Revisiting logarithmic corrections to black hole entropy

Ashoke Sen

ICTS, Bengaluru, India

A.H. Anupam, P.V. Athira, Chandramouli Chowdhury, A.S., arXiv:2306.07322
A.H. Anupam, Chandramouli Chowdhury, A.S., arXiv:2308.00038

Copenhagen, August 2023

Bekenstein-Hawking formula for black hole entropy is universal

$$S_0 = \frac{A}{4} \qquad \text{in } \hbar = c = G_N = k_B = 1 \text{ unit}$$

A: area of the event horizon

In a generic theory of gravity coupled to multiple U(1) gauge fields and other fields in D dimensions, the black hole can carry

– U(1) charges Q_k

– angular momentum J_i in Cartan subalgebra of SO(D-1)

– mass M

Then

$$S_0 = f_0(Q, M, J)$$

Q, J have multiple components in general

In classical two derivative theories in D dimensions

$$f_0(\lambda^{D-3}Q, \lambda^{D-3}M, \lambda^{D-2}J) = \lambda^{D-2} f_0(Q, M, J)$$

To take macroscopic limit, we take

$$M \sim \lambda^{D-3}, \qquad Q \sim \lambda^{D-3}, \qquad J \sim \lambda^{D-2}$$

and take λ large

Then

$$S_0 \sim \lambda^{D-2}$$

The fields associated with the black hole also has simple dependence on λ, e.g.

$$g_{\mu\nu} \sim \lambda^2$$

Note: In D=4 we can also have magnetic charges scaling as λ, but they are topological and do not fluctuate

The Bekenstein-Hawking formula is expected to receive corrections due to stringy effects and quantum effects

General structure:

$$\mathsf{S} = \lambda^{\mathsf{D}-2}\, \mathsf{f_0} + \text{power suppressed terms} + \mathsf{C} \ln \lambda + \cdots$$

Focus of attention in today's lecture will be the terms $\propto \ln \lambda$

These are determined purely from IR physics

– spectrum of massless fields and their interactions

Nevertheless any UV complete theory that is able to count black hole microstates must reproduce these results in the large λ limit

Logarithmic corrections from gravitational path integral

Euclidean continuation of a black hole leads to a conical singularity at the horizon, unless

1. The euclidean time τ and the azimuthal angles ϕ are periodically identified as

$$(\tau, \phi) \equiv (\tau + \beta, \phi - \mathbf{i}\omega\beta)$$

2. The time components of the gauge fields take specific asymptotic values

$$\mathbf{A}_\tau = -\mathbf{i}\,\mu$$

β, ω, μ are fixed in terms of M,Q,J for classical black hole

Interpretation:

Gibbons, Hawking

$\beta = \frac{\partial \mathbf{S_0}}{\partial \mathbf{M}}$ **=inverse temperature,** $\omega = \frac{1}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{J}}$ **=angular velocity**

$\mu = \frac{1}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{Q}}$ **=chemical potential**

Scaling from $\mathbf{S_0} \sim \lambda^{\mathbf{D-2}}, \quad \mathbf{M}, \mathbf{Q} \sim \lambda^{\mathbf{D-3}}, \quad \mathbf{J} \sim \lambda^{\mathbf{D-3}}$

$$\beta \sim \frac{\partial \mathbf{S_0}}{\partial \mathbf{M}} \sim \lambda, \quad \mu \sim \frac{\mathbf{1}}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{Q}} \sim \mathbf{1}, \quad \omega \sim \frac{\mathbf{1}}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{J}} \sim \lambda^{-\mathbf{1}}$$

β, ω, μ are dominant modes at infinity

e.g. $\mathbf{A}_\tau \sim \mathbf{i}(\mu + \mathbf{Q}\,\mathbf{r}^{\mathbf{3-D}})$

In quantum theory we treat β, ω, μ as independent variables, providing boundary condition to the path integral

The path integral over all fields with these boundary conditions gives the grand canonical partition function:

$$\mathbf{Z}(\beta, \mu, \mathbf{J}) = \mathbf{Tr}\left[\mathbf{e}^{-\beta \mathbf{E} - \beta\mu.\mathbf{Q} - \beta\omega.\mathbf{J}}\right]$$

Origin of logarithmic corrections:

1. In the path integral, one loop contribution of massless fields generate $\ln\lambda$ corrections to $\ln Z$

Fursaev, Solodukhin, $\cdots$, Review: arXiv:1104.3712 by Solodukhin; A.S. arXiv:1205.0971

2. We need to construct the entropy from the grand canonical partition function by taking appropriate Laplace transform:

$$e^S = \int d\beta\, d\mu\, d\omega\, e^{\beta E + \beta\mu . Q + \beta\omega . J}\, Z(\beta, \mu, J)$$

– can also generate logarithmic corrections to the entropy

A.S. arXiv:1205.0971

Example: Result for Kerr black hole in pure gravity in D=4

$$\left(\frac{212}{45} - 1\right) \ln \lambda$$

Any quantum theory of gravity that can count black hole microstates should reproduce this result.

At present such a counting is not possible in string theory.

This can be remedied using supersymmetric black holes.

Supersymmetric (BPS) black holes have zero temperature

$\Rightarrow$ instead of having a single large length scale, we have two different large scales

M, Q $\sim \lambda^{D-3}$ and $\beta \equiv \frac{\partial S}{\partial M} \to \infty$

– difficult to extract log correction

Remedy: Work in the near horizon geometry:

$AdS_2 \times (\text{squashed})\, S^{D-2}$ Mann, Solodukhin hep-th/9604118; $\cdots$

$$ds^2 = v_1 \left(\frac{dr^2}{r^2-1} + (r^2-1)d\tau^2\right) + v_2\, ds^2_{D-2}$$

$v_1, v_2 \sim \lambda^2$

We can compute logarithmic correction to the partition function in this geometry following the same guidelines

Some differences:

1. The partition function computes the path integral at fixed mass, charge and angular momentum since these modes dominate as r $\to \infty$

$\Rightarrow$ the path integral directly computes the entropy in the microcanonical ensemble and no change of ensemble is needed.

2. We integrate over modes living in the near horizon geometry

– different set of eigenvalues and eigenfunctions than those in the full geometry

List of BPS black holes for which the logarithmic correction has been predicted / tested

The theory	scaling of charges	logarithmic contribution	microscopic
Type II on T^6 $(D=4, \mathcal{N}=8\,\text{SUSY})$	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	$-8 \ln \lambda$	$\surd$
$\mathcal{N}=6$ supersymmetric theories in $D=4$	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	$-4 \ln \lambda$	?
$\mathcal{N}=5$ supersymmetric theories in $D=4$	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	$-2 \ln \lambda$	?
$\mathcal{N}=4$ supersymmetric CHL models in D=4 and type II on $K3 \times T^2$ with n_v matter multiplet	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	0	$\surd$
$\mathcal{N}=3$ supersymmetric theories in $D=4$ with n_v matter multiplets	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	$2 \ln \lambda$	?
$\mathcal{N}=2$ supersymmetric theories in $D=4$ with n_V vector and n_H hyper multiplets	$Q_i \sim \lambda, \quad S_0 \sim \lambda^2$	$\frac{1}{6}(23+n_H-n_V) \ln \lambda$	?
BMPV in type IIB on $T^5/\mathbb{Z}_N$ or $K3 \times S^1/\mathbb{Z}_N$ with n_V vectors (D=5, $\mathcal{N}$ =4 or 2 SUSY)	$Q_1, Q_5, n \sim \lambda^2$, $J \sim \lambda^3, \quad S_0 \sim \lambda^3$	$-\frac{1}{4}(n_V-3) \ln \lambda$	$\surd$

Banerjee, Gupta, A.S.; Banerjee, Gupta, Mandal, A.S.; A.S.; Ferrara, Marrani; Charles, Larsen

Success of this procedure for BPS black holes suggests that similar agreement must hold also for non-BPS black holes

– any UV complete theory that is able to count black hole microstates must reproduce the IR results

However it is somewhat unsatisfactory that the methods used for non-BPS and BPS calculations differ in details

– modes living in the full geometry vs near horizon geometry

– use of grand canonical ensemble vs microcanonical ensemble

The goal of this talk will be to rectify this

Recent developments

Iliesiu, Kologlu and Turiaci described a procedure for computing supersymmetric index using full black hole geometry at finite temperature

Iliesiu, Kologlu, Turiaci arXiv:2107.09062

Earlier work on AdS by Cabo-Bizet, Cassani, Martelli, Murthy, arXiv:1810.11442

Our goal will be to use this formalism to compute logarithmic correction to the BPS black hole entropy

J_0: A Cartan generator of SO(D-1), k: momentum

J' will represent Cartan generators other than J_0

Supersymmetric index:

$$I = Tr_{Q,J',k=0}\left[e^{-\beta H}(-1)^F(2J_0)^n\right], \qquad (-1)^F \equiv e^{2\pi i J_0}$$

The trace is taken over states at fixed Q, J' and k=0

The trace gets contribution from only those states that break 2n (or less) J'-invariant supersymmetries

A generic non-BPS state will break all ($>$ 2n) supersymmetries and will not contribute to this index for sufficiently small n

This index is expected to pick up the degeneracy of the supersymmetric states with fixed $Q, J', k=0$

Bachas, Kiritsis hep-th/9611205; Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline hep-th/9708062
Dabholkar, Gomes, Murthy, A.S.

$$I = Tr_{Q,J',k=0}\left[e^{-\beta H}(-1)^F(2J_0)^n\right] \equiv e^{S_{BPS}-\beta\, M_{BPS}}$$

Examples:

1. In D=4 the rotation group is SU(2)

J_0 is the third generator of the rotation group, J' trivial

The corresponding $e^{S_{BPS}}$ has been counted in $N = 4, 8$ supersymmetric theories

2. In D=5 the rotation group is $SO(4) = SU(2)_L \times SU(2)_R$

We can take $J_0 = J_{3R}$, $J' = J_{3L}$

Index is a function of J_{3L} and electric charges

$e^{S_{BPS}}$ has been counted in N=2, 4 supersymmetric compactifications (BMPV black holes)

Macroscopic analysis: Index from gravitational path integral

The path integral in the full space-time geometry computes:

$$\mathbf{Z} = \mathbf{Tr}\left[\mathbf{e}^{-\beta \mathbf{E}-\beta\mu.\mathbf{Q}-\beta\omega.\mathbf{J}}\right]$$

Set $\beta\omega_0 = 2\pi i$ and insert $(2J_0)^n$ in the path integral

$$\mathbf{Z} = \mathbf{Tr}\left[\mathbf{e}^{-\beta \mathbf{H}-\mu\mathbf{Q}-\beta\omega'.\mathbf{J}'-2\pi i \mathbf{J}_0}(2\mathbf{J}_0)^n\right] = \mathbf{Tr}\left[\mathbf{e}^{-\beta \mathbf{H}-\mu\mathbf{Q}-\beta\omega'.\mathbf{J}'}(-1)^F(2\mathbf{J}_0)^n\right]$$

Compare this with the index

$$\mathbf{I} = \mathbf{e}^{S_{BPS}-\beta M_{BPS}} = \mathbf{Tr}_{Q,J',k=0}\left[\mathbf{e}^{-\beta\mathbf{H}}(-1)^F(2\mathbf{J}_0)^n\right]$$

In the index I we take the trace for fixed Q, k=0, J' while in Z we sum / integrate over Q, k, J' keeping μ, ω' fixed

Z can be regarded as a sum / integral over Q, k, J' with I as integrand

$$Z = \int d^{n_V}Q\, d^{n'_c}J'\, d^{n_T}k\, e^{\left[S_{BPS} - \beta M_{BPS} - \beta k^2/2M - \beta\omega'.J' - \beta\mu.Q\right]}$$

n'_c: number of generators J' = rank of SO(D-1) -1

n_V: number of U(1) gauge fields, i.e. dimension of Q

k: momenta invariant under $\omega'.J'$

$\Rightarrow$ an n_T dimensional space of momenta to integrate over

The integrand is sharply peaked around k=0 and Q, J' determined from $\partial S_{BPS}/\partial Q = \beta\mu,\ \partial S_{BPS}/\partial J' = \beta\omega'$

Gaussian integral around the saddle point produces λ dependent result

e.g. k integration gives $\sim (M/\beta)^{n_T/2} \sim e^{\frac{n_T}{2}(D-4)\ln\lambda}$

Q, J' integrals give $\left(\det\frac{\partial^2 S_{BPS}}{\partial Q^2}\right)^{-1/2}\left(\det\frac{\partial^2 S_{BPS}}{\partial J'^2}\right)^{-1/2} \sim \lambda^{\frac{n_V(D-4)+n'_c(D-2)}{2}}$

Net result:

$$S_{BPS} = \ln Z + \beta M_{BPS} + \beta\omega'.J' + \beta\mu.Q + C_E \ln\lambda$$

with J', Q evaluated at the saddle, and

$$C_E = -\frac{1}{2}\left[(n_v + n_T)(D-4) + n_c'(D-2)\right]$$

n_c': number of generators J'

n_v: number of U(1) gauge fields, i.e. dimension of Q

We now need to evaluate the logarithmic correction to ln Z by evaluating the gravitational path integral

Power counting $\Rightarrow$ such contributions come from one loop contribution of massless fields

K_b: Kinetic operator for massless bosonic fields

K_f: Kinetic operator for massless fermionic fields

One loop contribution to Z from massless fields:

$$(\det K_b)^{-1/2}(\det K_f)^{1/2}$$

Correction to ln Z:

$$\delta \ln Z = -\frac{1}{2}\ln\det K_b + \frac{1}{2}\ln\det K_f = -\frac{1}{2}\mathrm{Tr}\ln K_b + \frac{1}{2}\mathrm{Tr}\ln K_f$$

λ dependence arises from $K_b \sim \lambda^{-2}, \qquad K_f \sim \lambda^{-1}$

– can be evaluated using the heat kernel expansion

Result

$$\delta \ln \mathbf{Z} = \mathbf{C_L} \ln \lambda + \cdots$$

$$\mathbf{C_L} = \int \mathbf{d^4x\, K(x)}$$

K(x) can be computed from the knowledge of K_b and K_f

Seeley; DeWitt; $\cdots$ Vassilevich, hep-th/0306138

In N $\geq$ 2 supergravity in D=4, K(x) is proportional to the Gauss-Bonnet term Charles, Larsen arXiv:1505.01156; Karan, Panda arXiv:2012.12227

$\Rightarrow$ $\mathbf{C_L} \propto$ **Euler number**

For D odd, $\mathbf{C_L} = 0$

Zero mode contribution:

K_b and / or K_f may have zero eigenvalues arising from broken symmetries like translation, rotation, supersymmetry

– cannot be treated as part of the determinant

1. Remove their contribution from $\delta \ln Z$

e.g. a bosonic non-zero mode contributes $(1/\lambda^2)^{-1/2} \sim \lambda$ to Z

$\Rightarrow$ $\ln \lambda$ to $\ln Z$

We need to subtract ($\ln \lambda$) from $\delta \ln Z$ for each bosonic zero mode

Similarly we add ($\ln \lambda$)/2 to $\delta \ln Z$ for each fermionic zero mode

Example: Counting of the number of rotational zero modes n_R

– must be generated by rotation <u>outside</u> the Cartan subalgebra of the group so that it deforms the solution

– must be invariant under $e^{\beta\omega'.J'}$ so that it satisfies the required periodicity as we go around the euclidean time circle.

In D=4, J' is trivial and rotations about 1 and 2 axes generate zero modes

In D=5, $J' = J_{3L}$ and rotations about 1 and 2 axes of $SU(2)_R$ generate zero modes

$\Rightarrow$ in both D=4 and D=5, $n_R = 2$

$\Rightarrow$ we need to subtract $2 \ln \lambda$ from lnZ

Similar analysis can be done for counting the translational zero modes and broken supersymmetry zero modes.

2. We need to find the actual λ dependent contribution to Z from the zero mode integrals

Zero modes typically arise from some broken symmetries

We express the integral over the zero modes as integral over the broken symmetry parameters and carry out the integral

e.g. in D dimensions the integration measure over the metric fluctuations $h_{\mu\nu}$ is $Dh_{\mu\nu} \equiv \lambda^{(D-4)/2} dh_{\mu\nu}$

– ensures $\int Dh_{\mu\nu} \exp[-\int d^D x \sqrt{\det g}\, g^{\mu\rho} g^{\nu\sigma} h_{\mu\nu} h_{\rho\sigma}] = 1$

Rotational zero modes are of the form $h_{\mu\nu} = c(D_\mu f_\nu + D_\nu f_\mu)$

c: rotation parameter, f^μ: λ independent functions, $f_\mu \sim \lambda^2$

$\Rightarrow Dh_{\mu\nu} \sim \lambda^{2+(D-4)/2} dc \quad \Rightarrow \lambda^{D/2}$ in Z for each zero mode

Similar procedure can be used for other zero modes.

Fermion zero modes are associated with broken supersymmetry

– saturated by the zero mode part of $(2J_0)^n$

Net logarithmic correction from zero modes

$$C_Z \ln\lambda$$

$$C_Z \equiv \frac{1}{2}\, n_T\,(D-4) + \frac{1}{2}\, n_R\,(D-2)$$

Net logarithmic correction to S_{BPS}:

$$(C_E + C_L + C_Z)\ln\lambda = \left[\frac{1}{2} n_R\,(D-2) - \frac{1}{2} n_c'\,(D-2) - \frac{1}{2} n_V (D-4) + C_L\right] \ln\lambda$$

$$C_L = \int_{\text{Full geometry}} K(x)$$

$$C_E = -\frac{1}{2}\left[(n_v + n_T)(D-4) + n_c'(D-2)\right]$$

$n_c' = 0, 1$ in D=4,5, $C_L = 0$ in D=5, $n_R = 2$ in D=4,5.

This reproduces all the <u>macroscopic</u> results described in the earlier table

Logarithmic correction to the index, computed from the near horizon geometry and the full geometry gives the same result$\cdots$

$\cdots$ even though the intermediate steps are quite different

$\Rightarrow$ the index computed from the full geometry correctly reproduces the microscopic results when they are known

e.g. in theories with 16, 32 supersymmetries in D=4, 5

Conclusion

Although this analysis has only reproduced known results, the agreement is significant due to several reasons:

1. The computation using the full geometry uses integration over the same set of modes and same ensemble as that for non-supersymmetric black holes

– gives us confidence in the results for non-supersymmetric black holes for which there is no independent test of the formula

2. In principle, the computation using the full geometry can be used to take into account all configurations that contribute to the index

e.g. multi-centered black holes

3. This formalism may be better suited for exact computation of supersymmetric index from gravitational path integral, e.g. via localization