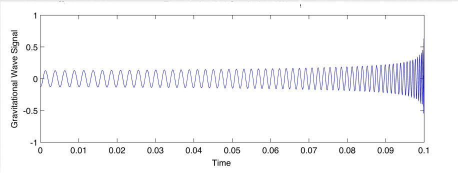

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The known 3.5PN parameters [Blanchet 2014 for a review]

They are computed with the Multipolar-post-Minkowskian-PN formalism

$$\varphi_{0\mathrm{PN}} = 1 \qquad \Longleftarrow \quad \text{Einstein quadrupole formula}$$

$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = -10\pi$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

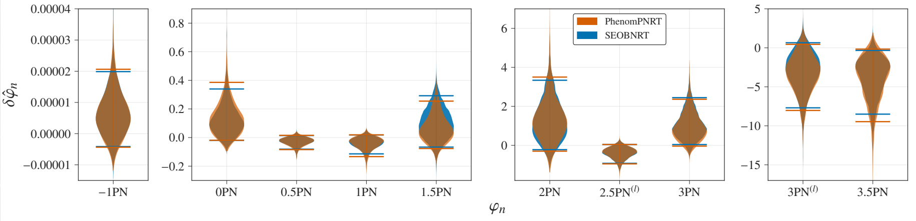
$$\varphi^{(l)}_{2.5\mathrm{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

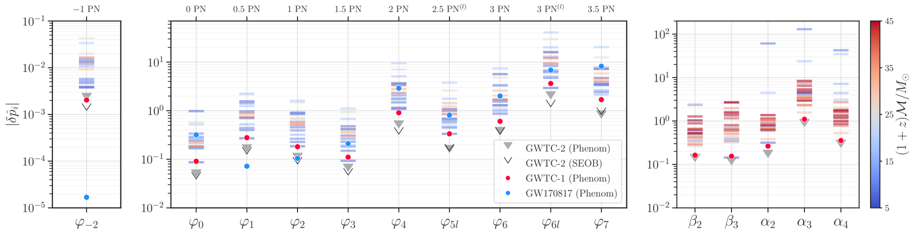
$$\varphi_{3\mathrm{PN}} = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{3424}{21}\ln 2 + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3$$

$$\varphi^{(l)}_{3\mathrm{PN}} = -\frac{856}{21}$$

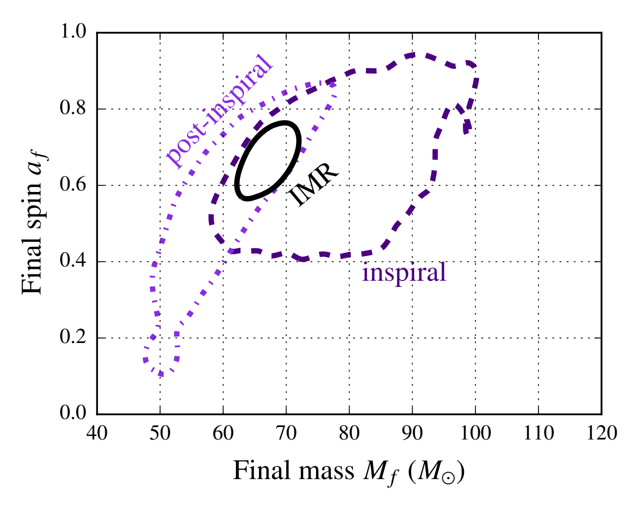
$$\varphi_{3.5\mathrm{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Measurement of PN parameters [LIGO/Virgo]



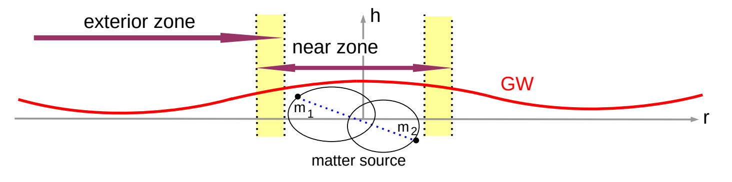


Inspiral-Merger-Ringdown consistency test [LIGO/Virgo]



GRAVITATIONAL WAVE GENERATION FORMALISM

Near zone/exterior zone split in PN expansions



- Multipole expansion in the exterior zone [Blanchet & Damour 1986]

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{F_L(t-r/c)}{r} \right\}}_{\substack{\text{general retarded homogeneous solution} \\ \text{(with no incoming radiation)}}}$$

- Post-Newtonian expansion in the near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{R_L(t-r/c) - R_L(t+r/c)}{r} \right\}}_{\substack{\text{general homogeneous retarded-advanced solution} \\ \text{(regular when } r \to 0)}}$$

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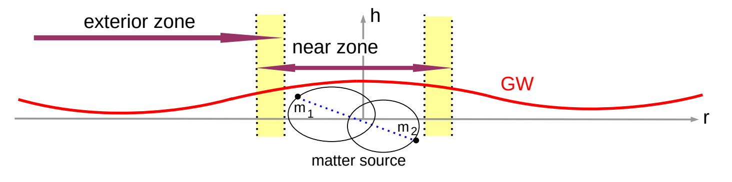

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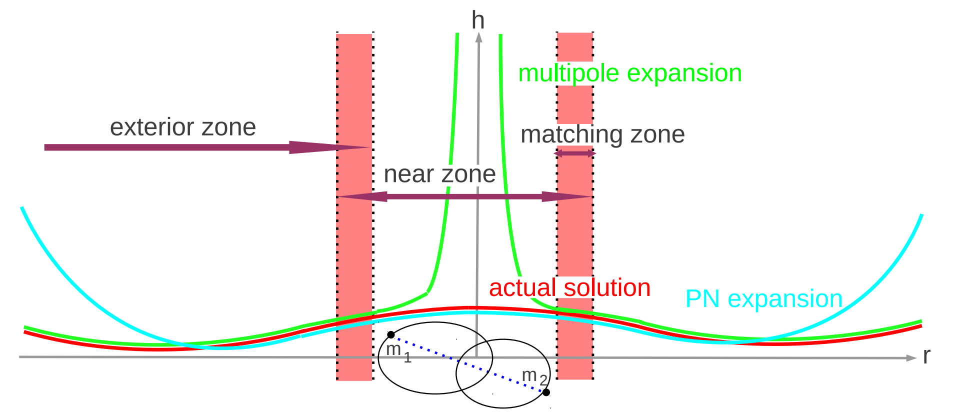
Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]



matching equation $\Longrightarrow$ $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Near-zone expansion of the multipole expansion

Lemma 1

$$\overline{\underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{ret}}\left[\left(\tfrac{r}{r_0}\right)^B\mathcal{M}(\Lambda)\right]} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\tfrac{r}{r_0}\right)^B\overline{\mathcal{M}(\Lambda)}\right] \underbrace{- \frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L(t-r/c)-\mathcal{R}_L(t+r/c)}{2r}\right\}}_{\text{antisymmetric type homogeneous solution}}$$

where the radiation reaction multipole moments are

$$\boxed{\mathcal{R}_L(u) = \underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\,\left(\tfrac{r}{r_0}\right)^B\hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau)(\mathbf{x},t-zr/c)}_{\text{multipole expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an UV regularization $(r\to 0)$

Far-zone expansion of the PN expansion

Lemma 2

$$\mathcal{M}\left(\underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\bar{\tau}\right]\right) = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\mathcal{M}(\bar{\tau})\right] \underbrace{- \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{F}_L(t-r/c)+\mathcal{F}_L(t+r/c)}{2r}\right\}}_{\text{symmetric type homogeneous solution}}$$

$$\boxed{\mathcal{F}_L(u) = \underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B \hat{x}_L \int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\,\underbrace{\bar{\tau}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an IR regularization $(r\to+\infty)$

General solution of the matching equation

[Blanchet 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

1. In the external zone

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] - \frac{4G}{c^4} \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{F}_L(t-r/c)}{r} \right\}}_{\text{source's multipole moments}}$$

2. In the near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] - \frac{4G}{c^4} \underbrace{\overline{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}}_{\text{non-local tail term (4PN+ order)}}$$

TOWARD 4.5PN PARAMETERS

Tail effects in PN parameters

tail terms

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$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = -10\pi$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

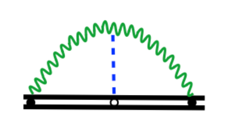
$$\varphi^{(l)}_{2.5\mathrm{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

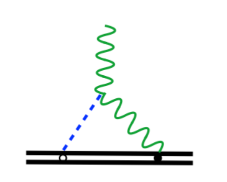
$$\varphi_{3\mathrm{PN}} = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{3424}{21}\ln 2 + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3$$

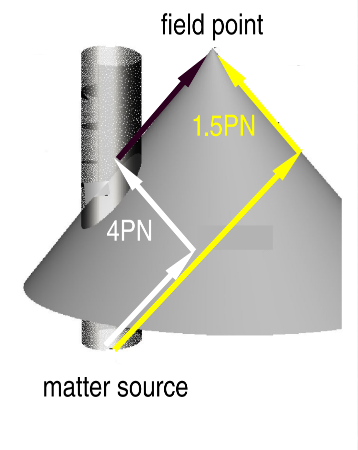
$$\varphi^{(l)}_{3\mathrm{PN}} = -\frac{856}{21}$$

$$\varphi_{3.5\mathrm{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

The gravitational wave tail effect [Blanchet & Damour 1988, 1992]







- In the near zone (4PN effect)

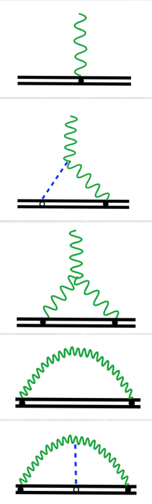
$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

- In the far zone (1.5PN effect)

$$h_{ij}^{\text{tail}} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{u} \mathrm{d}u'\, I_{ij}^{(4)}(u') \ln\left(\frac{u-u'}{P}\right)$$

Diagrammatic expansion in EFT

Effective Field Theory



$\Longleftrightarrow$

Post-Newtonian

- emission from a quadrupole source
- tail effect in radiation field (1.5PN)
- non-linear memory effect (2.5PN)
- radiation reaction (2.5PN)
- tail in radiation reaction (4PN)

The EFT is equivalent to the traditional PN at the level of tree diagrams

Tail effects in PN parameters

tail terms

tail-of-tail terms

$$\varphi_{0\mathrm{PN}} = 1$$

$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = -10\pi$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(l)}_{2.5\mathrm{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

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Toward 4.5PN parameters

- The 4.5PN term is also known and due to the 4.5PN tail-of-tail-of-tail integral for circular orbits [Marchand, Blanchet & Faye 2017; Messina & Nagar 2017]

$$\varphi_{4.5\text{PN}} = \Big(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_\text{E} + \frac{3424}{21}\ln 2 + \Big[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\Big]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3 \Big)\pi$$

$$\varphi^{(l)}_{4.5\text{PN}} = \frac{856}{21}\pi$$

tail-of-tail-of-tail terms

- However the 4PN term is only known from perturbative BH theory in the test-mass limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\varphi_{4\text{PN}} = \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_\text{E} - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu)$$

$$\varphi^{(l)}_{4\text{PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

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$$\varphi^{(l)}_{4\text{PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

The 4.5PN radiative quadrupole moment

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} \mathrm{d}\tau I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+ \frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} \mathrm{d}\tau I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+ \underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} \mathrm{d}\tau I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+ \underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} \mathrm{d}\tau I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+ \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

The source type multipole moments

Following the matching between the near zone and the exterior zone

$$I_L = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \bigg\{ \delta_\ell \, \hat{x}_L \, \overline{\Sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \, \delta_{\ell+1} \, \hat{x}_{iL} \, \overline{\Sigma}_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \, \delta_{\ell+2} \, \hat{x}_{ijL} \, \overline{\Sigma}_{ij}^{(2)} \bigg\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right)$$

$$J_L = \underset{B=0}{\mathrm{FP}} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \varepsilon_{ab\langle i_\ell} \int_{-1}^{1} \mathrm{d}z \bigg\{ \delta_\ell \hat{x}_{L-1\rangle a} \Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1\rangle ac} \overline{\Sigma}_{bc}^{(1)} \bigg\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right)$$

$$\overline{\Sigma} = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \qquad \overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c} \qquad \overline{\Sigma}_{ij} = \overline{\tau}^{ij}$$

where $\overline{\tau}^{\mu\nu}$ represents the PN expansion of the matter + gravitation stress-energy pseudo tensor (*a priori* valid only in the near zone)

The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet 2020]

- Using dimensional regularisation for UV but Hadamard regularization for IR

$$I_{ij} = \mu\, A\, x_{\langle i} x_{j\rangle} + \cdots + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$\begin{aligned}
A = {} & 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\
& + \underbrace{\gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu + \cdots\right)}_{\text{3PN terms}} \\
& + \underbrace{\gamma^4\left(-\frac{1023844001989}{12713500800} + \frac{31886}{2205}\ln\left(\frac{r}{r_0}\right) + \cdots\right)}_{\text{4PN terms}} \\
C = {} & \overbrace{\frac{48}{7} + \gamma\left(-\frac{4096}{315} - \frac{24512}{945}\nu\right)}^{\text{2.5PN and 3.5PN terms}}
\end{aligned}$$

- This result has to be completed by dimensional regularization for the IR

The 3.5PN gravitational-wave $(\ell, m) = (2,2)$ mode

$$h_+ - \mathrm{i}h_\times = \sum_{\ell=2}^{+\infty} \sum_{m=-\ell}^{\ell} h^{\ell m}\, Y_{-2}^{\ell m}(\Theta, \Phi)$$

- The modes can be compared directly with results from numerical relativity
- The dominant mass-type quadrupole mode is

$$\begin{aligned}
H^{22} &= 1 + x\Big(-\frac{107}{42} + \frac{55}{42}\nu\Big) + 2\pi x^{3/2} \\
&+ x^2\Big(-\frac{2173}{1512} - \frac{1069}{216}\nu + \frac{2047}{1512}\nu^2\Big) + x^{5/2}\Big(-\frac{107\pi}{21} - 24\,\mathrm{i}\,\nu + \frac{34\pi}{21}\nu\Big) \\
&+ x^3\Bigg(\frac{27027409}{646800} - \frac{856}{105}\,\gamma_\mathsf{E} + \frac{428\,\pi}{105}\mathrm{i} + \frac{2\pi^2}{3} \\
&\qquad + \Big(-\frac{278185}{33264} + \frac{41\pi^2}{96}\Big)\nu - \frac{20261}{2772}\nu^2 + \frac{114635}{99792}\nu^3 - \frac{428}{105}\ln(16x)\Bigg) \\
&+ x^{7/2}\Big(-\frac{2173\pi}{756} + \Big(-\frac{2495\pi}{378} + \frac{14333}{162}\mathrm{i}\Big)\,\nu + \Big(\frac{40\pi}{27} - \frac{4066}{945}\mathrm{i}\Big)\,\nu^2\Big)
\end{aligned}$$

- The $(\ell, m) = (2,2)$ mode at 4PN order is in progress

The 3PN current type quadrupole moment

[Henry, Faye & Blanchet 2021]

- We need dimensional regularisation for the UV but Hadamard regularization is sufficient for the IR
- To apply dimensional regularization we define the decomposition of a tensor into irreducible pieces in d dimensions (where we do not have the usual ε_{ijk} to define the current moment)
- The mass moment I_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $J_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

$$I_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline \end{array}$$

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i \\ \cline{1-1} \end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i \\ \cline{1-2} \end{array}$$

- The tensor $K_{ij|L}$ is absent in 3 dimensions

$$\sharp(\text{components}) = \frac{(d-3)d(d-1)_{\ell-2}(2\ell+d-2)(\ell+d-1)}{2\ell(\ell+1)(\ell-2)!}$$

The 3PN current type quadrupole moment

[Henry, Faye & Blanchet 2021]

- After dimensional regularization and renormalization

$$J_{ij} = -\mu\Delta\left[A\,L^{\langle i}x^{j\rangle} + \cdots\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$A = 1 + \gamma\left(\tfrac{67}{28} - \tfrac{2}{7}\nu\right) + \gamma^2\left(\tfrac{13}{9} - \tfrac{4651}{252}\nu - \tfrac{\nu^2}{168}\right) + \cdots$$

- The corresponding $(\ell, m) = (2,1)$ mode at 3.5PN order reads

$$\begin{aligned} H^{21} = \frac{\mathrm{i}}{3}\,\Delta\Big[& x^{1/2} + x^{3/2}\left(-\tfrac{17}{28} + \tfrac{5\nu}{7}\right) + x^2\left(\pi + \mathrm{i}\left[-\tfrac{1}{2} - 2\ln 2\right]\right) \\ & + x^{5/2}\left(-\tfrac{43}{126} - \tfrac{509\nu}{126} + \tfrac{79\nu^2}{168}\right) \\ & + x^3\left(\pi\left[-\tfrac{17}{28} + \tfrac{3\nu}{14}\right] + \mathrm{i}\left[\tfrac{17}{56} + \nu\left(-\tfrac{353}{28} - \tfrac{3}{7}\ln 2\right) + \tfrac{17}{14}\ln 2\right]\right) \\ & + x^{7/2}\bigg(\tfrac{15223771}{1455300} + \tfrac{\pi^2}{6} - \tfrac{214}{105}\gamma_\mathsf{E} - \tfrac{107}{105}\ln(4x) - \ln 2 - 2(\ln 2)^2 \\ & \quad + \nu\left[-\tfrac{102119}{2376} + \tfrac{205}{128}\pi^2\right] - \tfrac{4211}{8316}\nu^2 + \tfrac{2263}{8316}\nu^3 + \mathrm{i}\pi\left[\tfrac{109}{210} - 2\ln 2\right]\bigg)\Big] \end{aligned}$$