NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Non-relativistic symmetries in 2+1 dimensions and the Nappi-Witten algebra

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Motivation

- Non-relativistic (NR) symmetries provide non-AdS scenarios to study the AdS/CFT correspondence.

- They play an important role in condensed matter systems where some aspects of the holographic conjecture could be experimentally tested.

- Newton-Cartan Geometry is the natural geometric framework for these systems.

- Three-dimensional models of non-relativistic gravity provide interesting tractable models.

- Extensions of Non-relativistic symmetries, such as the NR Maxwell algebra, incorporate constant background fields.

- They define extended Newton-Cartan geometries and it is worth to analyze their implications

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- Non-relativistic systems are characterized by Galilean symmetries
- The Galilei algebra

$$[\boldsymbol{G}_a, \boldsymbol{H}] = \boldsymbol{P}_a, \qquad [\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = 4\delta_{[a[c}\boldsymbol{J}_{d]b]},$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{G}_c] = 2\delta_{c[b}\boldsymbol{G}_{a]}, \quad [\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = 2\delta_{c[b}\boldsymbol{P}_{a]}$$

- It is the $c \to \infty$ NR contraction of the Poincaré algebra

$$\boldsymbol{J}_{ab} = \mathcal{J}_{ab}, \qquad \boldsymbol{H} = c\,\mathcal{P}_0,$$
$$\boldsymbol{G}_a = \frac{1}{c}\mathcal{J}_{0a}, \quad \boldsymbol{P}_a = \mathcal{P}_a$$

where $\mathcal{J}_{AB} = \left\{\mathcal{J}_{0a}, \mathcal{J}_{ab}\right\}$, $\mathcal{P}_A = \left\{\mathcal{P}_0, \mathcal{P}_a\right\}$, are the Poincaré generators

$$[\mathcal{J}_{AB}, \mathcal{P}_C] = 2\eta_{C[B}\mathcal{P}_{A]}, \qquad\qquad [\mathcal{J}_{AB}, \mathcal{J}_{CD}] = 4\eta_{[A[C}\mathcal{J}_{D]B]},$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- Non-relativistic systems are characterized by Galilean symmetries
- The Bargmann algebra

$$[\boldsymbol{G}_a, \boldsymbol{H}] = \boldsymbol{P}_a \,, \qquad [\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = 4\delta_{[a[c}\boldsymbol{J}_{d]b]} \,,$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{G}_c] = 2\delta_{c[b}\boldsymbol{G}_{a]} \,, \quad [\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = 2\delta_{c[b}\boldsymbol{P}_{a]} \qquad [\boldsymbol{G}_a, \boldsymbol{P}_b] = \delta_{ab}\boldsymbol{M}$$

- as the $c \to \infty$ NR contraction of the Poincaré$\times\mathfrak{u}(1)$ *algebra*

$$\boldsymbol{J}_{ab} = \mathcal{J}_{ab} \,, \qquad \boldsymbol{H} = c\left(\mathcal{P}_0 + \mathcal{X}\right) \,, \qquad \boldsymbol{M} = \frac{1}{2c}\left(\mathcal{P}_0 - \mathcal{X}\right)$$
$$\boldsymbol{G}_a = \frac{1}{c}\mathcal{J}_{0a} \,, \quad \boldsymbol{P}_a = \mathcal{P}_a$$

where $\mathcal{J}_{AB} = \left\{\mathcal{J}_{0a}, \mathcal{J}_{ab}\right\}$, $\mathcal{P}_A = \left\{\mathcal{P}_0, \mathcal{P}_a\right\}$, are the Poincaré generators

$$[\mathcal{J}_{AB}, \mathcal{P}_C] = 2\eta_{C[B}\mathcal{P}_{A]} \,, \qquad [\mathcal{J}_{AB}, \mathcal{J}_{CD}] = 4\eta_{[A[C}\mathcal{J}_{D]B]} \,,$$

- An Extended Bargmann algebra can be defined in 2+1 dimensions
  [Lévy-Leblond(1993), Bose(1995), Grigore(1996)]

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b \,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,,$$
$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab}\boldsymbol{S} \,, \quad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab}\boldsymbol{M} \,,$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- In analogy with AdS→ Poincaré, the Newton-Hooke algebra generalizes the Galilei algebra to include cosmological constant.

$$[\boldsymbol{G}_a, \boldsymbol{H}] = \boldsymbol{P}_a\,, \qquad\qquad [\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = 4\delta_{[a[c}\boldsymbol{J}_{d]b]}\,,$$
$$[\boldsymbol{J}_{ab}, \boldsymbol{G}_c] = 2\delta_{c[b}\boldsymbol{G}_{a]}\,, \quad [\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = 2\delta_{c[b}\boldsymbol{P}_{a]}\,,$$
$$[\boldsymbol{P}_a, \boldsymbol{H}] = \Lambda\boldsymbol{G}_a\,, \qquad\qquad [\boldsymbol{P}_a, \boldsymbol{P}_b] = -\Lambda\boldsymbol{J}_{ab}$$

- In 2+1 dimensions This algebra can be similarly extended

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b\boldsymbol{G}_b\,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b\boldsymbol{P}_b\,, \qquad\quad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b\boldsymbol{P}_b\,,$$
$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab}\boldsymbol{S}\,, \quad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab}\boldsymbol{M}\,,$$
$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = \Lambda\epsilon_{ab}\,\boldsymbol{S}\,, \quad [\boldsymbol{H}, \boldsymbol{P}_a] = -\Lambda\epsilon_a{}^b\boldsymbol{G}_b\,.$$

- It reduces to the Bargmann algebra when $\Lambda \to 0$
- Newton-Cartan geometry can be obtained from gauging the Bargmann algebra
  [Andringa,Bergshoeff,Panda,Roo(2011)]
- Supersymmetric and Higher-Spin extensions of these algebras have been found
  [Bergshoeff,Rosel(2016)][Bergshoeff,Grumiller,Prohazka,Rosseel(2016)]
- Schrödinger extension of these symmetries is related to Horava-Lifshitz gravity
  [Hartong,Lei,Obers(2016)]

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Maxwell algebra

- The Maxwell algebra is an extension of the Poincaré symmetry
  [Schrader(1972)]

$$[\mathcal{J}_{AB}, \mathcal{P}_C] = 2\eta_{C[B}\mathcal{P}_{A]}, \qquad [\mathcal{J}_{AB}, \mathcal{J}_{CD}] = 4\eta_{[A[C}\mathcal{J}_{D]B]},$$

$$[\mathcal{P}_A, \mathcal{P}_B] = \mathcal{Z}_{AB}, \qquad [\mathcal{J}_{AB}, \mathcal{Z}_{CD}] = 4\eta_{[A[C}\mathcal{Z}_{D]B]},$$

- It describes particle systems in the presence of a constant electromagnetic field
  [Bacry(1970)][Negro,Olmo(1990)] [Gibbons,Gomis,Pope(2010)] [Gomis,Kleinschmidt(2017)]

- It has been used to construct gravity theories that extend GR
  [de Azcarraga,Kamimura,Lukierski(2011)][Concha,Peñafiel,Rodríguez,Salgado(2013)]

- Extensions to Supergravity and Higher-Spin theories have been formulated
  [Bonanos,Gomis,Kamimura,Lukierski(2010)][Fedoruk,Lukierski(2013)][de Azcarraga,Izquierdo,Lukierski,Woronowicz(2913]
  [Caroca,Concha,Fierro,Rodríguez,Salgado-Rebolledo(2018)][Concha,Ravera,Rodríguez(2019)]

- 2+1-dimensional Maxwell-invariant gravity as a Chern-Simons theory
  [Salgado,Szabo,Valdivia(2014)]

- The asymptotic symmetry of the theory is an extension of the $\mathfrak{bms}_3$ algebra
  [Concha,Merino,Miskovic,Rodríguez, Salgado-Rebolledo,Valdivia(2011)]

- First order formulation of Callan-Giddings-Harvey-Strominger dilaton gravity
  [Cangemi,Jackiw(1992)][Afshar,Gonzalez,Grumiller,Vassilevich(2020)]

- Geometric description of a topological insulator [Palumbo(2017)]

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- A similar $c \to \infty$ contraction leads to the NR Maxwell algebra
  [González,Rubio,Salgado,Salgado(2020)] [Gomis,Kleinschmidt,Palmkvist(2020)]

$$
\begin{aligned}
\boldsymbol{J}_{ab} &= \mathcal{J}_{ab}\,, &\boldsymbol{H} &= c\,\mathcal{P}_0\,, &\boldsymbol{Z}_{ab} &= c^2\,\mathcal{Z}_{ab}\,, \\
\boldsymbol{G}_a &= \frac{1}{c}\mathcal{J}_{0a}\,, &\boldsymbol{P}_a &= \mathcal{P}_a\,, &\boldsymbol{Z}_a &= c\,\mathcal{Z}_{0a}
\end{aligned}
$$

where $\mathcal{J}_{AB} = \left\{\mathcal{J}_{0a}, \mathcal{J}_{ab}\right\}, \mathcal{P}_A = \left\{\mathcal{P}_0, \mathcal{P}_a\right\}, \mathcal{Z}_{AB} = \left\{\mathcal{Z}_{0a}, \mathcal{Z}_{ab}\right\},$ are the Maxwell generators

$$
\begin{array}{lll}
[\boldsymbol{G}_a, \boldsymbol{H}] = \boldsymbol{P}_a\,, & [\boldsymbol{J}_{ab}, \boldsymbol{J}_{cd}] = 4\delta_{[a[c}\boldsymbol{J}_{d]b]}\,, & \\
[\boldsymbol{J}_{ab}, \boldsymbol{G}_c] = 2\delta_{c[b}\boldsymbol{G}_{a]}\,, & [\boldsymbol{J}_{ab}, \boldsymbol{P}_c] = 2\delta_{c[b}\boldsymbol{P}_{a]}\,, & [\boldsymbol{J}_{ab}, \boldsymbol{Z}_c] = 2\delta_{c[b}\boldsymbol{Z}_{a]} \\
[\boldsymbol{P}_a, \boldsymbol{H}] = \boldsymbol{Z}_a\,, & [\boldsymbol{J}_{ab}, \boldsymbol{Z}_{cd}] = 4\delta_{[a[c}\boldsymbol{Z}_{d]b]}\,, & [\boldsymbol{Z}_{ab}, \boldsymbol{G}_c] = 2\delta_{c[b}\boldsymbol{Z}_{a]}
\end{array}
$$

- Other NR Maxwell algebras exist [Becker,Hussin(1983)][Gomis,Kleinschmidt,Palmkvist(2020)]
- In 2+1 dimensions the algebra can be extended

$$
\begin{aligned}
[\boldsymbol{J}, \boldsymbol{G}_a] &= \epsilon_a{}^b\boldsymbol{G}_b\,, & [\boldsymbol{J}, \boldsymbol{P}_a] &= \epsilon_a{}^b\boldsymbol{P}_b\,, & [\boldsymbol{H}, \boldsymbol{G}_a] &= \epsilon_a{}^b\boldsymbol{P}_b\,, \\
[\boldsymbol{G}_a, \boldsymbol{G}_b] &= -\epsilon_{ab}\boldsymbol{S}\,, & [\boldsymbol{G}_a, \boldsymbol{P}_b] &= -\epsilon_{ab}\boldsymbol{M}\,, & & \\
[\boldsymbol{J}, \boldsymbol{Z}_a] &= \epsilon_a{}^b\boldsymbol{Z}_b\,, & [\boldsymbol{H}, \boldsymbol{P}_a] &= \epsilon_a{}^b\boldsymbol{Z}_b\,, & [\boldsymbol{Z}, \boldsymbol{G}_a] &= \epsilon_a{}^b\boldsymbol{Z}_a\,, \\
[\boldsymbol{P}_a, \boldsymbol{P}_b] &= -\epsilon_{ab}\boldsymbol{T}\,, & [\boldsymbol{G}_a, \boldsymbol{Z}_b] &= -\epsilon_{ab}\boldsymbol{T}\,. & &
\end{aligned}
$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The corresponding Chern-Simons theory leads to NR Maxwell gravity
  [Avilés,Frodden,Gomis,Hidalgo,Zanelli(2018)]

- The NR Maxwell algebra naturally appears in the description of anyons
  [Duval,Horvathy(2000)],[Horvathy,Martina,Stichel(2005)]

- The Maxwell algebra can be extended to include a cosmological constant:
  (A)dS-Lorentz algebra [Soroka,Soroka(2009)][Gibbons,Gomis,Pope(2010)]

$$[\mathcal{J}_{AB}, \mathcal{P}_C] = 2\eta_{C[B}\mathcal{P}_{A]}, \quad [\mathcal{J}_{AB}, \mathcal{J}_{CD}] = 4\eta_{[A[C}\mathcal{J}_{D]B]},$$

$$[\mathcal{P}_A, \mathcal{P}_B] = \mathcal{Z}_{AB}, \quad [\mathcal{J}_{AB}, \mathcal{Z}_{CD}] = 4\eta_{[A[C}\mathcal{Z}_{D]B]},$$

$$[\mathcal{Z}_A, \mathcal{P}_B] = -\Lambda 2\eta_{C[B}\mathcal{P}_{A]}, \quad [\mathcal{Z}_{AB}, \mathcal{Z}_{CD}] = -\Lambda 4\eta_{[A[C}\mathcal{Z}_{D]B]},$$

- Analogy with AdS → Poincaré

- It reduces to the Maxwell algebra in the limit $\Lambda \to 0$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- It has been used to define Supergravity and Higher-spin theories
  [Fierro,Izaurieta,Salgado,Valdivia(2014)][Banaudi,Ravera(2018)][Caroca,Concha,Fierro,Rodríguez,Salgado-Rebolledo(2018)]

- Extension of the $\mathfrak{bms}_3$ algebra [Concha,Merino,Rodríguez, Salgado-Rebolledo,Valdivia(2011)]

- The (A)dSL symmetry allows to define relativistic extensions of the Wen-Zee term.
  [Durka,Kowalski-Gilkman(2019)]

- A NR limit of the (A)dSL algebra can be defined

- In 2+1 dimensions it can be further centrally extended

$$[\boldsymbol{J},\boldsymbol{G}_a] = \epsilon_a{}^b\boldsymbol{G}_b\,, \qquad [\boldsymbol{J},\boldsymbol{P}_a] = \epsilon_a{}^b\boldsymbol{P}_b\,, \qquad [\boldsymbol{H},\boldsymbol{G}_a] = \epsilon_a{}^b\boldsymbol{P}_b\,,$$

$$[\boldsymbol{G}_a,\boldsymbol{G}_b] = -\epsilon_{ab}\boldsymbol{S}\,, \qquad [\boldsymbol{G}_a,\boldsymbol{P}_b] = -\epsilon_{ab}\boldsymbol{M}\,,$$

$$[\boldsymbol{J},\boldsymbol{Z}_a] = \epsilon_a{}^b\boldsymbol{Z}_b\,, \qquad [\boldsymbol{H},\boldsymbol{P}_a] = \epsilon_a{}^b\boldsymbol{Z}_b\,, \qquad [\boldsymbol{Z},\boldsymbol{G}_a] = \epsilon_a{}^b\boldsymbol{Z}_a\,,$$

$$[\boldsymbol{P}_a,\boldsymbol{P}_b] = -\epsilon_{ab}\boldsymbol{T}\,, \qquad [\boldsymbol{G}_a,\boldsymbol{Z}_b] = -\epsilon_{ab}\boldsymbol{T}\,.$$

$$[\boldsymbol{Z},\boldsymbol{Z}_a] = -\Lambda\epsilon_a{}^b\boldsymbol{Z}_b\,, \qquad [\boldsymbol{P}_a,\boldsymbol{Z}_b] = \Lambda\epsilon_{ab}\boldsymbol{M}\,, \qquad [\boldsymbol{Z}_a,\boldsymbol{Z}_b] = \Lambda\epsilon_{ab}\boldsymbol{T}\,,$$

$$[\boldsymbol{H},\boldsymbol{Z}_a] = -\Lambda\epsilon_a{}^b\boldsymbol{P}_b\,, \qquad [\boldsymbol{Z},\boldsymbol{P}_a] = -\Lambda\epsilon_a{}^b\boldsymbol{P}_b\,.$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- Extended Bargmann/ Newton-Hooke$^-$ in 2+1 dimensions

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b\,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b\,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b\,,$$

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S}\,, \qquad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M}\,,$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\tfrac{1}{\ell^2} \epsilon_{ab}\, \boldsymbol{S}\,, \quad [\boldsymbol{H}, \boldsymbol{P}_a] = \tfrac{1}{\ell^2} \epsilon_a{}^b \boldsymbol{G}_b$$

- Extended NR Maxwell/ NR AdSL in 2+1 dimensions

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b\,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b\,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b\,,$$

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S}\,, \qquad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M}\,,$$

$$[\boldsymbol{J}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b\,, \qquad [\boldsymbol{H}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{Z}_b\,, \qquad [\boldsymbol{Z}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{Z}_a\,,$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T}\,, \qquad [\boldsymbol{G}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T}\,.$$

$$[\boldsymbol{Z}, \boldsymbol{Z}_a] = \frac{1}{\ell^2} \epsilon_a{}^b \boldsymbol{Z}_b\,, \qquad [\boldsymbol{P}_a, \boldsymbol{Z}_b] = -\frac{1}{\ell^2} \epsilon_{ab} \boldsymbol{M}\,, \quad [\boldsymbol{Z}_a, \boldsymbol{Z}_b] = -\frac{1}{\ell^2} \epsilon_{ab} \boldsymbol{T}\,,$$

$$[\boldsymbol{H}, \boldsymbol{Z}_a] = \frac{1}{\ell^2} \epsilon_a{}^b \boldsymbol{P}_b\,, \qquad [\boldsymbol{Z}, \boldsymbol{P}_a] = \frac{1}{\ell^2} \epsilon_a{}^b \boldsymbol{P}_b$$

- These algebras and some generalizations can be systematically constructed using the Nappi-Witten symmetry

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Outline

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Nappi-Witten algebra

- Nappi-Witten ($\mathfrak{nw}$) algebra: central extension of the $iso(1,1)$ algebra
  [Cangemi,Jackiw(1992)][Nappi,Witten(1993)]

- We will consider the analogous extension of the E_2 algebra
  [Figueora-O'Farrill,Stanciu (1994)]

$$[\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \qquad [\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T}$$

- Isomorphic to the Newton-Hooke$_2$ algebra
  [Alvarez,Gomis,Kamimura,Plyushchay(2008)]

  and to the Maxwell algebra in $1+1$ dimensions
  [Afshar,Gonzalez,Grumaller,Vassalevach(2020)]

- Invariant bi-linear form

$$\langle \boldsymbol{J}\boldsymbol{J} \rangle = \mu_0 \,, \ \ \langle \boldsymbol{P}_a \boldsymbol{P}_b \rangle = \mu_1 \delta_{ab} \,, \ \ \langle \boldsymbol{J}\boldsymbol{T} \rangle = -\mu_1$$

- $\mathfrak{nw}$ can be used as a building block of NR symmetries in $2+1$ dimensions

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- First example:

$$\mathfrak{nw} \times \mathfrak{nw} \cong \text{Ext. Newton} - \text{Hooke}_3^-$$

- Consider two copies of $\mathfrak{nw}$

$$[\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b, \quad [\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T} \quad \longrightarrow \quad [\boldsymbol{t}, \boldsymbol{t}_a] = \epsilon_a{}^b \boldsymbol{t}_b, \quad [\boldsymbol{t}_a, \boldsymbol{t}_b] = -\epsilon_{ab} \boldsymbol{\tau}$$

$$\left[\boldsymbol{t}^{\pm}, \boldsymbol{t}_a^{\pm}\right] = \epsilon_a{}^b \boldsymbol{t}_b^{\pm}, \quad \left[\boldsymbol{t}_a^{\pm}, \boldsymbol{t}_b^{\pm}\right] = -\epsilon_{ab} \boldsymbol{\tau}^{\pm},$$

and the redefinition

$$\boldsymbol{t}_a^{\pm} = \frac{1}{2} \left(\boldsymbol{G}_a \pm \ell \boldsymbol{P}_a\right), \quad \boldsymbol{t}^{\pm} = \frac{1}{2} \left(\boldsymbol{J} \pm \ell \boldsymbol{H}\right), \quad \boldsymbol{\tau}^{\pm} = \frac{1}{2} \left(\boldsymbol{S} \pm \ell \boldsymbol{M}\right),$$

- The new generators satisfy the Extended Newton-Hooke$_3^-$ algebra

[Alvarez,Gomis,Kamimura,Plyushchay(2008)][Papageorgiou,Schroers(2010)] [Hartong,Lei,Oling,Obers(2017)]

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b,$$

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S}, \qquad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M},$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\frac{1}{\ell^2}\epsilon_{ab} \boldsymbol{S}, \quad [\boldsymbol{H}, \boldsymbol{P}_a] = \frac{1}{\ell^2}\epsilon_a{}^b \boldsymbol{G}_b.$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- In the limit $\ell \to \infty$ we find the Extended Bargmann algebra

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b,$$
$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S}, \quad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M},$$

- The Extended Bargmann algebra can also be constructed as

$$\mathfrak{nw} \ltimes_{ad} \mathfrak{nw}_{\mathrm{Ab}} \cong \mathrm{Ext.\ Bargman}_3$$

- The bracket of a semidirect sum algebra has the general form

$$[(X, \alpha), (Y, \beta)] = \Big([X, Y],\ [X, \beta] - [Y, \alpha]\Big).$$

- The Extended Bargmann algebra is recovered when setting

$$\boldsymbol{J} \equiv (\boldsymbol{t}, 0), \qquad \boldsymbol{H} \equiv (0, \boldsymbol{t}),$$
$$\boldsymbol{G}_a \equiv (\boldsymbol{t}_a, 0), \qquad \boldsymbol{P}_a \equiv (0, \boldsymbol{t}_a),$$
$$\boldsymbol{S} \equiv (\boldsymbol{\tau}, 0), \qquad \boldsymbol{M} \equiv (0, \boldsymbol{\tau})$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- One can generalize the previous result

$$\mathfrak{nw} \times \mathfrak{nw} \times \mathfrak{nw} \cong \text{Ext. NR AdSL}_3$$

- Consider three copies of $\mathfrak{nw}$, $\{t^{\pm}, t_a^{\pm}, \tau^{\pm}\}$, $\{\hat{t}, \hat{t}_a, \hat{\tau}\}$ and define the new basis

$$J = t^+ + t^- + \hat{t} \qquad H = \frac{1}{\ell}\left(t^+ - t^-\right), \qquad Z = \frac{1}{\ell^2}\left(t^+ + t^-\right),$$

$$G_a = t_a^+ + t_a^- + \hat{t}_a, \qquad P_a = \frac{1}{\ell}\left(t_a^+ - t_a^-\right), \qquad Z_a = \frac{1}{\ell^2}\left(t_a^+ + t_a^-\right),$$

$$S = \tau^+ + \tau^- + \hat{\tau}, \qquad M = \frac{1}{\ell}\left(\tau^+ - \tau^-\right), \qquad T = \frac{1}{\ell^2}\left(\tau^+ + \tau^-\right),$$

which satisfies the Extended NR-AdSL algebra

$$
\begin{array}{lll}
[J, G_a] = \epsilon_a{}^b G_b, & [J, P_a] = \epsilon_a{}^b P_b, & [H, G_a] = \epsilon_a{}^b P_b, \\
[G_a, G_b] = -\epsilon_{ab} S, & [G_a, P_b] = -\epsilon_{ab} M, & \\
[J, Z_a] = \epsilon_a{}^b Z_b, & [H, P_a] = \epsilon_a{}^b Z_b, & [Z, G_a] = \epsilon_a{}^b Z_a, \\
[P_a, P_b] = -\epsilon_{ab} T, & [G_a, Z_b] = -\epsilon_{ab} T. & \\
[Z, Z_a] = \frac{1}{\ell^2}\epsilon_a{}^b Z_b, & [P_a, Z_b] = -\frac{1}{\ell^2}\epsilon_{ab} M, & [Z_a, Z_b] = -\frac{1}{\ell^2}\epsilon_{ab} T, \\
[H, Z_a] = \frac{1}{\ell^2}\epsilon_a{}^b P_b, & [Z, P_a] = \frac{1}{\ell^2}\epsilon_a{}^b P_b. &
\end{array}
$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- In the limit $\ell \to \infty$ this reduces to the Extended NR Maxwell symmetry

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b \,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,,$$
$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S} \,, \quad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M} \,,$$
$$[\boldsymbol{J}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{Z}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{Z}_a \,,$$
$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T} \,, \quad [\boldsymbol{G}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T} \,.$$

- The Extended NR Maxwell algebra can also be constructed as

$$\mathfrak{nw} \ltimes_{ad} \mathfrak{nw}_{\mathrm{Ab}}^{\mathrm{Ext}} \cong \text{Ext. NR Maxwell}_3$$
$$\mathfrak{nw}_{\mathrm{Ab}}^{\mathrm{Ext}} : \qquad (\alpha, a) \,\hat{+}\, (\beta, b) = \left(\alpha + \beta, \; a + b + \frac{1}{2}[\alpha, \beta] \right) \,.$$

- The bracket of this algebra has the general form [PSR(2019)]

$$[(X, \alpha, a), (Y, \beta, b)] = \left([X, Y], \; [X, \beta] - [Y, \alpha], \; [X, b] - [Y, a] + [\alpha, \beta] \right) \,.$$

- The Extended NR Maxwell algebra is recovered when setting

$$\{\boldsymbol{J}, \boldsymbol{G}_a, \boldsymbol{S}\} \equiv (\{\boldsymbol{t}, \boldsymbol{t}_a, \boldsymbol{\tau}\}, 0, 0) \,, \quad \{\boldsymbol{H}, \boldsymbol{P}_a, \boldsymbol{M}\} \equiv (0, \{\boldsymbol{t}, \boldsymbol{t}_a, \boldsymbol{\tau}\}, 0) \,,$$
$$\{\boldsymbol{Z}, \boldsymbol{Z}_a, \boldsymbol{T}\} \equiv (0, 0, \{\boldsymbol{t}, \boldsymbol{t}_a, \boldsymbol{\tau}\})$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Expansions of $\mathfrak{nw}$

- The previous results can be alternatively obtained by means of Lie algebra expansions

  [Hatsuda,Sakaguchi(2003)] [de Azcárraga, Picón, Varela(2003)] [Izaurieta,Rodríguez,Salgado(2006)]

- Consider once again the $\mathfrak{nw}$ algebra

$$[\boldsymbol{t}, \boldsymbol{t}_a] = \epsilon_a{}^b \, \boldsymbol{t}_b \, , \quad [\boldsymbol{t}_a, \boldsymbol{t}_b] = -\epsilon_{ab} \, \boldsymbol{\tau} \, ,$$

- Semigroup

$$\lambda_i \cdot \lambda_j = \lambda_{i \diamond j} \, ,$$

- The expanded algebra as given by

$$S \times \mathfrak{nw} = \left\{ S \otimes \{\boldsymbol{t}, \boldsymbol{t}_a, \boldsymbol{\tau}\} \right\} \, ,$$

- Expanded generators

$$\boldsymbol{t}_a^{(i)} = \lambda_i \otimes \boldsymbol{t}_a \, , \quad \boldsymbol{t}^{(i)} = \lambda_i \otimes \boldsymbol{t} \, , \quad \boldsymbol{\tau}^{(i)} = \lambda_i \otimes \boldsymbol{\tau} \, ,$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The expanded algebra has the form

$$\left[\boldsymbol{t}^{(i)}, \boldsymbol{t}_a^{(j)}\right] = \epsilon_a{}^b \boldsymbol{t}_b^{(i \diamond j)}, \quad \left[\boldsymbol{t}_a^{(i)}, \boldsymbol{t}_b^{(j)}\right] = -\epsilon_{ab} \boldsymbol{\tau}^{(i \diamond j)}.$$

- Consider the semigroups

$$S_M^{(n)} = \{\lambda_0, \ldots, \lambda_n\} \qquad i \diamond j = \begin{cases} i+j & \text{if} \quad i+j \leqslant n, \\ i+j-2\left[\frac{n+1}{2}\right] & \text{if} \quad i+j > n. \end{cases}$$

$$S_E^{(n)} = \{\lambda_0, \ldots, \lambda_{n+1}\} \qquad i \diamond j = \begin{cases} i+j & \text{if} \quad i+j \leqslant n, \\ n+1 & \text{if} \quad i+j > n. \end{cases}$$

- $n = 1$: Ext. Newton-Hooke / Ext. Bargmann

$$\begin{aligned} \boldsymbol{G}_a &= \boldsymbol{t}_a^{(0)}, & \boldsymbol{J} &= \boldsymbol{t}^{(0)}, & \boldsymbol{S} &= \boldsymbol{\tau}^{(0)}, \\ \boldsymbol{P}_a &= \boldsymbol{t}_a^{(1)}, & \boldsymbol{H} &= \boldsymbol{t}^{(1)}, & \boldsymbol{M} &= \boldsymbol{\tau}^{(1)} \end{aligned}$$

- $n = 2$: Ext. NR AdSL / Ext. NR Maxwell

$$\boldsymbol{Z}_a = \boldsymbol{t}_a^{(2)}, \quad \boldsymbol{Z} = \boldsymbol{t}^{(2)}, \quad \boldsymbol{T} = \boldsymbol{\tau}^{(2)}$$

- Reduction [Izaurieta,Rodríguez,Salgado(2006)]

$$\boldsymbol{t}^{(n+1)} = \boldsymbol{t}_a^{(n+1)} = \boldsymbol{\tau}^{(n+1)} \equiv 0,$$

- $n > 2$ More general NR symmetries

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The method allows to construct NR gravity theories invariant under these NR symmetries in a systematic way
- This can be done by defining a connection on $S \times \mathfrak{nw}$,

$$\mathbb{A} = \sum_{i=0}^{N} \left(\theta_{(i)}^{a} \, \boldsymbol{t}_{a}^{(i)} + \alpha_{(i)} \, \boldsymbol{t}^{(i)} + \beta_{(i)} \, \boldsymbol{\tau}^{(i)} \right) ,$$

and defining the Chern-Simons action

$$S = \int \left\langle \mathbb{A} \wedge d\mathbb{A} + \frac{2}{3} \, \mathbb{A} \wedge \mathbb{A} \wedge \mathbb{A} \right\rangle ,$$

- The general result is

$$\mathcal{L}_{S \times \mathfrak{nw}} = \sum_{i,j=0}^{N} \mu_{i \diamond j} \left(\delta_{ab} \theta_{(i)}^{a} \wedge R_{(j)}^{b} - \alpha_{(i)} \wedge \beta_{(j)} - \beta_{(i)} \wedge \alpha_{(j)} \right)$$

$$+ \sum_{i,j=0}^{N} \nu_{i \diamond j} \alpha_{(i)} \wedge \alpha_{(j)} ,$$

where we have defined $R_{(i)}^{a} = \theta_{(i)}^{a} + \sum_{j,k=0}^{N} \epsilon^{a}_{\ b} \, \delta_{i}^{j \diamond k} \theta_{(k)}^{b} \wedge \alpha_{(j)}$

- Supersymmetric extension [Concha,Ipinza,Ravera,Rodríguez (2020)]

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The $n = 1$ case reproduces Extended Bargmann and Extended Newton-Hooke gravity in 2+1 dimensions
  [Papageorgiou,Schroers(2009)][Bergshoeff,Rosseel(2016)][Hartong,Lei,Obers(2016)]

$$\theta_{(0)}^a = \omega^a, \qquad \alpha_{(0)} = \omega, \qquad \beta_{(0)} = s,$$
$$\theta_{(1)}^a = e^a, \qquad \alpha_{(1)} = h, \qquad \beta_{(1)} = m,$$

- Using the semigroup S_E, the $n = 2$ case reproduces Extended NR Maxwell gravity
  [Avilés,Frodden,Gomis,Hidalgo,Zanelli(2018)]

$$\theta_{(2)}^a = k^a, \quad \alpha_{(2)} = k, \quad \beta_{(2)} = -t.$$

- Using the semigroup S_M, the $n = 2$ case corresponds to Extended NR AdSL gravity, which reduces to the Maxwellian case in the cosmological constant vanishes [Peñafiel,Salgado-Rebolledo(2019)][Concha,Rodríguez(2019)]

- For $n > 2$, more general NR gravity theories in 2+1 dimensions can be constructed

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Quantum Hall effect

- The quantum Hall effect is the quantization of the conductance in two-dimensional electron systems under the presence of a strong external magnetic field
- Hall law

$$J_i = -\frac{\nu}{2\pi}\epsilon_{ij}E_j$$

- It can be obtained as the field equation of a Chern-Simons action

$$S = \frac{\nu}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}A_{\mu\nu}A_\rho + J_\mu A^\mu$$

- The conductance ν is restricted to take integer values
- It was later discovered that ν can take fractional values: Fractional Quantum Hall Effect

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The values $v = 1/2p + 1$ correspond to Laughling states
- An effective description can be obtained by introducing an emergent field a and considering the effective action
  [Wen(2007)]

$$S = \int \left(-\frac{q}{4\pi} a da + \frac{1}{2\pi} A da \right)$$

- Integrating out the field a leads to a Chern-Simons theory with fractional conductance

$$S = \frac{\nu}{4\pi} \int A dA, \qquad \nu = \frac{1}{q}$$

- When the QHE takes places on a curved surface, the Wen-Zee and a gravitational Chern-Simons term are relevant
  [Wen,Zee(1993)][Fröhlich,Studer(1993)][Gromov,Cho,You,Abanov,Fradkin(2015)][Moroz,Hoyos,Radzihovsky(2015)] [Bradlyn,Read(2015)]
  [Gromov,Jensen,Abanov(2016)][Capelli,Randellini(2016)]

$$S = \int \left[\left(\frac{\nu \bar{s}^2}{4\pi} - \frac{c}{48\pi} \right) \omega d\omega + \frac{\nu}{4\pi} A dA + \frac{\nu \bar{s}}{2\pi} A d\omega \right]$$

- The Quantum Hall effect can be described in terms of Newton-Cartan geometry
  [Son(2013)][Geracie,Son,Wu,Wu(2015)]

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The Extended NR Maxwell/ NR AdSL algebra in 2+1 dimensions is expected to play a role in the description of NR systems in the presence of a constant electromagnetic field

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b \,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,,$$

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S} \,, \quad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M} \,,$$

$$[\boldsymbol{J}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{Z}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{Z}_a \,,$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T} \,, \quad [\boldsymbol{G}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T} \,.$$

$$[\boldsymbol{Z}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{P}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{M} \,, \quad [\boldsymbol{Z}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T} \,,$$

$$[\boldsymbol{H}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{Z}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b$$

- It contains a Nappi-Witten subalgebra $\{\boldsymbol{J}, \boldsymbol{P}_a, \boldsymbol{T}\}$
- Based on the previous discussion one can associate

$$\omega \to \boldsymbol{J} \,, \qquad A \to \boldsymbol{T}$$

- The field a is missing

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Extended Nappi-Witten algebra

- Consider once again the $\mathfrak{nw}$ algebra

$$[\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \qquad [\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T}$$

- We add a $\mathfrak{u}(1)$ generator Y associated to an emergent gauge field a

$$\mathbb{A} = \omega \boldsymbol{J} + e^a \boldsymbol{P}_a + A\boldsymbol{T} + a\boldsymbol{Y}.$$

- Admits a non-degenerate invariant bilinear form

$$\langle JJ \rangle = \mu_0 \,, \quad \langle P_a P_b \rangle = \mu_1 \delta_{ab} \,, \quad \langle JT \rangle = -\mu_1 \,, \quad \langle YJ \rangle = \rho_0 \,, \quad \langle YY \rangle = \rho_1 \,.$$

- The corresponding Chern-Simons action reads

$$S = \frac{k}{4\pi} \int \left[\mu_0 \, \omega d\omega + \mu_1 e^a \left(de_a + \epsilon_{ab} e^b \omega \right) - 2\mu_1 A d\omega + 2\rho_0 a d\omega + 2\rho_1 a da \right]$$

- This action does not contain the term Ada, which is crucial to describe the FQHE.

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- This term can be generated by introducing the shift

$$\omega \to \omega + \beta a \qquad \mathbb{A} = (\omega + \beta a)\boldsymbol{J} + e^a \boldsymbol{P}_a + A \boldsymbol{T} + a \boldsymbol{Y}.$$

- The action takes the form

$$S = \frac{k}{4\pi} \int \left[\mu_0 \, \omega d\omega + \mu_1 \, e^a \left(d e_a + \epsilon_{ab} e^b \omega + \beta \epsilon_{ab} e^b a \right) \right.$$

$$\left. - 2\mu_1 \, A d\omega - 2\mu_1 \beta \, A da + 2(\rho_0 + \beta\mu_0) \, a d\omega + (2\rho_1 + 2\rho_0\beta + \beta^2) \, a da \right],$$

- Varying with respect to e^a we find

$$T^a + \beta \epsilon^a{}_b e^b a = 0$$

- Using this condition we can eliminate the zweibein from the action.

- Renaming

$$\nu_0 = \rho_0 + \beta\mu_0 \,, \qquad \nu_1 = 2\rho_1 + 2\rho_0\beta + \beta^2$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The action now looks like

$$
S = \frac{k}{4\pi} \int \left[\mu_0 \, \omega d\omega - 2\mu_1 \, Ad\omega - 2\mu_1\beta \, Ada + 2\nu_0 \, ad\omega + \nu_1 \, ada \right],
$$

- Note that the connection can be put in the form

$$
\mathbb{A} = \omega J + e^a P_a + AT + aZ
$$

where we have defined

$$
\boldsymbol{Z} = \boldsymbol{Y} + \beta \boldsymbol{J}
$$

- This leads to the Extended nw algebra

$$
[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T}, \quad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b, \quad [\boldsymbol{Z}, \boldsymbol{P}_a] = \beta \epsilon_a{}^b \boldsymbol{P}_b,
$$

- Integrating out the field a and using the following identifications of the parameters

$$
k = \mu_3 = 1, \quad \mu_1 = 2\bar{s}\nu, \quad \mu_0 = \frac{c}{12},
$$

$$
\mu_2 = \bar{s}\sqrt{\nu}, \quad \beta = \frac{1}{2\bar{s}\sqrt{\nu}},
$$

we obtain the effective action that we considered before

$$
S = \int \left[\left(\frac{\nu\bar{s}^2}{4\pi} - \frac{c}{48\pi} \right) \omega d\omega + \frac{\nu}{4\pi} AdA + \frac{\nu\bar{s}}{2\pi} Ad\omega \right],
$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

- The Extended $\mathfrak{nw}$ can be naturally embedded in the NR AdSL symmetry

$$[\boldsymbol{J}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{G}_b \,, \qquad [\boldsymbol{J}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,,$$

$$[\boldsymbol{G}_a, \boldsymbol{G}_b] = -\epsilon_{ab} \boldsymbol{S} \,, \qquad [\boldsymbol{G}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{M} \,,$$

$$[\boldsymbol{J}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{H}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{Z}, \boldsymbol{G}_a] = \epsilon_a{}^b \boldsymbol{Z}_a \,,$$

$$[\boldsymbol{P}_a, \boldsymbol{P}_b] = -\epsilon_{ab} \boldsymbol{T} \,, \qquad [\boldsymbol{G}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T} \,.$$

$$[\boldsymbol{Z}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{Z}_b \,, \qquad [\boldsymbol{P}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{M} \,, \quad [\boldsymbol{Z}_a, \boldsymbol{Z}_b] = -\epsilon_{ab} \boldsymbol{T} \,,$$

$$[\boldsymbol{H}, \boldsymbol{Z}_a] = \epsilon_a{}^b \boldsymbol{P}_b \,, \qquad [\boldsymbol{Z}, \boldsymbol{P}_a] = \epsilon_a{}^b \boldsymbol{P}_b$$

- A Chern-Simons theory for the NR AdSL algebra involves a connection of the form

$$\mathbb{A} = \omega \boldsymbol{J} + \tau \boldsymbol{H} + a \boldsymbol{Z} + \omega^a \boldsymbol{G}_a + e^a \boldsymbol{P}_a + k^a \boldsymbol{Z}_a + m \boldsymbol{M} + s \boldsymbol{S} + A \boldsymbol{T}$$

- The geometric model previously constructed is recovered when fixing the gauge

$$\tau_\mu = \delta_\mu^0 \,, \qquad \omega_\mu^a = 0$$

- and setting

$$k_\mu^a = 0 \,, \qquad m_\mu = 0 = s_\mu \,.$$

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Conclusions and future directions

- We have shown how NR symmetries in 2+1 dimensions can be constructed out of the Nappi-Witten symmetry
- In particular we have found the NR version of the AdSL algebra
- Expansions of the $\mathfrak{nw}$ algebra lead to two different families of NR symmetries
- The method allows to find NR Chern-Simons gravity theories that generalize Extended Bargmann$_3$ and Extended Newton-Hooke$_3$ gravities.
- An extension of the $\mathfrak{nw}$ algebra allows to reproduce a geometric model for the FQHE.
- The Extended $\mathfrak{nw}$ symmetry can be naturally embedded in the NR AdSL algebra
- Bi-metric theories and Higher-Spin fields have been considered in the description of the FQHE.
  [Capelli,Randellini(2015)][Gromov,Son(2017)]
- FD: To generalize of the previous results to bi-metric theories. Analyze the role of Z_a or go to $n > 2$ in the expansion.
- Higher-Spin Extensions of the Maxwell and AdS-Lorentz algebra in 2+1 dimensions and Chern-Simons theories have been previously studied
  [Caroca,Concha,Fierro,Rodríguez,Salgado-Rebolledo(2018)] [Salgado-Rebolledo(2019)]

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Conclusions and future directions

- These theories generalize previous Higher-Spin gravity theories based on the AdS or Poincaré algebras [Campoleoni,Fredenhagen,Pfenninger,Theisen(2010)][Gonzalez,Pino(2014)]

- FD: To consider the following novel Higher-Spin Extension of the $\mathfrak{nw}$ algebra (Work in progress) [Caroca,Palumbo,Peñafiel,Salgado-Rebolledo]

$$[t, \mathcal{T}_i] = \epsilon_i{}^j \mathcal{T}_j, \qquad [t, \mathcal{T}_{ij}] = \epsilon_a{}^k \mathcal{T}_{j)k}$$

$$[t_i, \mathcal{T}_{jk}] = -\epsilon_{i(jk)}, \qquad [\tau, \mathcal{T}_i] = \epsilon_i{}^j{}_j, \qquad [i, \mathcal{T}_j] = \epsilon_{ij}\tau$$

$$[\mathcal{T}_i, \mathcal{T}_j] = \epsilon_{ij}t, \qquad [t_{,i}] = \epsilon_i{}^j{}_j, \qquad [\mathcal{T}_{ij}, \mathcal{T}_{kl}] = \left(\eta_{i(k}\epsilon_{l)j} + \eta_{j(k}\epsilon_{l)i}\right)\tau$$

$$[t_i, \mathcal{T}_j] = -\epsilon_i{}^k \mathcal{T}_{jk} - \epsilon_{ij}\eta^{kl}\mathcal{T}_{kl}, \qquad [\mathcal{T}_{ij}, \mathcal{T}_k] = -\eta_{k(i}\epsilon_{j)}{}^l t_l$$

- The simplest expansions of this Higher-Spin $\mathfrak{nw}$ algebra reproduces Higher-Spin Extensions of Extended Bargmann$_3$ and Extended Newton-Hooke$_3$.

  [Bergshoeff,Grumiller,Prohazka,Rosseel(2016)]

- One can continue to obtain Higher-Spin Extensions of NR Maxwell and AdSL in 2+1 dimensions and evaluate possible applications.

NR symmetries Nappi-Witten algebra
Expansions of Nappi-Witten
Extended Nappi-Witten algebra and FQHE

Thank you!