Scale-Free Distributions In Nature:

An Overview of Self-Organized Criticality

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OUTLINE

Power-law distributions in nature

The challenge for statistical physics

Self-Organized Criticality: scaling without tuning

Sandpile models: activity threshold + slow drive

Connection with absorbing-state phase transitions

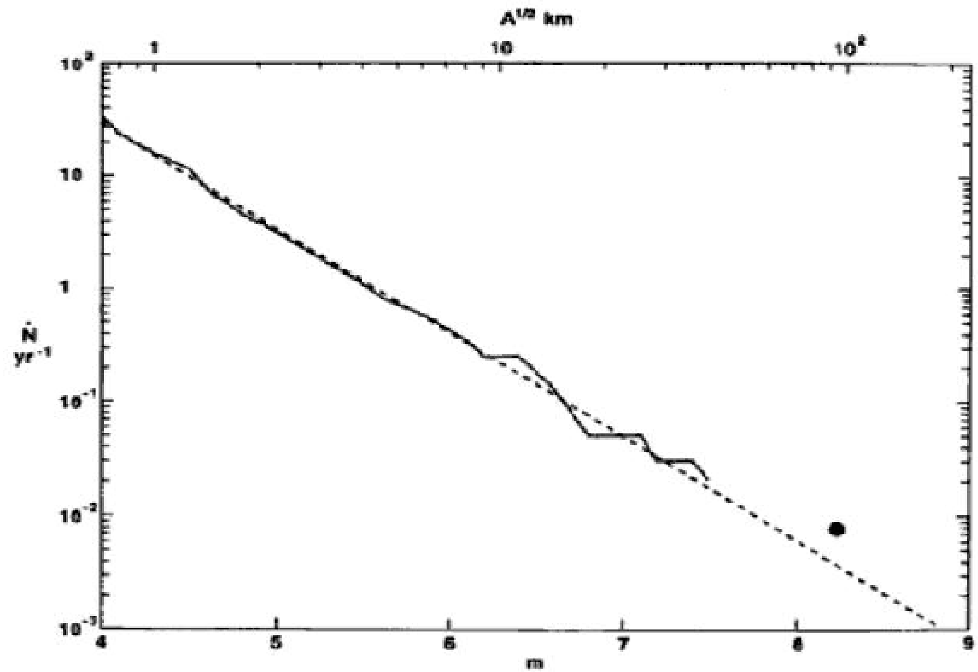
Alternatives to SOC

Power-law distributions in nature

Distributions of event sizes for many natural phenomena appear to follow power laws, in some cases over many orders of magnitude

Examples: earthquakes $N \sim m^{-b}$

Figure 4.2. Number of earthquakes per year $\dot{N}$ occurring in southern California with magnitudes greater than m as a function of m. The solid line is the data from the southern California earthquake network for the period 1932–1994. The straight dashed line is the correlation with (4.1) taking $b = 0.923$ ($D = 1.846$) and $\dot{a} = 1.4 \times 10^5$. The solid circle is the observed rate of occurrence of great earthquakes in southern California (Sieh *et al.*, 1989).



D Turcotte, ***Fractals and Chaos in Geology and Geophysics***, Cambridge, 1997

Where do scale-free distributions occur?

- In systems of many interacting nonlinear elements, characterized by periods of intense activity (***avalanches***) separated by inactive intervals.

- The distributions of avalanche sizes, durations, etc., follow power laws [$p(s) \sim s^{-\tau}$], which are intrinsically scale-free

- This is the key property suggesting that a system exhibits self-organized criticality

- A few examples: earthquakes, forest fires, ***rain intensity, drought duration***, magnetic domain-wall motion, neural activity, solar flares...

Why is the observation of power-law avalanche distributions in nature a problem?

Systems of many interacting unitsare described by ***statistical mechanics***

Statistical mechanics shows that most systems generically exhibit ***non***-power-law distributions (i.e., Poisson or exponential), with scale-invariant behavior ***only at a critical point*** (a continuous phase transition)

To reach criticality, one or more parameters (temperature, pressure...) must be adjusted precisely

Who is adjusting the parameters of Earth's tectonic plates or of forest-fire propagation?

Self-Organized Criticality (SOC)

Bak, Tang and Wiesenfeld (Phys Rev Lett,1987) argued that scale-invariant distributions and 1/f noise could arise without tuning in systems far from equilibrium, with a threshold for activity and transfer between active elements, when subject to a slow external drive.

Threshold dynamics: The response of each element to perturbations below a certain value is minimal; strong response or activity above this threshold

Coupling: When an active element relaxes, it perturbs its neighbors, which may themselves become active → ***avalanches***

Slow loss mechanism: When activity reaches the edge of the system, some is lost

Slow external drive: In the absence of activity, the system is excited at a rate << rate of relaxation/propagation

Sandpile Models: The BTW Sandpile

Square lattice of LxL sites

Each site (i,j) harbors z(i,j) particles or “sand grains” (z is called “height”)

$z(i,j) = 0, 1, 2, 3,$ or 4

If $z(i,j) = 4$, the site “topples”, transferring one grain to each neighbor:

$z(i,j) \rightarrow 0$ and $z(i+1,j) \rightarrow z(i+1,j) + 1$ and similarly for the other three neighbors of site (i,j)

This may cause other neighbors to topple, and so on (avalanche)

When a site at the edge of the system topples, one (or more) grains are lost

When all sites have $z(i,j) < 4$, a new grain is added at a randomly chosen site – infinitely slow drive

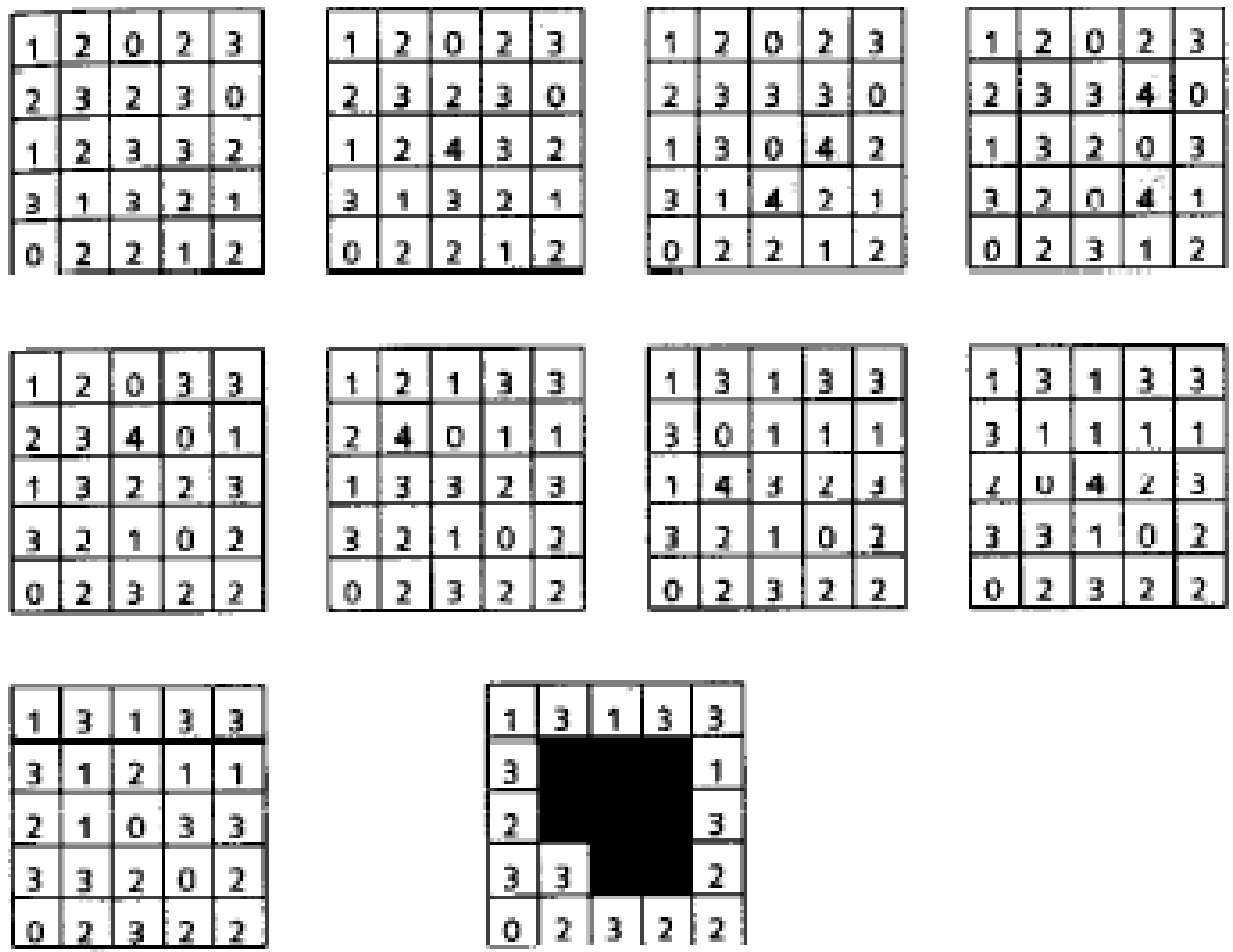


Figure 12. Illustration of toppling avalanche in a small sandpile. A grain falling at the site with height 3 at the center of the grid leads to an avalanche composed of nine toppling events, with a duration of seven update steps. The avalanche has a size $s = 9$. The black squares indicate the eight sites that toppled. One site toppled twice.

P Bak, ***How Nature Works*** (Springer-Verlag, 1996)

The simple rules of the BTW sandpile give rise to a scale-invariant avalanche-size distribution in the stationary state, apparently without adjusting any parameters

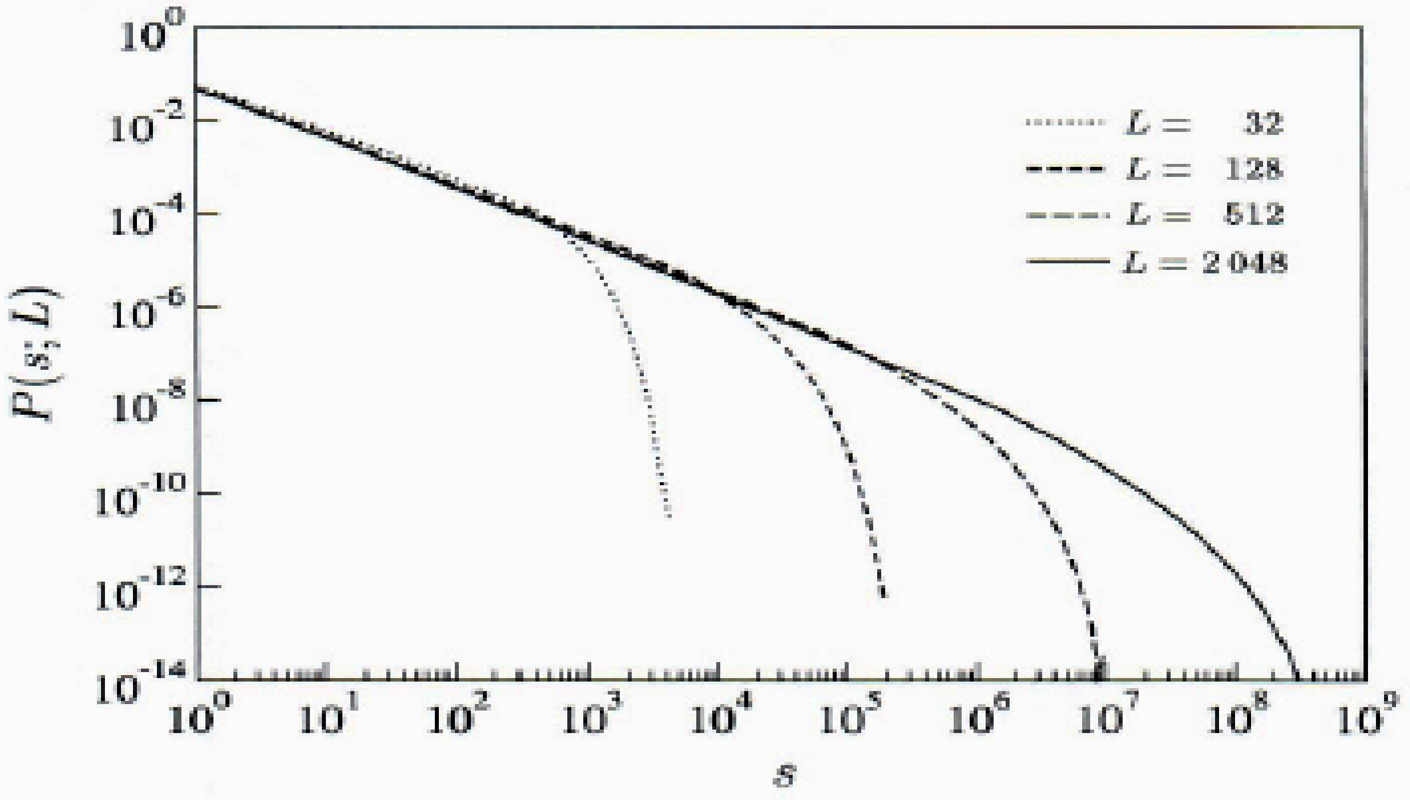


Fig. 3.13 Numerical results of the avalanche-size probability, $P(s; L)$, versus the avalanche size, s, for the two-dimensional BTW model on square lattices of size $L = 32, 128, 512, 2\,048$ marked with lines of increasing dash length. The frequency of avalanches decays with size. There is no typical size of an avalanche except for a cutoff avalanche size which increases with system size.

K Christensen and N R Moloney, ***Complexity and Criticality***, Imperial Coll. Press, 2005

Stochastic Sandpile Models

Scaling behavior appears to be simpler in the ***stochastic*** sandpile (Manna 1991)

Here $z(i,j) = 0, 1,$ or 2. Sites with $z = 2$ topple, sending two grains to ***randomly chosen*** neighbors

The stochastic sandpile again features loss of grains at the edges and addition when there are no toppling sites

This model also produces scale-invariant avalanche distributions, with somewhat different exponents than the BTW model

1. Natural systems exhibiting scale-free behavior are far from equilibrium. Critical phenomena in far from equilibrium systems? -Yes!

2. How can such critical phenomena appear without our having to tune parameters? -The parameters are hidden!

Connection with absorbing-state phase transitions:

The protocol of grain addition (in absence of activity) and loss of grains at boundaries pins the system at a critical point

(RD, M. A. Muñoz, A. Vespignani and S. Zapperi, Braz. J. Phys., 2000)

Phase Transitions

Examples: liquid-vapor, magnetic, binary mixtures...

Formal definition: singular dependence of macroscopic properties (e.g., density) on control parameters (temperature, pressure) in a system with a very large number of degrees of freedom

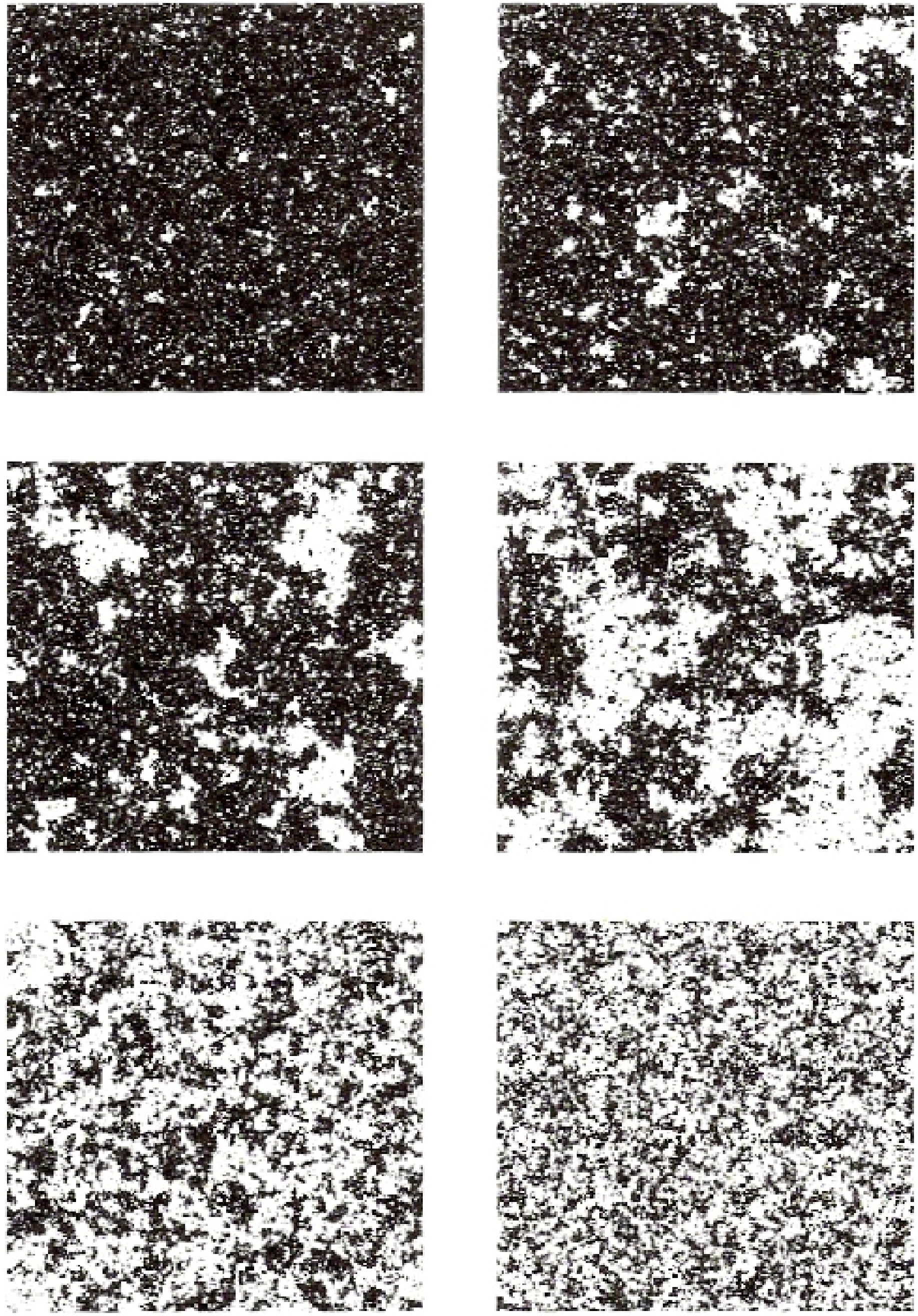
Example: magnetic systems (ferromagnetic/paramegnetic transition)
Control parameters are temperature (T) and external magnetic field (H)
Order parameter: magnetization

At the ***critical point*** (zero field, T=Tc) there are long-range correlations:

The correlation function of the local magnetization, m(r),

$$C(r) = \mathrm{cov}[m(r), m(0)]$$

decays as a power law; distribution of cluster sizes also follows a power law



Ising model: typical configurations at various temperatures

Examples of absorbing-state phase transitions:

Directed percolation* (DP) (contact process)

Parity-conserving (branching-annihilating random walks)

Conserved DP (conserved stochastic sandpile)**

*Experiment:Takeuchi et al, Phys Rev Lett **99** 234503 (2007)

**Experiment: L Corté, P M Chaikin, J P Gollub and D J Pine, Nature Phys 2008
Transition between reversible and irreversible deformation in sheared colloidal suspension

General references on absoring-state phase transitions:
J Marro and R Dickman, *Nonequilibrium Phase Transitions in Lattice Models*, (Cambridge Press, 1999).
H Hinrichsen, Adv. Phys. **49** 815 (2000).
G Odór, Rev. Mod. Phys. **76**, 663 (2004)

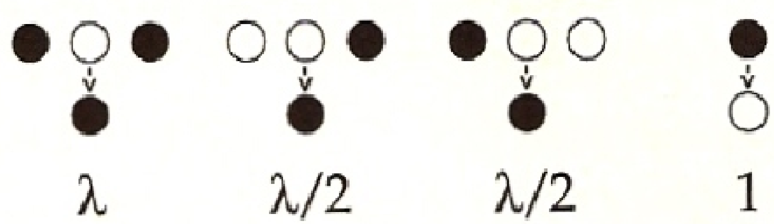
Contact Process (Harris 1972): a birth-and-death process with spatial structure

Lattice of L^d sites in d dimensions

Each site can be either active ($\sigma_i = 1$) or inactive ($\sigma_i = 0$)
An active site represents an organism

Active sites become inactive at a rate of unity, indep. of neighbors
An inactive site becomes active at a rate of λ times the fraction of active neighbors

The state with all sites inactive is absorbing



Rates for the one-dimensional CP.

Contact Process: order parameter ρ is fraction of active sites

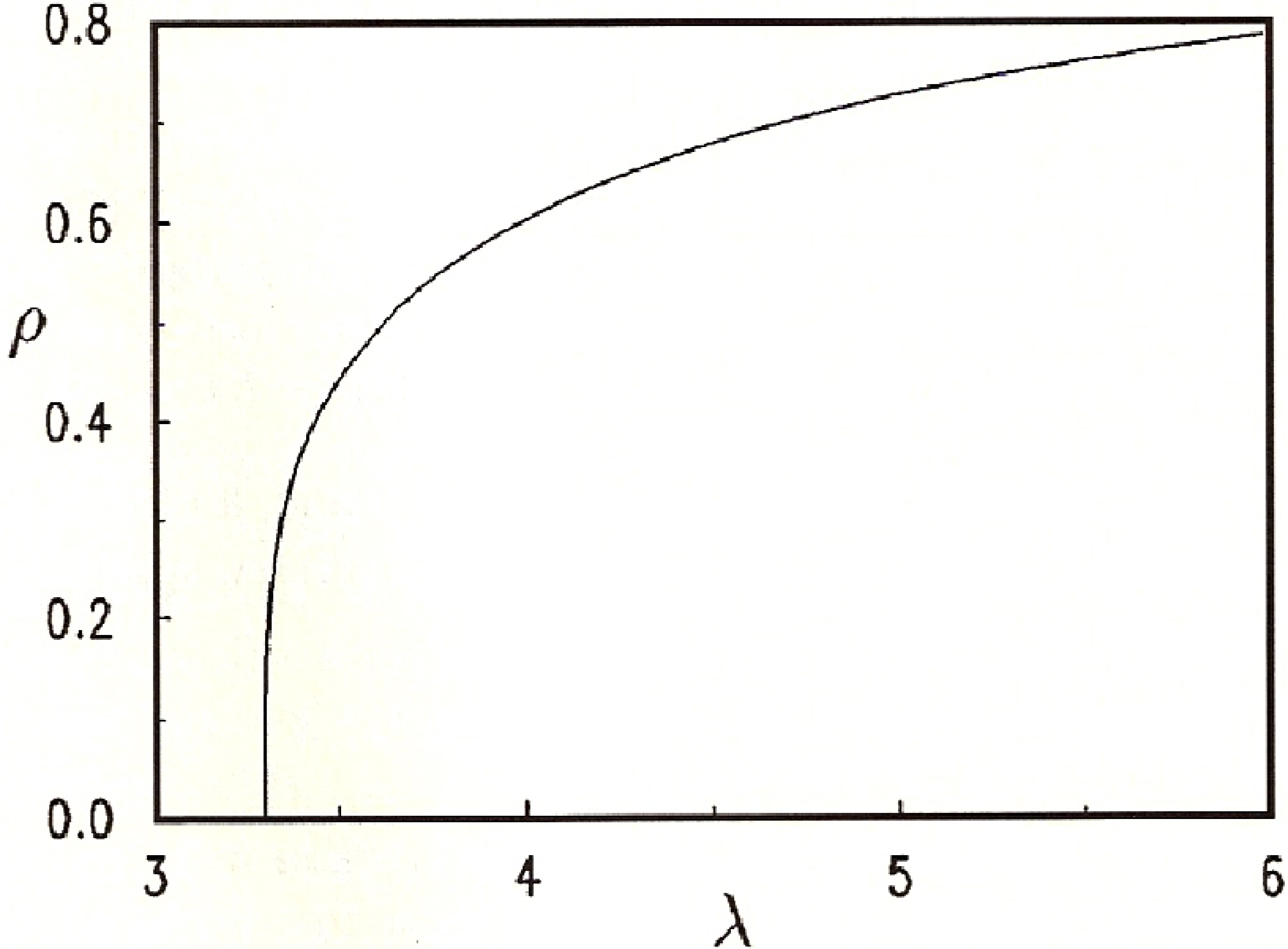
Rigorous results: continuous phase transition between active and absorbing state for d ≥ 1, at some λ_c (Harris, Grimmet...)

Order parameter: $\rho \sim (\lambda - \lambda_c)^{\beta}$

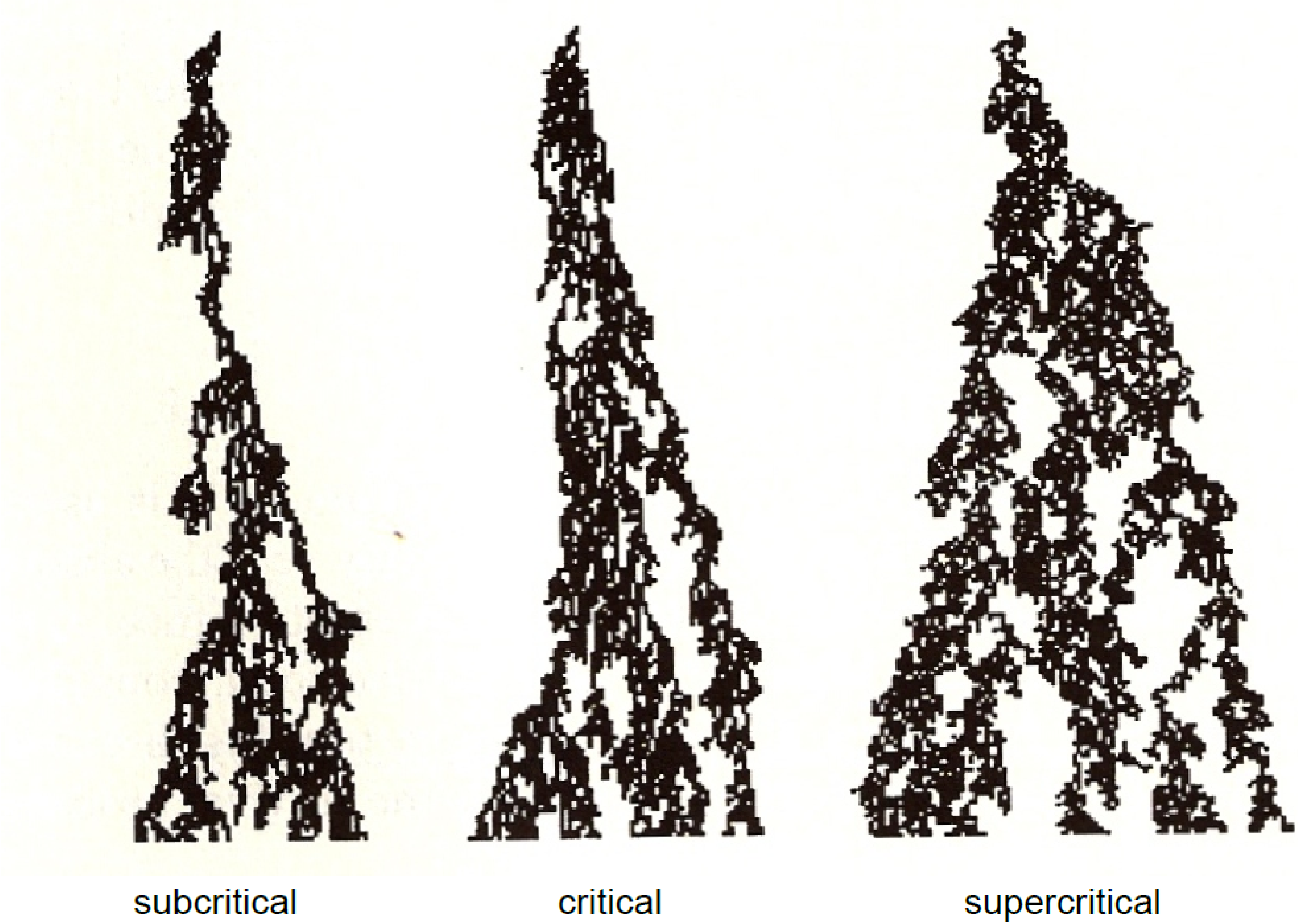
(Mean-field theory: $\lambda_c = 1$, $\beta = 1$)

Results for λ_c, critical exponents: series expansion, simulation, analysis of the master equation, ε-expansion...

Types of critical behavior:
- Static
- Dynamic
- Spread of activity: Avalanches!



Order parameter in the one-dimensional contact process: series expansion analysis



Spread of activity in contact process (avalanches)
At critical point avalanche-size distribution is power-law

The contact process is a good example of a critical point in a far from equilibrium system, but to observe power-law scaling we must adjust the creation rate to its critical value

Let's consider another simple model, ***activated random walkers***

A Markov process defined on a lattice of $\mathbf{L}^{\mathbf{d}}$ sites with ***periodic boundaries***

Particles perform random walks on the lattice
Let n_i denote the number of particles at site i (n_i = 0, 1, 2,...)

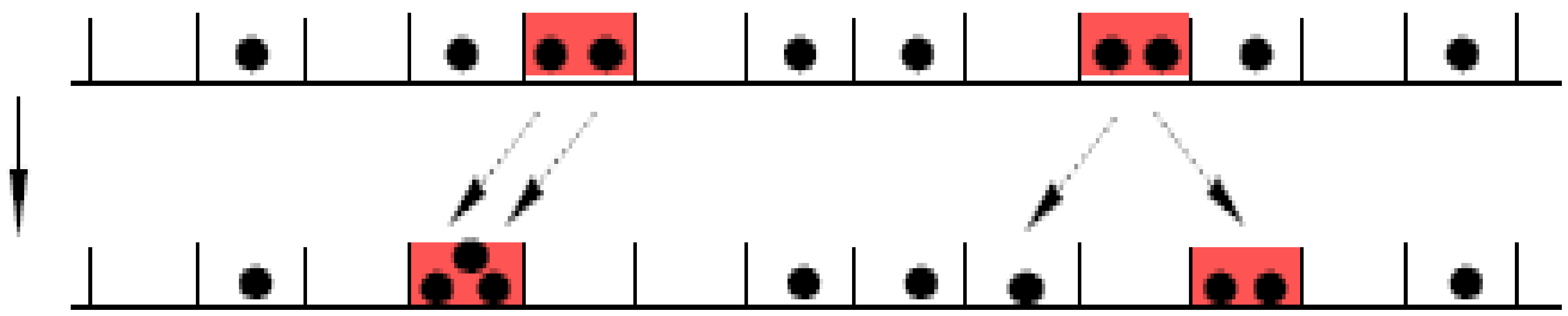
Initially **N** particles are distributed randomly over the lattice

Dynamics: any site with $n_i \geq 2$ is ***active***
Active sites ***topple*** at a rate of unity, sending two particles to randomly chosen neighbors

The number of particles remains constant throughout the evolution
$\zeta = N/L^{\mathbf{d}}$ is a control parameter

Activated random walkers (ARW): when site i topples two particles jump from i to a nearest neighbor, independently

Examples of topplings in one dimension

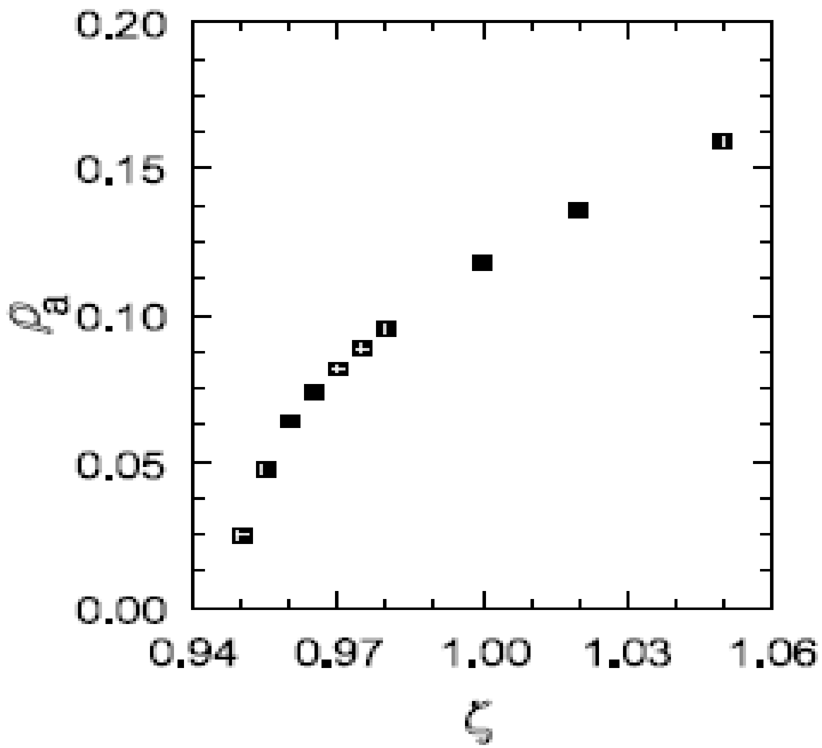


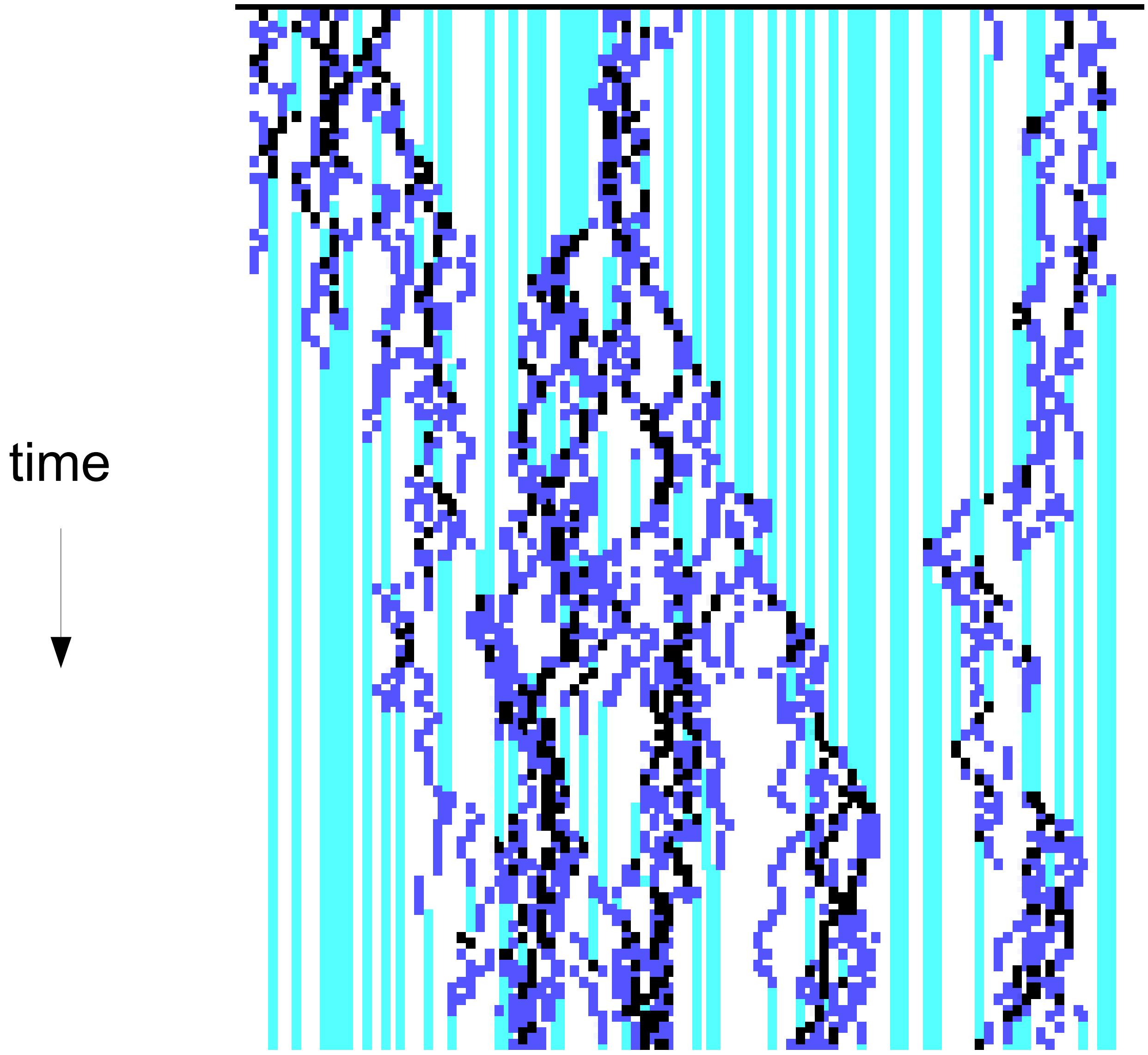
Activated random walkers: any configuration with without active sites is absorbing

Such configurations exist for $\zeta < 1$

There is an absorbing-state phase transition at $\zeta = \zeta_c$ (= 0.94885 in one dimension)

Order parameter: ρ, the fraction of active sites





Typical evolution of ARW process

As in the contact process, the activated random walkers process exhibits scale-invariance at the critical point, but to reach this point we must tune ζ to its critical value.

Now we make two simple changes in the process:

1. Replace the periodic boundary condition with open boundaries
   When a site at the edge topples, particles may be lost

2. Eventually the system reaches an absorbing configuration
   When this happens a new particle is added at a randomly chosen site

This converts the ARW process into the Manna sandpile!

These changes ***force*** the ARW process to its critical point:

If $\zeta > \zeta_c$ there is activity and ζ can only decrease

If $\zeta < \zeta_c$ activity will stop and ζ will then increase

Conclusion: ζ is the hidden parameter whose value is tuned by the sandpile dynamics!

Absorbing-state mechanism for SOC: self-organized criticality in a slowly driven system corresponds to an absorbing-state phase transition in the model with the same local dynamics, but with strict conservation

Avalanche exponents are related to critical exponents of conserved model.

Simulations confirm that the critical exponents in SOC and in the absorbing phase transition are related [Muñoz et al, Phys. Rev. E 59, 6175 (1999)]

As the system size increases, the fluctuations of ζ in the ***driven*** sandpile are restricted to an ever smaller region centered on the critical density of the ***conserved*** model

The SOC and absorbing "ensembles" are however distinct
(Pruessner and Peters, Phys. Rev. E, 2006, arXiv:0912.2305)

In ***deterministic*** sandpiles, the critical density in the conserved version is a ***tiny*** bit higher than in the SOC version. (Fey et al., Phys Rev Lett, 2010)

This was subsequently explained as being due to the choice of initial condition. (Poghosyan et al., Phys. Rev. E 84, 066119)

In sandpile and related models, an infinite timescale separation between activity (toppling) and driving is realized by prohibiting addition while activity is in progress

In natural systems, we can't expect the driving mechanism to “wait” for all activity to cease before perturbing the system

If the driving rate rate **h** is very small, scale-free distributions can be generated over a finite range: the avalanche duration distribution is cut off at a time **~1/h**

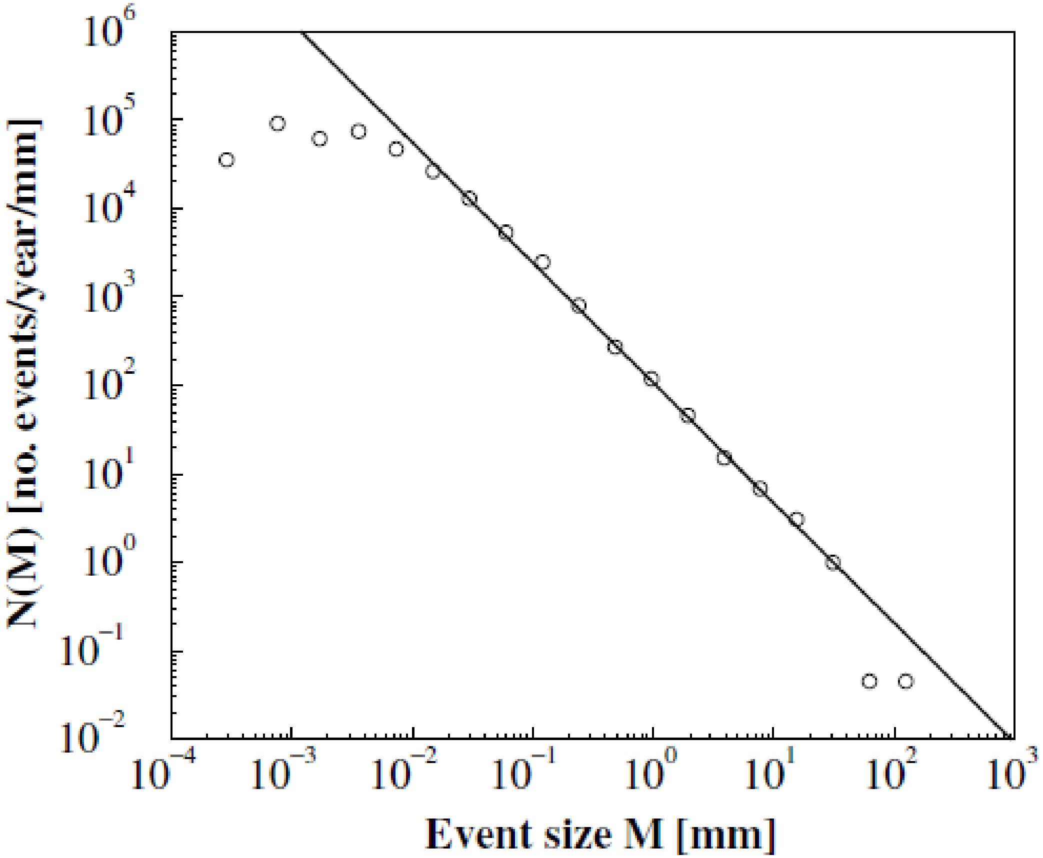
This should be fine from an empirical viewpoint!

Alternatives to SOC

In some instances, the validity of power-law distributions have been questioned; in others, alternative explanations have been proposed

Example: Scale-invariant rain and drought distributions
Rain event intensity: integrated precipitation over a rainy period
The intensity distribution follows a power law

Peters, Hertlein, and Christensen, Phys. Rev. Lett., 2002



Peters et al. suggest the observed power laws are evidence of SOC in the dynamics of evaporation and condensation in Earth's atmosphere

Activity: Rapid condensation above a threshold value of humidity

Slow drive: Energy influx from Sun, causing evaporation

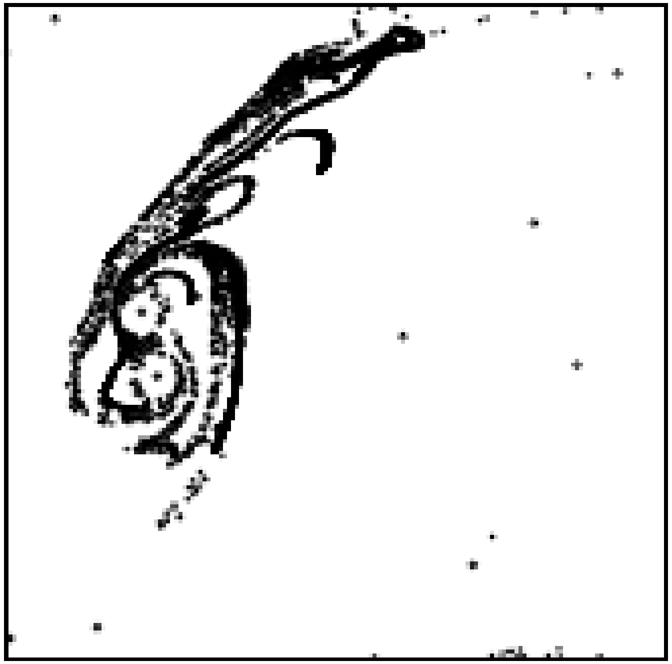
Loss: Rain falling to Earth

What is the coupling mechanism?

Alternative model (RD Phys Rev Lett, 2003)

If condensation occurs in localized regions, chaotic advection can generate power-law distributions of rain intensities at fixed observation sites

Simple model: two-dimensional fluid with rain treated as passive tracers in a velocity field generated by a system of ideal vortices



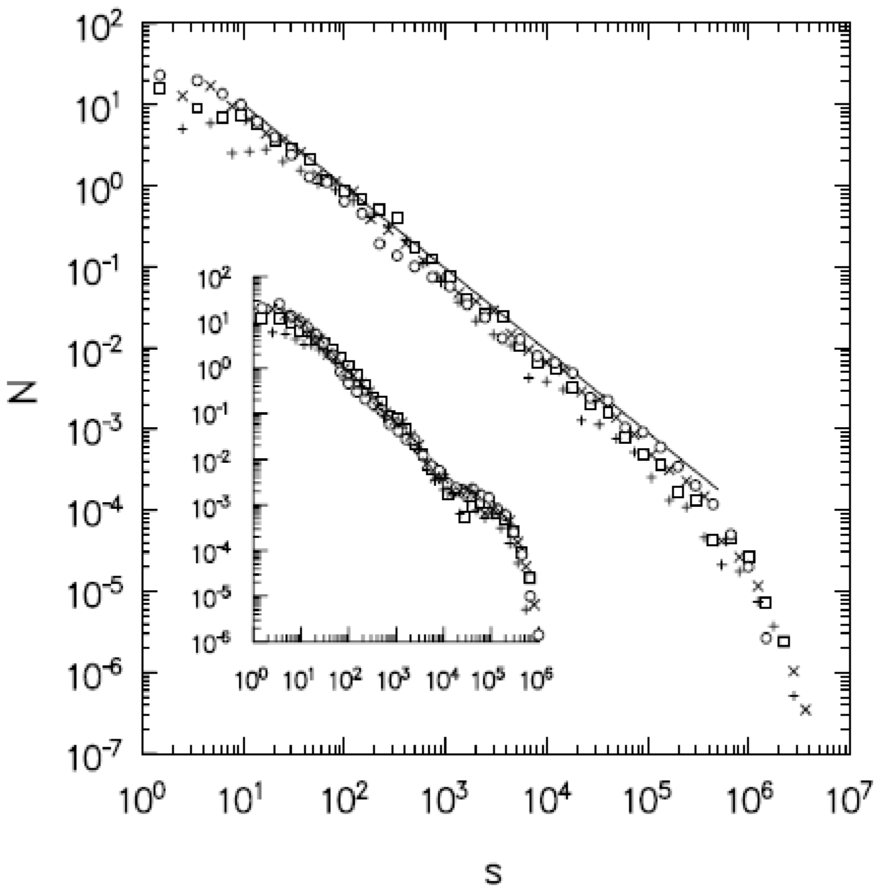


Figure 4. Rain-size (main graph) and drought-duration (inset) distributions in systems of vortices of equal strength, $T \simeq 0.85\tau_c$. $\circ$: $N_V = 10$; $\times$: $N_V = 20$; $\Box$: $N_V = 50$; $+$: $N_V = 100$. The vortex intensity K is scaled $\sim 1/\sqrt{N_V}$ in these studies. The straight lines have slopes of -1.01 (rain size) and -1.13 (drought).

This simple two-dimensional model yields power-law distributions but does not reproduce the observed exponents (1.36 and 1.42 for rain intensity and drought duration, resp.)
It does raise the possibility that the observed power laws are due to chaotic advection

SUMMARY

Power-law distributions are observed in many natural and social systems

SOC provides a mechanism for generating scale-invariant behavior without parameter tuning

The essential ingredients are: (1) a system of many coupled nonlinear elements having a threshold for activity; (2) a slow loss mechanism; (3) an even slower external drive

SOC works by forcing the system to an absorbing-state critical point

In some instances, alternatives to SOC have been proposed