1. **Three-flavor neutrino oscillations**
In this problem, we derive approximate expressions for the 3-flavor neutrino oscillation probabilities. Crucial to this is the observation that
$$\Delta m^2_{21} \ll \Delta m^2_{31}, \Delta m^2_{32} \tag{1}$$
Indeed, global fits show that $\Delta m^2_{21} \simeq 8 \times 10^{-5}$ eV^2, $\Delta m^2_{31}, \Delta m^2_{32} \simeq 2 \times 10^{-3}$ eV^2.
   (a) Consider oscillations over baselines such that
   $$\frac{\Delta m^2_{21} L}{2E} \ll 1\,. \tag{2}$$
   Since Δm^2_{21} appears only in the combination $\Delta m^2_{21} L/2E$ in the oscillation probabilities, this implies that we can effectively take $\Delta m^2_{21} \to 0$. Eq. (2) is relevant for atmospheric, accelerator, and most reactor neutrino experiments. Show that, if (2) holds, the oscillation probabilities are given by
   $$P_{\alpha\beta} \simeq 4|U_{\alpha 3}|^2 |U_{\beta 3}|^2 \sin^2 \frac{\Delta m^2_{31} L}{4E}\,. \tag{3}$$
   (b) Using the explicit parameterization of U from the lecture, derive expressions for $P_{e\mu}$ and P_{ee} in the limit (2).
   (c) Consider now the limit
   $$\frac{\Delta m^2_{31} L}{2E} \simeq \frac{\Delta m^2_{32} L}{2E} \gg 1\,, \tag{4}$$
   which is relevant for instance to very long baseline reactor neutrino experiments. In this case, oscillations driven by Δm^2_{31}, Δm^2_{32} are very fast, and only their averaged effect can be probed experimentally due to finite experimental energy and position resolutions. Show that, in the limit (4), P_{ee} is given by
   $$P_{ee} = \sin^4\theta_{13} + \cos^4\theta_{13}\left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m^2_{21}}{4E}\right)\,. \tag{5}$$

2. Imagine a world in which neutrinos are massive, but charged leptons are massless. Will neutrinos oscillate in such a world?

3. Do neutrinos produced in the decay $Z^0 \to \bar{\nu}\nu$ oscillate? If so, describe a gedanken-experiment in which these oscillations could be observed.

4. **Atmospheric neutrinos**
Explain why the initial flavor ratio of atmospheric neutrinos at energies $\lesssim$ few GeV is $\overset{(-)}{\nu}_e : \overset{(-)}{\nu}_\mu : \overset{(-)}{\nu}_\tau \sim 1:2:0$. Consider the main products of the interactions of high-energy cosmic ray protons with nuclei in the Earth's atmosphere.

5. Discuss the advantages and disadvantages of a hypothetical neutrino oscillation experiment using a mono-energetic beam of $\mathcal{O}$(GeV) neutrinos. Can you think of a possible realization of such an experiment?

6. Imagine there is a fourth neutrino flavor in nature, with a mass of order $m_4 \sim 1$ eV, and with $U_{e4} \simeq U_{\mu 4} \simeq 0.1$. (Such scenarios will be one of the main topics of tomorrow's lecture.) Discuss the phenomenological consequences. How would you test this hypothesis experimentally? Describe at least two possible experimental setups.