Classical observables of General Relativity from scattering amplitudes

Paolo Di Vecchia

Niels Bohr Institute, Copenhagen and Nordita, Stockholm

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Foreword

This talk is based on many papers together with

C. Heissenberg, R. Russo and G. Veneziano

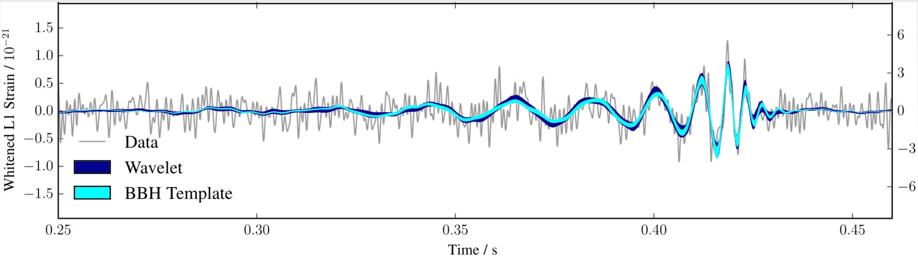
See the review submitted to Phys. Rep. (2306.16488)

and, for the work on spin, on the paper with F. Alessio, 2203.13272

Plan of the talk

Waveform Templates

LIGO Scientific Collaboration '16



Inspiral	Merger	Ringdown
Weak gravity	Strong gravity	Small oscillations

- The discovery of gravitational waves at LIGO, generated by black hole merging, poses the problem of computing very precisely the dynamics of binary black hole merging.
- and extract from the theory the waveform of the gravitational waves to be compared with what is observed at LIGO/VIRGO.
- In the past this has mostly been done by solving Einstein's equations in the presence of the two black holes.
- Mostly using the Post-Newtonian (PN) expansion.
- It is an expansion for small G_N and small velocity v

$$\frac{2G_N m}{rc^2} \sim \frac{v^2}{c^2} << 1$$

- Recently a complementary approach has been used thanks also to [Damour, 1710.10599].
- Extract classical quantities from the quantum scattering amplitude using the Post-Minkowskian (PM) expansion.
- When the two black holes are far away from each other, one can use perturbation theory expanding in powers of the Newton's constant G_N: Post-Minkowskian (PM) expansion.

- When they get closer to each other, their interaction becomes very strong and one must use Numerical Relativity (NR) [Petrorius, 2306.03797].
- Another very useful approach is the so-called Effective One Body (EOB) formalism introduced by Buonanno and Damour (1999).
- One constructs an effective Hamiltonian for a particle with mass $\mu = \frac{m_1 m_2}{m_1+m_2}$ in an external metrics that is fixed by requiring that the effective dynamics be the same as the original dynamics.
- It leads to results faster than NR. See the recent papers by [Damour and Rettegno, 2211.01399] and [Rettegno, Pratten, Thomas, Schmidt and Damour, 2307.06999]
- EOB waveforms are an important class of inspiral-merger-ringdown waveforms models employed by the LIGO/VIRGO searches.
- Self-force approach: start from $m_1 >> m_2$ and a metric generated by the large mass and then compute corrections to this metric produced by the small mass.

- In this seminar we will be describing the two black holes with two spinless particles with mass m_1 and m_2 and we will consider their scattering rather than their merging.
- There are techniques that allow to go from the scattering to the merging.
- We will consider both the elastic scattering and the inelastic scattering with the production of extra gravitons.
- In the case of the elastic scattering we will show that a physical observable as the classical deflection angle can be extracted from a classical quantity, called the eikonal, that can be computed from the elastic scattering amplitude.
- In the inelastic case the eikonal becomes an operator containing the graviton creation and annihilation operators.
- We will start discussing the simpler case in which the emitted graviton is soft.
- Then we will generalise it to the case where the graviton has arbitrary frequency.
- We will compute in both cases inelastic observables as the linear and angular momentum of both particles and field.

The leading and sub-leading eikonal

- We start from the tree-level scattering amplitude with one-graviton exchange:

$$\mathcal{A}_0(\sigma, q^2) = \frac{8\pi G_N}{q^2}\left[4m_1^2 m_2^2\left(\sigma^2 - \frac{1}{D-2}\right)\right] + \ldots \Longrightarrow \frac{8\pi G_N s^2}{q^2}$$

where $\ldots$ stand for powers of q, $\sigma = -\frac{p_1 p_2}{m_1 m_2} = \frac{s-m_1^2-m_2^2}{2m_1 m_2}$ and $t = -q^2$. In the last step we took the high-energy limit.
- The process above involves the exchange of a single quantum.
- We can go to impact parameter space

$$2\delta_0(\sigma, b) = \widetilde{\mathcal{A}}_0(\sigma, b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}_0(\sigma, q^2)}{4Ep} e^{ibq}$$

getting the leading eikonal

$$2\delta_0 = \frac{2G\, m_1 m_2 \left(\sigma^2 - \frac{1}{D-2}\right) \Gamma\left(\frac{D-4}{2}\right)}{\hbar\sqrt{\sigma^2 - 1}(\pi b^2)^{\frac{D-4}{2}}}$$

- In the classical limit, it is natural to take b, σ and the length scale $R^{D-3} \sim G_N\sqrt{m_1 m_2}$ (in analogy with the Schwarzschild radius) as classical quantities characterising the collision.
- In terms of these classical quantities the eikonal becomes:

$$2\delta_0 = \frac{2\left(\sigma^2 - \frac{1}{D-2}\right)\Gamma\left(\frac{D-4}{2}\right)}{\sqrt{\sigma^2-1}\pi^{\frac{D-4}{2}}}\left(\frac{R}{b}\right)^{D-3}\frac{b\sqrt{m_1 m_2}}{\hbar}$$

- The regime we are describing is the one in which

$$\frac{\hbar}{\sqrt{s}} << R << b$$

  corresponding to classical regime on the left and perturbative regime on the right.
- $2\delta_0$ is a big quantity and the factor $1/\hbar$ signals that this quantity should appear in an exponential $\mathrm{e}^{2i\delta_0}$, so it can describe the value of the classical action.

- By summing ladder diagrams with many exchanged gravitons it has been shown that the previous quantity exponentiates Kabat and Ortiz, hep-th/9203082

- Conversely, the hypothesis that the eikonal exponentiates fixes the leading high energy behaviour of the multiloop diagrams.
- After the eikonal resummation the leading contribution to the S-matrix is captured by the phase $e^{2i\delta_0}$, which effectively resums infinitely many exchanges.
- This can be seen by rewriting the resumed amplitude in momentum space:

$$S^{(M)}(\sigma, Q) \simeq \int d^{D-2}b\, e^{-i\frac{bQ}{\hbar}}\, e^{2i\delta_0(\sigma,b)}$$

- The Fourier transform above is dominated by the saddle point

$$Q_s^\mu = \hbar \frac{\partial(2\delta_0)}{\partial b^\mu}; \quad N_s \simeq \frac{|Q_s|}{|q|} \simeq \frac{4Gm_1m_2\left(\sigma^2 - \frac{\zeta}{D-2}\right)\Gamma\left(\frac{D-2}{2}\right)}{\hbar\sqrt{\sigma^2-1}\,\pi^{\frac{D-4}{2}}b^{D-4}}$$

  Q_s represents the momentum exchanged in the classical deflection. It is also called the impulse.
- N_s is the number of soft particles exchanged during the scattering obtained dividing Q_s by the typical momentum of each soft particle $q \sim \frac{\hbar}{b}$.
- N_s is large and becomes infinite in the strict classical limit.
- Then, from the relation $Q_s^2 = (p_1 - p_4)^2 = (\vec{p}_1 - \vec{p}_4)^2 = 2p^2(1 - \cos\Theta_s)$ one gets the deflection angle

$$\sin\frac{\Theta_s}{2} = \frac{|Q_s|}{2p} = -\frac{\hbar}{2p}\frac{\partial(2\delta_0)}{\partial b} = \frac{2G_N(\sigma^2 - \frac{1}{2})E}{(\sigma^2-1)b}$$

  p is the momentum of each particle and E is the total energy, both in the center of mass frame.

- In conclusion at 1PM we get

$$p\Theta_s \simeq Q_{1PM} = \frac{2G_N m_1 m_2(2\sigma^2 - 1)}{b\sqrt{\sigma^2 - 1}}$$

- Straightforward to formally generalise this discussion beyond the case of the 1PM elastic eikonal.
- One just needs to use the full eikonal and write the long-range elastic S-matrix as follows

$$S^{(M)}(\sigma, Q) = \int d^{D-2}b\, e^{-i\frac{bQ}{\hbar}}\, (1 + 2i\Delta(\sigma, b))\, e^{2i\delta(\sigma,b)}$$

where Δ represents quantum corrections that must be subtracted from the full amplitude to isolate the classical eikonal $\delta(b, \sigma)$.
- Again the classical deflection angle Θ_s is derived from the momentum $|Q_s|$ by a saddle point now related to δ instead of δ_0

$$Q_s^{\mu} = \hbar\frac{\partial(2\,\mathrm{Re}\,\delta)}{\partial b_{\mu}}\,, \qquad \sin\frac{\Theta_s}{2} = \frac{|Q_s|}{2p}$$

- The previous exponentiation is certainly correct up to two loops but there is no proof that it continues to be valid for higher loops.

- Problems with exponentiation at 3-loops in $\mathcal{N}=8$ supergravity [Naculich, Russo, Veneziano, White, DV, 1911.11716]
- The sub-leading eikonal is extracted from one-loop diagrams.
- Expanding the exponentiated expression one gets at order G_N^2 (2PM):

$$i\tilde{A}_1 = \frac{1}{2}(2i\delta_0)^2 + 2i\delta_1 + 2i\Delta_1$$

- For $D=4$ the sub-leading eikonal turns out to be:

$$2\delta_1 = \frac{3\pi G_N^2 m_1 m_2 (m_1+m_2)(5\sigma^2-1)}{4b\hbar\sqrt{\sigma^2-1}}$$

- One gets the following deflection angle:

$$\Theta_s = \frac{4G_N(\sigma^2-\frac{1}{2})E}{(\sigma^2-1)b} + \frac{3\pi G_N^2 E(m_1+m_2)(5\sigma^2-1)}{4b^2(\sigma^2-1)}$$

- The sub-leading eikonal contributes to smaller values of b.
- From the second term we get the 2PM impulse:

$$Q_{2PM} = \frac{3\pi m_1 m_2 G^2(m_1+m_2)(5\sigma^2-1)}{4b^2\sqrt{\sigma^2-1}}$$

The sub-sub-leading eikonal from the 3-particle cut

- The sub-sub-leading eikonal $2\delta_2$ comes from two-loop diagrams and it has an imaginary part.
- It is computed from the 3-particle cut in the unitarity relation:

$$2[\text{Im}\, A_2]_{3pc} = \int \frac{d^{D-1}k_1}{(2\pi)^{D-1}2k_1^0} \frac{d^{D-1}k_2}{(2\pi)^{D-1}2k_2^0} \frac{d^{D-1}k}{(2\pi)^{D-1}2k^0}$$

$$\times A_5^{MN}(P_1, P_2, K_1, K_2, k) \left[\sum_i \epsilon_{MN}^{(i)} \epsilon_{RS}^{(i)}\right]$$

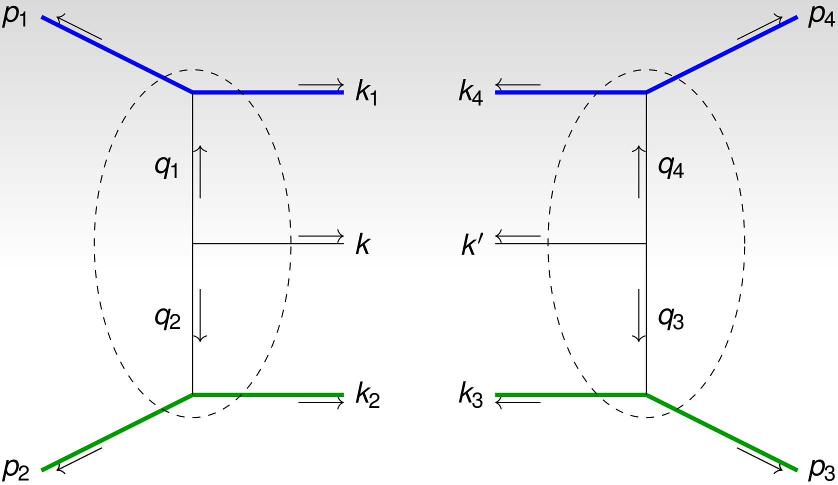
$$\times A_5^{RS}(P_4, P_3, -K_1, -K_2, -k)(2\pi)^D \delta^{(D)}(p_1 + p_2 + k_1 + k_2 + k)$$

- In $N = 8$ supergravity the indices are 10-dim

$$\sum_i \epsilon_{MN}^{(i)} \epsilon_{RS}^{(i)} = \eta_{MR}\eta_{NS}$$

while in GR they are 4-dim

$$\sum_i \epsilon_{\mu\nu}^{(i)} \epsilon_{\rho\sigma}^{(i)} = \frac{1}{2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\right) - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}$$


p_1
p_4
k_1
k_4
q_1
q_4
k
k'
q_2
q_3
k_2
k_3
p_2
p_3

- In $N=8$ it is convenient to choose the following 10-dim kinematics:

$$
\begin{aligned}
P_1 &= (p_1; 0,0,0,0,0,m_1) \qquad & P_1^2 &= 0 \\
P_2 &= (p_2; 0,0,0,0,m_2,0) \qquad & P_2^2 &= 0 \\
K_1 &= (k_1; 0,0,0,0,0,-m_1) \qquad & K_1^2 &= 0 \\
K_2 &= (k_2; 0,0,0,0,-m_2,0) \qquad & K_2^2 &= 0
\end{aligned}
$$

while in GR all momenta are 4-dim:

$$
\begin{aligned}
P_1 &= (p_1; 0,0,0,0,0,0) \qquad & p_1^2 &= -m_1^2 \\
P_2 &= (p_2; 0,0,0,0,0,0) \qquad & p_2^2 &= -m_2^2 \\
K_1 &= (k_1; 0,0,0,0,0,0) \qquad & k_1^2 &= -m_1^2 \\
K_2 &= (k_2; 0,0,0,0,0,0) \qquad & k_2^2 &= -m_2^2
\end{aligned}
$$

and

$$
\beta^{\mathcal{N}=8} = 2\sigma^2 \quad ; \quad \beta^{GR} = 2\sigma^2 - \frac{2}{D-2} \quad ; \quad \sigma = -\frac{p_1 p_2}{m_1 m_2}
$$

▶ The 5-point classical amplitude is given by

$$A_5^{MN} = (8\pi G)^{\frac{3}{2}} \Bigg\{ \frac{8\left(P_1kP_2^M - P_2kP_1^M\right)\left(P_1kP_2^N - P_2kP_1^N\right)}{q_1^2 q_2^2}$$

$$+8P_1P_2\left[\frac{P_1^M P_1^N \frac{kP_2}{kP_1} - P_1^{(M}P_2^{N)}}{q_2^2} + \frac{P_2^M P_2^N \frac{kP_1}{kP_2} - P_1^{(M}P_2^{N)}}{q_1^2}\right.$$

$$\left. -2\frac{P_1kP_2^{(M}q_1^{N)} - P_2kP_1^{(M}q_1^{N)}}{q_1^2 q_2^2}\right]$$

$$+2m_1^2m_2^2\beta\left[-\frac{P_1^M P_1^N (kq_1)}{(P_1k)^2 q_2^2} - \frac{P_2^M P_2^N (kq_2)}{(P_2k)^2 q_1^2}\right.$$

$$\left.\left. +2\left(\frac{P_1^{(M}q_1^{N)}}{(P_1k)q_2^2} - \frac{P_2^{(M}q_1^{N)}}{(P_2k)q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2}\right)\right]\right\} \; ; \; k_M A_5^{MN} = k_N A_5^{MN} = 0$$

W. Goldberger and A. Ridgway, 1611.03493
A. Luna, I. Nicholson, D. 'O Connell and C. White, 1711.03901
G. Mogull, J. Plefka and J. Steinhoff, 2010.02865.

- We can go to impact parameter space and we get

$$2\,\mathrm{Im}\,2\delta_2(b,s) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2\omega} \sum_i |\tilde{A}_{5i}(b,\vec{k})|^2$$

just in terms of the classical tree five-point amplitude in impact parameter space

$$\tilde{A}_{5i}(b,\vec{k}) = \int \frac{d^{D-2}q_1\, d^{D-2}q_2}{(2\pi)^{D-2}} \delta^{(D-2)}(q_1+q_2+k) \frac{e^{i\frac{b}{2}(q_1-q_2)}}{4Ep}$$
$$\times\, A_5^{MN}(P_1, P_2, K_1, K_2, k)\epsilon_{MN}^{(i)}$$

- The previous expression is very powerful because it allows to compute $Im(2\delta_2)$ directly from unitarity without needing to know the complete two-loop amplitude.
- In GR sum over i means a sum over the two graviton polarisations.
- In $\mathcal{N}=8$ massive sugra is a sum over all massless degrees of freedom (graviton, dilaton, 2 scalars and 2 vectors).

- $\mathrm{Im}(2\delta_2)$ is infrared divergent as $\frac{1}{\epsilon}$.
- It turns out that the infrared divergent part of $Im(2\delta_2)$ is completely fixed by the leading soft term of the 5-point amplitude .
- This is the quantity we want to compute.
- In the soft graviton limit the 5-point amplitude drastically simplifies

$$A_5^{\mu\nu} \simeq \kappa \sum_{i=1}^{4} \frac{p_i^\mu p_i^\nu}{p_i k} A_0(p_i) \qquad\qquad p_1 + p_4 = -p_2 - p_3 = q$$

$$\simeq \kappa \left[\left(\frac{p_1^\mu p_1^\nu}{(p_1 k)^2} - \frac{p_2^\mu p_2^\nu}{(p_2 k)^2} \right) (qk) - \frac{p_1^\mu q^\nu + p_1^\nu q^\mu}{(p_1 k)} + \frac{p_2^\mu q^\nu + p_2^\nu q^\mu}{(p_2 k)} \right] A_0$$

in terms of a product of a soft factor times the four-point amplitude without the graviton. Keep only linear term in q in classical limit.

- Inserting it in the 3-particle cut one gets:

$$(\mathrm{Im}\, 2\delta_2)_{gr}(\sigma, b) \simeq -\frac{G}{2\pi\epsilon}\left(\frac{2m_1 m_2 G(2\sigma^2-1)}{b\sqrt{\sigma^2-1}}\right)^2 \frac{1}{2}\mathcal{I}(\sigma)$$

  where

$$\frac{1}{2}\mathcal{I}(\sigma) = \frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma(2\sigma^2-3)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)$$

- Because of the imaginary part the eikonal is not unitary anymore.
- This divergence implies that the elastic process is suppressed.
- It emerges from the fact that, in the elastic process, we have neglected the soft-graviton emission.
- In the following we extend our analysis to the emission of extra gravitons and the *c*-number eikonal becomes an operator.
- Before doing that, let us extract $Re(2\delta_2^{(rr)})$ from the divergent part of $\mathrm{Im}\, 2\delta_2$

- Using arguments based on real analyticity, we argue that the contribution to radiation reaction should appear in the following combination:

$$\left[1 + \frac{i}{\pi}\left(-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right)\right] Re(2\delta_2^{(rr)})$$

- The part in the round bracket comes from the integral over the frequency of the graviton given by

$$\int_0^{\overline{\omega b}} \frac{d\omega}{\omega}(\omega b)^{-2\epsilon} = -\frac{(\overline{\omega b})^{-2\epsilon}}{2\epsilon} = -\frac{1}{2\epsilon} + \log(\overline{\omega b})$$

- Then we argue that $\overline{\omega b} = \sqrt{\sigma^2 - 1}$.
- Then real analyticity implies the connection with the real part

$$\log(1 - \sigma^2) = \log(\sigma^2 - 1) - i\pi$$

- In this way we extracted $Re(2\delta_2^{(rr)})$ from the divergent part of $Im(2\delta_2)$, finding in GR agreement with T. Damour, 2010.01641.
- For the complete amplitude we have to use the technique of differential equations and master integrals.

- and we get

$$
\begin{aligned}
\mathrm{Im}\, 2\delta_2^{(gr)} &= \frac{G}{2\pi}\left(\frac{2m_1 m_2 G(2\sigma^2-1)}{b\sqrt{\sigma^2-1}}\right)^2 \frac{1}{(\sigma^2-1)} \\
&\times\Bigg\{-\frac{1}{\epsilon}\left[\frac{8-5\sigma^2}{3}-\frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\
&+\left(\log(4(\sigma^2-1))-3\log(\pi b^2 e^{\gamma_E})\right) \\
&\times\left[\frac{8-5\sigma^2}{3}-\frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\
&+(\cosh^{-1}(\sigma))^2\left[\frac{\sigma\left(3-2\sigma^2\right)}{(\sigma^2-1)^{\frac{1}{2}}}-2\frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{\left(2\sigma^2-1\right)^2}\right] \\
&+\cosh^{-1}(\sigma)\left[\frac{\sigma\left(88\sigma^6-240\sigma^4+240\sigma^2-97\right)}{3\left(2\sigma^2-1\right)^2\left(\sigma^2-1\right)^{\frac{1}{2}}}\right] \\
&+\frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}}\mathrm{Li}_2(1-z^2)+\frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9\left(2\sigma^2-1\right)^2}\Bigg\}
\end{aligned}
$$

- The divergent term reproduces the one obtained using the leading soft term of the amplitude.
- The divergent term and the term proportional to $\log(\sigma^2 - 1)$ are related precisely as argued above.
- It behaves as $\log s$ at high energy as predicted in ACV90.
- The complete $Re(2\delta_2)$ is then given by

$$\mathrm{Re}\, 2\delta_2^{(gr)} = \frac{4G^3 m_1^2 m_2^2}{b^2} \left\{ \frac{(2\sigma^2-1)^2(8-5\sigma^2)}{6(\sigma^2-1)^2} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} \right.$$
$$+ \frac{s(12\sigma^4-10\sigma^2+1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} + \cosh^{-1}\sigma$$
$$\left. \times \left[\frac{\sigma(2\sigma^2-1)^2(2\sigma^2-3)}{2(\sigma^2-1)^{\frac{5}{2}}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1} \right] \right\}$$

- At high energy we get the same behaviour for GR and $\mathcal{N}=8$ supergravity: universality.

$$\Theta = \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{j\sqrt{\sigma^2 - 1}} + \frac{3\pi G^2 m_1^2 m_2^2 (m_1 + m_2)\left(5\sigma^2 - 1\right)}{4Ej^2}$$

$$+ \frac{8G^3 m_1^4 m_2^4}{sj^3} \left\{ \frac{\left(2\sigma^2 - 1\right)^2 \left(8 - 5\sigma^2\right)}{6\left(\sigma^2 - 1\right)} - \frac{\sigma\left(14\sigma^2 + 25\right)}{3(\sigma^2 - 1)^{-\frac{1}{2}}} \right.$$

$$- \frac{s\left(12\sigma^4 - 10\sigma^2 + 1\right)}{2m_1 m_2 \sqrt{\sigma^2 - 1}}$$

$$\left. + \operatorname{arccosh}(\sigma) \left[\frac{\sigma\left(2\sigma^2 - 1\right)^2 \left(2\sigma^2 - 3\right)}{2\left(\sigma^2 - 1\right)^{\frac{3}{2}}} - 4\sigma^4 + 12\sigma^2 + 3 \right] \right\}$$

$$- \frac{2G^3 m_1^3 m_2^3 \left(2\sigma^2 - 1\right)^3}{3j^3\left(\sigma^2 - 1\right)^{3/2}} \quad ; \quad j = pb$$

- Terms in black: potential gravitons. Terms in green: probe limit.
- Terms in blue: soft gravitons: radiation reaction
- The terms in green and black were computed by [Bern, Cheung, Roiban, Shen and Solon, 1901.04424]

- Anyway, at this point, there is no doubt that the contribution of radiation reaction correctly completes the conservative contribution of the classical amplitude, as shown in recent beautiful papers by
  N.E.J. Bjerrum-Bohr, P.H. Damgaard, L. Planté and P. Vanhove, 2104.04510, 2105.05218.
  N.E.J. Bjerrum-Bohr, L. Planté and P. Vanhove, 2111.02976
  A. Brandhuber, G. Chen, G. Travaglini and C. Wen, 2108.04216.
  E. Herrmann, J. Parra-Martinez, M. Ruf, M. Zeng, 2104.03957.
- They managed to extract from the quantum amplitude the complete classical integrand (including both the conservative part and the part due to radiation reaction).
- They confirmed the previous results with a direct calculation.
- By now there also complete results at 4PM.
- For the conservative part: [Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng, 2112.10750] and [Dlapa, Kälin, Liu, Porto, 2112.11296] Including radiation: [Dlapa, Kälin, Liu, Neef, Porto, 2210.05541], [Damgaard, Planté, Vanhove, 2307.04746] and to come from the Berlin group.

- If we don't integrate over the momentum of the graviton we get the differential spectrum of the number of emitted gravitons according

$$dN_{\rm gr} = \sum_i \left|\tilde{A}_{5,{\rm gr,i}}(b,\vec{k})\right|^2 \frac{d^3k}{\hbar(2\pi)^3 2\omega}$$

that, because of a factor $\frac{1}{\hbar}$, is divergent in the classical limit.

- By multiplying it with $\hbar\omega$ we get the differential spectrum of the energy:

$$dE_{gr} = \hbar\omega dN_{gr} = \frac{1}{2}\sum_i \left|\tilde{A}_{5,{\rm gr,i}}(b,\vec{k})\right|^2 \frac{d^3k}{(2\pi)^3}$$

that is a classical quantity.

- Integrating over the angles we get the spectrum $\frac{dE}{d\omega}(\omega)$ of emitted energy.
- For $\omega = 0$ we get the ZFL that we are now going to compute.

The inelastic case: the soft eikonal operator (no static)

- The S-matrix element for the emission of N soft gravitons factorises as the matrix element $S^{(M)}(\sigma, Q)$ for the background elastic process and N universal factors $w_j(k)$

$$S^{(M)}_{s.r.,N} = \prod_{r=1}^{N} w_{j_r}(k_r)\, S^{(M)}(\sigma, Q)\ ;\ \ w_j(k) = \varepsilon_j^{*\mu\nu}(k) w_{\mu\nu}(k)$$

$$w^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

- We want to write an eikonal operator that reproduces the previous equation.
- Then we will use it to compute inelastic observables (ZFL, linear and angular momentum) up to 3PM.

- We introduce the creation and annihilation operators for the gravitons satisfying the following commutation relation:

$$[a_i(k), a_j^\dagger(k')] = \delta(\vec{k}, \vec{k}\,')\delta_{ij}$$

  and

$$\delta(\vec{k}, \vec{k}') = 2\hbar\omega(2\pi)^{D-1}\delta^{D-1}(\vec{k} - \vec{k}')$$

- We restrict ourselves to soft gravitons: $\omega = |\vec{k}| \leq \omega_*$.
- ω_* is a frequency scale below which the soft approximation is valid ($\frac{\omega^* b}{v} < 1$). v is the relative velocity given by $\sigma = \frac{1}{\sqrt{1-v^2}}$.
- Following the approach of Bloch-Nordsieck, we can write the S-matrix for the emission of soft gravitons as a product of two terms: one describing the emission of gravitons and the other the elastic scattering amplitude.

- We introduce the operator

$$e^{2i\hat{\delta}_{s.r.}} = \exp\left(\frac{1}{\hbar}\int_{\vec{k}}\sum_{j}\left[w_j(k)\,a_j^{\dagger}(k) - w_j^{*}(k)\,a_j(k)\right]\right)$$

  where

$$\int_{\vec{k}} \equiv \int_0^{\omega_*} \frac{d^{D-1}\vec{k}}{2\omega(2\pi)^{D-1}}$$

- Then the S-matrix for the emission of soft gravitons is given by

$$S^{(M)}_{s.r.} = e^{2i\hat{\delta}_{s.r.}}\,\frac{S^{(M)}(\sigma, Q)}{\langle 0|e^{2i\hat{\delta}_{s.r.}}|0\rangle}$$

- The amplitude for the emission of N soft gravitons is obtained from the following matrix element:

$$S^{(M)}_{s.r.,N} = \langle 0|a_{j_1}(k_1)\cdots a_{j_N}(k_N)S^{(M)}_{s.r.}|0\rangle$$

- We can go to impact parameter space getting

$$\tilde{S}_{s.r.}(\sigma, b; a, a^\dagger) = \exp\left(\frac{1}{\hbar}\int_{\vec{k}}\sum_j\left[w_j(k)a_j^\dagger(k) - w_j^*(k)a_j(k)\right]\right) \times [1 + 2i\Delta(\sigma, b)]\, e^{2i\,\mathrm{Re}\,\delta(\sigma,b)}$$

- Because of the factor $\langle 0|e^{2i\hat{\delta}_{s.r.}}|0\rangle$ in the denominator one gets only the real part of the eikonal in the exponent.
- Remember

$$w_j(k) = \varepsilon_j^{*\mu\nu}(k)w_{\mu\nu}(k) \;\; ; \;\; w^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

- Remember that $w_j(k)$ depends on the momenta of the massive particles that are given by:

$$-p_1^\mu = \bar{p}_1^\mu - \frac{Q^\mu}{2} \; ; \quad -p_2^\mu = \bar{p}_2^\mu + \frac{Q^\mu}{2} \; ; \quad \bar{p}^\mu_{1,2} = \overline{m}_{1,2} u^\mu_{1,2}$$
$$p_4^\mu = \bar{p}_1^\mu + \frac{Q^\mu}{2} \; ; \quad p_3^\mu = \bar{p}_2^\mu - \frac{Q^\mu}{2} \; ; \quad \bar{p}_i Q = 0$$

where $\overline{m}_i^2 = m_i^2 + \frac{Q^2}{4}$ from mass-shell conditions.
- Going to Fourier space we can trade each Q^μ with a derivative

$$Q^\mu \to -i\hbar \frac{\partial}{\partial b_\mu} \to \hbar \frac{\partial \, 2 \, \mathrm{Re}\, \delta}{\partial b_\mu} = \hat{b}^\mu \, 2p \sin \frac{\Theta_s}{2}$$

where $\hat{b}^\mu = b^\mu / |b|$.

- $\hbar\partial_b \,\mathrm{Re}\, 2\delta \sim \mathcal{O}(\hbar^0)$, while if we act on $\mathrm{Re}\, 2\delta$ more than once with $\hbar\partial_b$, we would only produce terms of higher order in $\hbar$.
- Then in $w_j(k)$ we should use the following momenta for the external hard particles ($\hat{b}^\mu = \frac{b^\mu}{b}$):

$$p_1^\mu = -\,\overline{m}_1 u_1^\mu + \hat{b}^\mu p \sin\frac{\Theta_s}{2} \;\;;\;\; p_2^\mu = -\overline{m}_2 u_2^\mu - \hat{b}^\mu p \sin\frac{\Theta_s}{2}$$
$$p_4^\mu = \overline{m}_1 u_1^\mu + \hat{b}^\mu p \sin\frac{\Theta_s}{2} \;\;;\;\; p_3^\mu = \overline{m}_2 u_2^\mu - \hat{b}^\mu p \sin\frac{\Theta_s}{2}$$

that are the initial and final momenta in the classical elastic scattering.

- The S-matrix is now unitary.
- Since the soft factor in the classical limit is proportional to Q, the term 1 from the expansion of the exponential does not contribute.
- The exponential with the graviton oscillators can be regarded as a soft dressing of the initial and final states.
- To this end, it is sufficient to define

$$w_j^{\text{out/in}}(k) = \varepsilon^*_{j\mu\nu}(k) \sum_{n\in\text{out/in}} \eta_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

  with $\eta_n = +1$ ($\eta_n = -1$) if n is a final (initial) state of the background process
- Then introduce the dressed states

$$|\text{out/in}\rangle = e^{\int_k^* \left(w_j^{\text{out/in}}(k) a_j^\dagger(k) - w_j^{\text{out/in}*}(k) a_j(k) \right)} |\Psi_{\text{out/in}}\rangle\,, \qquad (1)$$

- $|\Psi_{\text{out/in}}\rangle$ only involve massive (hard) states and are related by $|\Psi_{\text{out}}\rangle = e^{i\,\text{Re}\,2\delta}|\Psi_{\text{in}}\rangle$.

- By rewriting this relation in terms of the previous dressed states

$$\begin{aligned}|\text{out}\rangle =& e^{\int_k^* \left(w_j^{\text{out}}(k)a_j^\dagger(k) - w_j^{\text{out}*}(k)a_j(k)\right)} e^{\int_k^* \left(-w_j^{\text{in}}(k)a_j^\dagger(k) + w_j^{\text{in}*}(k)a_j(k)\right)} \\ &\times e^{i\,\text{Re}\,2\delta}|\text{in}\rangle \\ =& e^{\int_k^* \left((w_j^{\text{out}}(k) - w_j^{\text{in}}(k))a_j^\dagger(k) - (w_j^{\text{out}}(k) - w_j^{\text{in}}(k))^* a_j(k)\right)} e^{i\,\text{Re}\,2\delta}|\text{in}\rangle\end{aligned}$$

  one can check that the two dressings for initial and final states commute as operators, owing to the reality of the combinations $w_j^{\text{out/in}}(k)$ themselves.
- One obtains a total dressed state with $w_j(k) = w_j^{\text{out}}(k) - w_j^{\text{in}}(k)$.
- In this way, if $|\Psi_{\text{out}}\rangle = e^{i\,\text{Re}\,2\delta}|\Psi_{\text{in}}\rangle$, then $|\text{out}\rangle = S_{s.r.}|\text{in}\rangle$ with $S_{s.r.}$ precisely taking the overall dressing factor into account.

- In the present construction the real part of the eikonal is already present in $S_{s.r.}$ while the imaginary part comes from reordering of the graviton oscillators using the BCH formula: $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$.
- One gets

$$\langle \Psi_{\rm in}|S_{s.r.}|\Psi_{\rm in}\rangle = \exp\left[-\frac{1}{2}\int_k^* w^*_{\mu\nu}(k)\Pi^{\mu\nu,\rho\sigma}(k)w_{\rho\sigma}(k)\right] e^{i\,{\rm Re}\,2\delta(b)}$$
$$= e^{i2\delta(b)}$$

where

$$\operatorname{Im} 2\delta(b) = \frac{1}{2}\int_k^* w^*_{\mu\nu}(k)\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right) w_{\rho\sigma}(k)$$

- It can be computed and one gets:

$$\mathrm{Im}\, 2\delta(b) = \left[\frac{(\omega^*)^{-2\epsilon}}{-2\epsilon}\right] \frac{G}{\pi} \sum_{n,m} m_n m_m \left(\sigma_{nm}^2 - \tfrac{1}{2}\right) F_{nm} + \mathcal{O}(\epsilon^0)$$

where

$$F_{nm} = \frac{\eta_n \eta_m \operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}}\,, \qquad \sigma_{nm} = -v_n \cdot v_m$$

- For $2 \to 2$ scattering one gets the same expression obtained from the 3-particle cut.
- One can also compute the momentum of the field:

$$P^\alpha = \int_k k^\alpha\, a_j^\dagger(k) a_j(k)\,, \qquad \boldsymbol{P}^\alpha = \langle \psi_{in} | S_{s.r.}^\dagger P^\alpha S_{s.r.} | \psi_{in} \rangle$$

that is equal to

$$\boldsymbol{P}^\alpha = \int_k^* k^\alpha w_{\mu\nu}^*(k) \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\, \eta^{\mu\nu}\eta^{\rho\sigma}\right) w_{\rho\sigma}(k)$$

- The term in the round bracket is there to compute the sum over the two graviton polarisations.

- From it we can compute the ZFL of the energy emitted spectrum:

$$\lim_{\omega\to 0}\frac{dE}{d\omega}\equiv\frac{\partial \boldsymbol{P}^0}{\partial\omega^*}=\lim_{\epsilon\to 0}2\omega^*\frac{\partial}{\partial\omega^*}\,\mathrm{Im}\,2\delta(b)=\lim_{\epsilon\to 0}\left[-4\epsilon\,\mathrm{Im}\,2\delta(b)\right]$$

$$=\frac{2G}{\pi}\sum_{n,m}m_n m_m\left(\sigma_{nm}^2-\tfrac{1}{2}\right)F_{nm}\ ;\ F_{nm}=\frac{\eta_n\eta_m\cosh^{-1}(\sigma_{nm})}{\sqrt{\sigma_{nm}^2-1}}$$

- For the case $2\to 2$ we must use $\sigma_{nn}=1=F_{nn}$ together with

$$\sigma_{12}=\sigma_{34}=\sigma\,,\quad \sigma_{13}=\sigma_{24}=\sigma_Q\,,\quad \sigma_{14}=1+\frac{Q^2}{2m_1^2}\,,\quad \sigma_{23}=1+\frac{Q^2}{2m_2^2}$$

where

$$\sigma_Q=\sigma-\frac{Q^2}{2m_1m_2}$$

- Finally we get

$$\lim_{\omega\to 0}\frac{dE}{d\omega}=\frac{4G}{\pi}\Bigg[2m_1m_2\left(\sigma^2-\tfrac{1}{2}\right)\frac{\text{arccosh}\,\sigma}{\sqrt{\sigma^2-1}}$$

$$-2m_1m_2\left(\sigma_Q^2-\tfrac{1}{2}\right)\frac{\text{arccosh}\,\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+\sum_{i=1,2}\left[\frac{m_i^2}{2}-m_i^2\left(\left(1+\tfrac{Q^2}{2m_i^2}\right)^2-\tfrac{1}{2}\right)\frac{\text{arccosh}\left(1+\frac{Q^2}{2m_i^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_i^2}\right)^2-1}}\right]$$

where $Q\to 2p\sin\frac{\Theta_s}{2}$.

- Assuming that $Q^2 << m_i^2$ we get

$$\lim_{\omega\to 0}\frac{dE}{d\omega}\simeq\frac{2G}{\pi}Q_{1PM}^2\frac{1}{2}\mathcal{I}(\sigma)\ \ ;\ \ Q_{1PM}=\frac{2Gm_1m_2(2\sigma^2-1)}{b\sqrt{\sigma^2-1}}$$

where

$$\frac{1}{2}\mathcal{I}(\sigma)=\left[\frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{(2\sigma^2-3)\sigma\,\text{arccosh}\,\sigma}{(\sigma^2-1)^{3/2}}\right]$$

The inelastic case: the soft eikonal operator (static)

- The previous soft eikonal operator is based on the standard Weinberg soft theorem, which includes soft gravitons with low but nonzero frequency.
- It does not include effects that arise due to exactly static fields, whose Fourier transform is localized at zero frequency.
- To include them it is sufficient to replace the standard soft factor by

$$f_j(k) = \varepsilon_{j\mu\nu}(k)^* F^{\mu\nu}(k)\,, \qquad F^{\mu\nu}(k) = \sum_n \frac{\sqrt{8\pi G}\, p_n^\mu p_n^\nu}{p_n \cdot k - i0}\,.$$

- and to consider the following operator

$$\mathcal{S}_{s.r.} = e^{\int_k^* \left[f_j\,(k)\, a_j^\dagger(k) - f_j^*(k)\, a_j\,(k)\right]} e^{2i\tilde{\delta}(b)}$$

where $2\tilde{\delta}$ has to be specified.

- By including the $-i0$ above prescription, even for real emissions of gravitons, we are now dressing the full S-matrix, including the identity term.
- Thus we include possible "emissions" localized at $\omega = 0$ from disconnected pieces of the hard matrix element.
- To see how this modifies the definition of the dressed states, compared to the one discussed in the previous subsection, let us now consider

$$f_j^{\text{out/in}}(k) = \varepsilon^*_{j\mu\nu}(k) \sum_{n \in \text{out/in}} \eta_n \frac{\sqrt{8\pi G} p_n^\mu p_n^\nu}{p_n \cdot k - i0}$$

and

$$\big|\text{OUT/IN}\big\rangle = e^{\int_k^* \left(f_j^{\text{out/in}}(k) a_j^\dagger(k) - f_j^{\text{out/in}*}(k) a_j(k) \right)} \big|\Psi_{\text{out/in}}\big\rangle$$

- If we start again from $|\Psi_{\text{out}}\rangle = e^{i\,\text{Re}\,2\delta} |\Psi_{\text{in}}\rangle$ and we rewrite it in terms of the *in* and *out* states we get

$$\big|\text{OUT}\big\rangle = e^{\int_k^* \left(f_j^{\text{out}}(k) a_j^\dagger(k) - f_j^{\text{out}*}(k) a_j(k) \right)} e^{-\int_k^* \left(f_j^{\text{in}}(k) a_j^\dagger(k) - f_j^{\text{in}*}(k) a_j(k) \right)} e^{i\,\text{Re}\,2\delta} \big|\text{IN}\big\rangle$$

- In this new setup, the two dressings for initial and final states no longer commute, and using the Baker–Campbell–Hausdorff formula $e^A e^B = e^{A+B} e^{+\frac{1}{2}[A,B]}$ one obtains

$$|\mathrm{OUT}\rangle = e^{\int_k^* \left(f_j(k) a_j^\dagger(k) - f_j^*(k) a_j(k) \right)} e^{\frac{1}{2} \int_k^* \left(f_j^{\mathrm{out}*}(k) f_j^{\mathrm{in}}(k) - f_j^{\mathrm{out}}(k) f_j^{\mathrm{in}*}(k) \right) + i\,\mathrm{Re}\, 2\delta} |\mathrm{IN}\rangle$$

  where $f_j(k) = f_j^{\mathrm{out}} - f_j^{\mathrm{in}}$.
- Comparing with what we computed before, we see that $|\mathrm{OUT}\rangle = \mathcal{S}_{s.r}|\mathrm{IN}\rangle$ provided the phase takes the value

$$2i\tilde{\delta} = i\,\mathrm{Re}\, 2\delta - 2i\delta^{\mathrm{dr.}} \;\; ; \;\; 2i\delta^{\mathrm{dr.}} = -\frac{1}{2} \int_k^* \left(f_j^{\mathrm{out}*}(k) f_j^{\mathrm{in}}(k) - f_j^{\mathrm{out}}(k) f_j^{\mathrm{in}*}(k) \right)$$

- It can be computed and one gets:

$$2i\delta^{\mathrm{dr.}} = iG \sum_{\substack{n \in \mathrm{out} \\ m \in \mathrm{in}}} m_n m_m (\sigma_{nm}^2 - \tfrac{1}{2}) \frac{\operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}}$$

- Expanding for small deflections $Q = Q_{1\text{PM}} + \mathcal{O}(G^2)$,

$$2i\delta^{\text{dr.}} = \frac{iGQ_{1\text{PM}}^2}{2}\frac{1}{2}\mathcal{I}(\sigma) = i\,\text{Re}\,2\delta_2^{\text{RR}} + \mathcal{O}(G^4)$$

$$\frac{1}{2}\mathcal{I}(\sigma) = \frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma(2\sigma^2-3)\,\text{arccosh}\,\sigma}{(\sigma^2-1)^{\frac{3}{2}}}$$

- In conclusion, the overall phase $2i\tilde{\delta}(b)$ contains only the conservative part up to 3PM and not the radiation reaction.
- One can compute classical observables by

$$\langle\mathcal{O}\rangle = \langle\Psi_{in}|S^{\dagger}_{s.r.}\mathcal{O}S_{s.r.}|\psi_{in}\rangle$$

- In the case of the angular momentum of the field one must insert

$$J_{\alpha\beta} = -i\int_{\vec{k}} a^{\dagger}_{\mu\nu}(k)\left(P^{\mu\nu,\rho\sigma}k_{[\alpha}\frac{\overset{\leftrightarrow}{\partial}}{\partial k^{\beta]}} + 2\eta^{\mu\rho}\delta^{\nu}_{[\alpha}\delta^{\sigma}_{\beta]}\right)a_{\rho\sigma}(k)$$

where

$$P^{\mu\nu,\rho\sigma} = \frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)$$

- In the case of $2 \to 2$ scattering, one gets

$$\mathcal{J}^{\alpha\beta} = -\frac{G}{2}(p_1 - p_2)^{[\alpha} Q^{\beta]} \mathcal{I}(\sigma) + \mathcal{O}(G^4)$$

- Using

$$Q^\beta = -\frac{b^\beta}{b} Q_{1PM} \;\; ; \;\; Q_{1PM} = \frac{2Gm_1 m_2}{b} \frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}}$$

we get

$$\mathcal{J}_2^{\alpha\beta} = \frac{G^2 m_1 m_2}{b^2} \frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} (p_1 - p_2)^{[\alpha} b^{\beta]} \mathcal{I}(\sigma)$$

that agrees with [Damour, 2010.01641].

- Using instead

$$Q^\beta = -\frac{b^\beta}{b} Q_{2PM} \quad ; \quad Q_{2PM} = \frac{3\pi G^2 m_1 m_2 (m_1 + m_2)(5\sigma^2 - 1)}{4b^2\sqrt{\sigma^2 - 1}}$$

one gets

$$\mathcal{J}_3^{\alpha\beta} = \frac{G^3 m_1 m_2 (m_1 + m_2) 3\pi (5\sigma^2 - 1)}{8b^3\sqrt{\sigma^2 - 1}} (p_1 - p_2)^{[\alpha} b^{\beta]} \mathcal{I}(\sigma)$$

that agrees with the corresponding static term of [Manohar, Ridgway and Shen, 2203.04283].
- These static quantities come out naturally from an amplitude approach and are physical quantities.
- In the framework of GR people are still debating about their physical meaning: [Veneziano and Vilkovisky, 2201.11607], [Javadinezhad and Porrati, 2211.06538] [Riva, Vernizzi and Wong, 2302. 09065].

Universality at high energy?

- In gravity the massless particle with the highest spin is the graviton and a theory with a massless particle with spin 2 is consistent only if it is invariant under any choice of coordinates.
- Since we expect that, at high energy, the massless particle with the highest spin dominates, we should get, in this limit, a universal behaviour of the various observables.
- We have seen that this happens in the elastic process up to 3PM.
- Will this also happen at 4PM?
- Is it valid also for inelastic processes with extra gravitons?
- We will limit ourselves to the case in which the graviton is soft.
- We will see that, in the case of the inelastic processes, universality is also recovered at high energy, but in a very not trivial way.

- It turns out that the PM approximation can break down even when Θ_s is small and the energy is high enough.
- This happens when

$$\frac{Q}{\sqrt{2}m_i} \gtrsim 1 \quad \Rightarrow \quad \frac{\sqrt{2}p}{m_i}\sin\frac{\Theta_s}{2} \gtrsim 1 \ ; \quad Q = 2p\sin\frac{\Theta_s}{2}$$

discussed by D'Eath (1978) and by Kovacs and Thorne (1978).

- For the ZFL we got

$$\lim_{\omega\to 0}\frac{dE}{d\omega} = \frac{4G}{\pi}\Bigg[2m_1m_2\left(\sigma^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}}$$

$$-2m_1m_2\left(\sigma_Q^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$

$$+\sum_{i=1,2}\left[\frac{m_i^2}{2}-m_j^2\left(\left(1+\tfrac{Q^2}{2m_i^2}\right)^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_i^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_i^2}\right)^2-1}}\right]\Bigg]$$

- The last line starts to diverge when $Q^2 = -4m_i^2$ for $i = 1$ or 2.

- In the standard relativistic regime requiring that $Q^2 \sim (p\Theta_s)^2 << m_i^2$ we get

$$\lim_{\omega\to 0} \frac{dE}{d\omega} \simeq \frac{2G}{\pi} Q^2_{1PM} \frac{1}{2}\mathcal{I}(\sigma) \;\; ; \quad Q_{1PM} = \frac{2Gm_1 m_2(2\sigma^2 - 1)}{b\sqrt{\sigma^2 - 1}}$$

where

$$\frac{1}{2}\mathcal{I}(\sigma) = \left[\frac{8 - 5\sigma^2}{3(\sigma^2 - 1)} + \frac{(2\sigma^2 - 3)\sigma \operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}}\right]$$

- In $\mathcal{N} = 8$ massive supergravity one gets instead:

$$\lim_{\omega\to 0} \frac{dE}{d\omega} = \frac{2GQ^2_{1PM}}{\pi} \left[\frac{\sigma^2}{\sigma^2 - 1} + \frac{\sigma(\sigma^2 - 1)}{(\sigma^2 - 1)^{3/2}} \cosh^{-1}(\sigma)\right]$$

where $Q^2_{1PM} = \frac{2Gm_1 m_2(2\sigma^2)}{b\sqrt{\sigma^2-1}}$

- It looks universal at high energy, but the factor $\log \frac{s}{m_1 m_2}$ is singular for zero mass.

- Focusing on the extreme ultrarelativistic regime, or equivalently the massless limit, where $2p \to \sqrt{s}$ and $m_1, m_2 \ll Q = \sqrt{s} \sin \frac{\Theta_s}{2}$ we get instead

$$\frac{dE^{\rm rad}}{d\omega}(\omega \to 0) \simeq \frac{4G}{\pi}\left[s \log \frac{s}{s-Q^2} + Q^2 \log \frac{s-Q^2}{Q^2}\right]_{Q=\sqrt{s}\sin\frac{\Theta_s}{2}}$$

- It is equal to

$$\frac{dE^{\rm rad}}{d\omega}(\omega \to 0) \simeq -\frac{4Gs}{\pi}\left[\cos^2\frac{\Theta_s}{2}\log\cos^2\frac{\Theta_s}{2} + \sin^2\frac{\Theta_s}{2}\log\sin^2\frac{\Theta_s}{2}\right]$$

that agrees with the leading soft limit of Sahoo and Sen, 2105.08739

- At leading order for $\Theta_s << 1$ we get

$$\frac{dE^{\rm rad}}{d\omega}(\omega \to 0) \simeq \frac{Gs\Theta_s^2}{\pi}\left[1 + \log\frac{4}{\Theta_s^2}\right]$$

- It reproduces the result obtained by Gruzinov and Veneziano, 1409.4555 within a classical GR approach.

- The same result has been obtained by Ciafaloni, Colferai and Veneziano, 1812.08137 from a scattering amplitude perspective.
- One obtains a quantity that, written in terms of classical quantities, is perfectly well defined in the UR limit:

$$\frac{1}{E}\frac{dE^{\text{rad}}}{d(\omega b)}(\omega \to 0) \simeq \frac{R}{b}\frac{\Theta_s^2}{\pi}\left[1 + \log\frac{4}{\Theta_s^2}\right] \quad ; \quad R \sim GE$$

- The same thing happens for the angular momentum:

$$\frac{\mathcal{J}^{xy}}{Eb} \simeq 2\frac{R}{b}\log\frac{4}{\Theta_s^2}$$

and also for the waveform.
- Needless to say that the same result holds also per $\mathcal{N} = 8$ supergravity: universality at high energy.
- In conclusion, going over the bound of D'Eath and Kovacs and Thorne one recovers a universal behaviour, but the PM expansion breaks down even when Θ_s is small.

Elastic case in KMOC formalism

- Here in the elastic case we follow [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola and White, 2112.07556].
- In the elastic scattering write the momenta

$$p_4 = \bar{p}_1 + \frac{Q}{2} \ ; \ p_3 = \bar{p}_2 - \frac{Q}{2} \ ; \ \bar{p}_{1,2} Q = 0$$
$$-p_1 = \bar{p}_1 - \frac{Q}{2} \ ; \ -p_2 = \bar{p}_2 + \frac{Q}{2}$$

- In KMOC ([Kosower, Maybe, O'Connell, 1811.10950]) one starts from an in state:

$$|\psi\rangle = \int \frac{d^D p_1}{(2\pi)^D} \left[(2\pi)\delta(p_1^2 + m_1^2)\theta(-p_1^0)\right] \Phi(-p_1) \int \frac{d^D p_2}{(2\pi)^D}$$
$$\times \left[(2\pi)\delta(p_2^2 + m_2^2)\theta(-p_2^0)\right] \Phi(-p_2) e^{ip_1 b_1 + ip_2 b_2} | -p_1, -p_2\rangle$$

  in terms of on-shell integrals.
- $\Phi(p)$ is the wave-packet that is peaked around the momentum p.
- $b_J = b_1 - b_2$ is the impact parameter that is orthogonal to $p_{1,2}$.

- It can be rewritten as follows:

$$\prod_{i=1,2}\left[\int \frac{d^D p_i}{(2\pi)^D}\right](2\pi)^2\delta(2Q\bar{p}_1)\delta(2Q\bar{p}_2)\Phi(-p_1)\Phi(-p_2)e^{ip_1b_1+ip_2b_2}$$
$$\times\,|-p_1,-p_2\rangle$$

- Then one introduces an out state

$$S|\psi\rangle = |\psi\rangle + iT|\psi\rangle$$

- where

$$iT|\psi\rangle = \int\prod_{i=3,4}\left(\frac{d^{D-1}p_i}{2E_i(2\pi)^{D-1}}\right)|p_3,p_4\rangle\int\prod_{i=1,2}\left(\frac{d^{D-1}p_i}{2E_i(2\pi)^{D-1}}\right)$$
$$\times\,\Phi(-p_1)\Phi(-p_2)e^{ip_1b_1+ip_2b_2}\langle p_3,p_4|iT|-p_1,-p_2\rangle$$

- with

$$\langle p_3, p_4 | iT | -p_1, -p_2 \rangle = \int \frac{d^D Q}{(2\pi)^D}$$

$$\times (2\pi)^D \delta^D(p_1 + p_4 - Q)(2\pi)^D \delta^D(p_2 + p_3 + Q) iA(s_{12}, Q^2)$$

where $s_{12} = -(p_1 + p_2)^2$.

- The integral over p_1 and p_2 can be done getting:

$$iT|\psi\rangle = \int \prod_{i=3,4} \left(\frac{d^{D-1} p_i}{2E_i (2\pi)^{D-1}} \right) |p_3, p_4\rangle e^{-ib_1 p_4} e^{-ib_2 p_3}$$

$$\times \int \frac{d^D Q}{(2\pi)^D} \Phi(p_4 - Q)\Phi(p_3 + Q) e^{iQ(b_1 - b_2)}$$

$$\times (2\pi)\delta(2\bar{p}_1 Q) 2\pi \delta(2\bar{p}_2 Q) iA(s_{12}, Q^2)$$

- Using the inverse Fourier transform of the eikonal result

$$i(2\pi)\delta(2\bar{p}_2 Q)(2\pi)\delta(2\bar{p}_1 Q) A(s_{12}, Q^2) = \int d^D x (e^{2i\delta(b)} - 1) e^{-ixQ}$$

and neglecting for simplicity the quantum part,

- we get

$$S|\psi\rangle = \left(\prod_{i=3}^{4} \frac{d^{D-1}p_i}{(2\pi)^{D-1}2E_i}\right) |p_3, p_4\rangle e^{-ib_1 p_4} e^{-ib_2 p_3} \int \frac{d^D Q}{(2\pi)^D} \int d^D x$$
$$\times e^{iQ(b_1-b_2)} e^{2i\delta(b)} e^{-ixQ} \Phi(p_4 - Q)\Phi(Q + p_3)$$

- In order to reproduce the δ-functions on the l.h.s. we need to impose that b does not depend on the component of x along $\bar{p}_i$.
- We introduce b to indicate the component of x orthogonal to $\bar{p}_i$.
- In this way we recover also that $b \cdot \bar{p}_i = 0$, while $b_J \cdot p_i = 0$.
- More explicitly we can write:

$$x^\mu = b^\mu + (\bar{p}_1 + \bar{p}_2)^\mu A_1 + (\bar{p}_1 - \bar{p}_2)^\mu A_2$$

  where $A_{1,2}$ can be determined by imposing that $b \cdot \bar{p}_{1,2} = 0$.
- We can perform the integrals by saddle point.

- We get two saddle-point equations:

$$Q^\mu = \frac{\partial 2\delta(b, s_{12})}{\partial x_\mu} = \frac{\partial 2\delta(b, s_{12})}{\partial b}\frac{b^\mu}{b} = -Q\frac{b^\mu}{b}$$

where we have used

$$\frac{\partial b}{\partial x^\mu} = \frac{b^\mu}{b} \quad ; \quad Q = -\frac{\partial 2\delta(b, s_{12})}{\partial b}$$

- The second saddle-point equation is:

$$(b_1 - b_2)^\mu - x^\mu = -\frac{\partial 2\delta(b, s_{12})}{\partial Q^\mu} = -\frac{\partial 2\delta(b, s_{12})}{\partial b}\frac{\partial b}{\partial Q^\mu}$$

$$\frac{\partial b}{\partial Q^\mu} = \frac{b^\nu}{b}\frac{\partial b^\nu}{\partial Q^\mu}$$

- Since we are integrating over Q and x they should be seen as independent variables and we need to compute $\frac{\partial b}{\partial Q^\mu}$ keeping x fixed. We keep also $p_{3,4}$ fixed.

- To make this more explicit we need to decompose x along b and $\bar{p}_{1,2}$

$$x^\mu = b^\mu + (\bar{p}_1 + \bar{p}_2)A_1 + (\bar{p}_1 - \bar{p}_2)A_2$$
$$= b^\mu + (p_3 + p_4)^\mu A_1 + (p_4 - p_3 - Q)^\mu A_2$$

- and we get

$$\frac{\partial b^\nu}{\partial Q^\mu} = -(\bar{p}_1 + \bar{p}_2)^\nu \frac{\partial A_1}{\partial Q^\mu} - (\bar{p}_1 - \bar{p}_2)^\nu \frac{\partial A_2}{\partial Q^\mu} + \delta^\nu_\mu A_2$$

- Using it in the second saddle point equation we see that only the last term contributes ($b \cdot \bar{p}_{1,2} = 0$)

$$b_J^\mu - x^\mu = Q\frac{b^\mu}{b}A_2 = -Q^\mu A_2 \;\; ; \;\; b_J = b_1 - b_2$$

-

$$b_J^\mu = b^\mu - (p_1 + p_2)^\mu A_1 - (p_1 - p_2)^\mu A_2$$
$$= b^\mu + (\bar{p}_1 + \bar{p}_2)^\mu A_1 + (\bar{p}_1 - \bar{p}_2 - Q)^\mu A_2$$

- We can fix A_1 and A_2 by imposing the conditions:

$$p_1 \cdot b_J = p_2 \cdot b_J = 0 \;\; ; \;\; \bar{p}_1 \cdot b = \bar{p}_2 \cdot b = 0$$

- We get

$$A_1 = \frac{(m_1^2 - m_2^2)|Q|b}{4m_1^2 m_2^2(\sigma^2 - 1)} \ ; \ A_2 = -\frac{s|Q|b}{4m_1^2 m_2^2(\sigma^2 - 1)}$$

- We can also compute:

$$b_J^2 = b_J \cdot b \ ; \ b_J \cdot b = b^2 - Q \cdot b A_2 \ ; \ Q^\mu = -\frac{b^\mu}{b} Q$$

- They imply

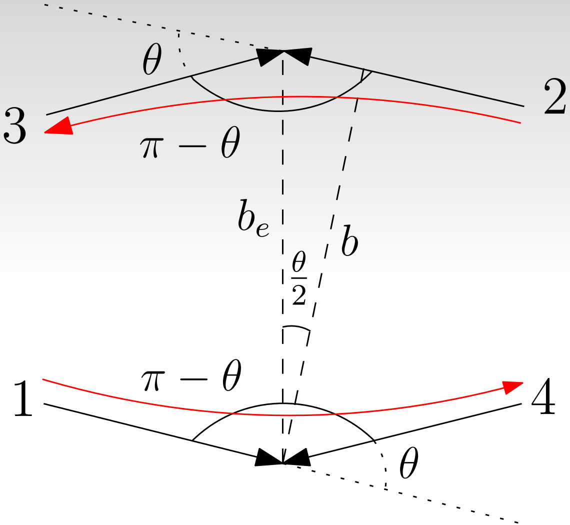
$$b_J^2 = b^2 \left(1 - \frac{sQ^2}{4m_1^2 m_2^2(\sigma^2 - 1)} \right)$$

- Taking into account that

$$Q = 2p \sin\frac{\Theta_s}{2} = 2\frac{m_1 m_2 \sqrt{\sigma^2 - 1}}{\sqrt{s}} \sin\frac{\Theta_s}{2}$$

we get

$$b_J^2 = b^2 \cos^2\frac{\Theta_s}{2} \Longrightarrow b_J = b \cos\frac{\Theta_s}{2}$$



θ
2
3
$\pi - \theta$
b_e
b
$\frac{\theta}{2}$
$\pi - \theta$
1
4
θ

Inelastic case

- In the inelastic case we have also a graviton in the final state.
- Treat the massive particles 1 and 4 independently from 2 and 3:

$$(x, Q) \Longrightarrow (x_1 Q_1) + (x_2, Q_2)$$

- We propose the following extension to the inelastic case

$$S|\psi\rangle \simeq \int_{p_3} \int_{p_4} e^{-ib_1 \cdot p_4 - ib_2 \cdot p_3} \\ \times \int \frac{d^D Q_1}{(2\pi)^D} \int \frac{d^D Q_2}{(2\pi)^D} \Phi_1(p_4 - Q_1) \Phi_2(p_3 - Q_2) \\ \times \int d^D x_1 \int d^D x_2 \, e^{i(b_1 - x_1) \cdot Q_1 + i(b_2 - x_2) \cdot Q_2} \, e^{2i\hat{\delta}(x_1, x_2)} |p_3, p_4, 0\rangle$$

  where

$$p_1 + p_4 = Q_1 \;\; ; \;\; p_2 + p_3 = Q_2$$

  that follow from the wave packets.
- We present two eikonal operators: one without and another with static modes.

- We construct them in order to reproduce all data up to 3PM.
- Both of them include the Fourier transform of the $2 \to 3$ scattering amplitude in the classical limit:

$$\tilde{\mathcal{A}}_5^{\mu\nu}(x_1, x_2, k) = \int \frac{d^D q_1}{(2\pi)^{D-2}} \delta(2p_1 \cdot q_1)\delta(2p_2 \cdot q_2) \times e^{ix_1 \cdot q_1 + ix_2 \cdot q_2} \mathcal{A}_5^{\mu\nu}(q_1, q_2, k)$$

where $q_1 + q_2 + k = 0$.

- It satisfies the important property:

$$x_{1,2} \to x_{1,2} + a \;\; ; \;\; \tilde{\mathcal{A}}_5^{\mu\nu}(x_1, x_2, k) \to e^{-ik\cdot a}\tilde{\mathcal{A}}_5^{\mu\nu}(x_1, x_2, k)$$

- We go from the soft eikonal operator valid for $\omega b < 1$ to the eikonal operator valid for arbitrary ω by changing the Fourier transform of soft factor with the FT of the classical 5-point amplitude.
- We call it $\mathcal{W}_j = \epsilon_j^{\mu\nu} \tilde{\mathcal{A}}_{5\mu\nu}$ and here we restrict ourselves to the tree level classical 5-point amplitude.
- Both of them contain the information of the 5-point amplitude and of the 4-point elastic amplitude through the c-number eikonal.

- To clarify a bit the meaning of the various integrations it is convenient to change variables as follows:

$$x_1 = x_+ + \frac{x_-}{2} \;\; ; \;\; x_2 = x_- - \frac{x_-}{2} \;\; ; \;\; Q_1 = Q_e - \frac{P}{2} \;\; ; \;\; Q_2 = -Q_e + \frac{P}{2}$$

- Rewritten in terms of these variables we see that $\tilde{\mathcal{A}}^{\mu\nu}$ depends on x_+ only through the factor e^{-ix_+k}.
- The integration over x_+ then implies that P is equal to the sum of the momenta of the emitted graviton, as one can see by expanding the exponential with the creation modes.

Inelastic without static modes

- The first one without static modes is

$$e^{2i\hat{\delta}(x_1,x_2)} = \int \frac{d^DQ}{(2\pi)^D} \int d^Dx\, e^{-iQ(x-x_1+x_2)}\, e^{2i\delta_s(b)}$$

$$\times\, e^{i\int_k \left[\mathcal{W}_j(x_1,x_2,k)a_j^\dagger(k)+\mathcal{W}_j^*(x_1,x_2,k)a_j(k)\right]}$$

where $\int_k = \int \frac{d^Dk}{(2\pi)^D}\, 2\pi\theta(k^0)\delta(k^2)\,.$

- It reduces to the elastic one without the last term.
- Classical unitarity imposes:

$$\langle\psi|S^\dagger S|\psi\rangle = \langle\psi|\psi\rangle$$

- See if the large phases cancel at the stationary point.

- The saddle point conditions are satisfied for:

$$x'_\mu = x_\mu = (x_1 - x_2)_\mu + \frac{\partial 2\delta_s(b)}{\partial Q^\mu} \ , \quad Q'_\mu = Q_\mu = \frac{\partial 2\delta_s(b)}{\partial x^\mu} \ ,$$

$$Q'_{i\mu} = Q_{i\mu} = (-1)^{i+1} Q_\mu - i \int_k \mathcal{W}_j^*(x_1, x_2, k) \frac{\overleftrightarrow{\partial}}{\partial x_i^\mu} \mathcal{W}_j(x_1, x_2, k)$$

$$(x'_i - b_i)_\mu = (x_i - b_i)_\mu = \frac{\partial 2\delta(b)}{\partial Q_i^\mu} - i \int_k \mathcal{W}_j^*(x_1, x_2, k) \frac{\overleftrightarrow{\partial}}{\partial Q_i^\mu} \mathcal{W}_j(x_1, x_2, k)$$

where

$$f \overleftrightarrow{\partial} g = (f \partial g - g \partial f)/2$$

- It turns out that, using the saddle point conditions, the large phase in the exponential cancel, consistently with classical unitarity.

Inelastic with static modes

- In this case we have:

$$e^{2i\hat{\delta}(x_1,x_2)} = \int \frac{d^DQ}{(2\pi)^D} \int d^Dx\, e^{-iQ(x-x_1+x_2)} e^{i2\delta_s(b)}$$
$$\times\, e^{\int_k \theta(\omega^*-k^0)\left[f_j^{\text{out}} a_j^\dagger - f_j^{\text{out}*} a_j\right]} e^{-\int_k \theta(\omega^*-k^0)\left[f_j^{\text{in}} a_j^\dagger - f_j^{\text{in}*} a_j\right]}$$
$$\times\, e^{i\int_k \theta(k^0-\omega^*)\left[\mathcal{W}_j(x_1,x_2,k)a_j^\dagger(k)+\mathcal{W}_j^*(x_1,x_2,k)a_j(k)\right]}$$

- Following the same steps as before we simply redefine the eikonal phase:

$$2i\tilde{\delta}(b) = 2i\delta_s(b) - 2i\delta^{\text{dr.}}(b)$$

with

$$2i\delta^{\text{dr.}}(b) = -\frac{1}{2}\int_k^{\omega^*}\left(f_j^{\text{out}*}(k)f_j^{\text{in}}(k) - f_j^{\text{in}*}(k)f_j^{\text{out}}(k)\right) = \frac{i}{4}GQ_{1PM}^2\mathcal{I}(\sigma)\,,$$

- $\tilde{\delta}(b)$ contains only the conservative part as in the soft case with static modes.

- Then we get

$$e^{2i\hat{\delta}(x_1,x_2)} = \int \frac{d^DQ}{(2\pi)^D} \int d^Dx\, e^{-iQ(x-x_1+x_2)} e^{i2\tilde{\delta}(b)}$$
$$\times\, e^{\int_k \theta(\omega^*-k^0)\left[f_j a_j(k)^\dagger - f_j^*(k) a_j(k)\right]} e^{i\int_k \theta(k^0-\omega^*)\left[\mathcal{W}_j(x_1,x_2,k) a_j^\dagger(k) + \mathcal{W}_j^*(x_1,x_2,k) a_j(k\right.}$$

where $f_j(k) = f_j^{\text{out}}(k) - f_j^{\text{in}}(k)$.

- We get the following saddle point conditions:

$$x_\mu = (x_1 - x_2)_\mu + \frac{\partial 2\tilde{\delta}(b)}{\partial Q^\mu}\,, \qquad Q_\mu = \frac{\partial 2\tilde{\delta}(b)}{\partial x^\mu}\,,$$

$$Q_{i\,\mu} = (-1)^{i+1} Q_\mu - i \int_k \mathcal{W}_j^*(x_1, x_2, k) \frac{\overleftrightarrow{\partial}}{\partial x_i^\mu} \mathcal{W}_j(x_1, x_2, k)\,,$$

$$(x_i - b_i)_\mu = \frac{\partial 2\tilde{\delta}(b)}{\partial Q_i} - i \int_k \theta(\omega^* - k^0)\, f_j^*(k) \frac{\overleftrightarrow{\partial}}{\partial Q_i^\mu} f_j(k)$$
$$- i \int_k \theta(k^0 - \omega^*)\, \mathcal{W}_j^*(x_1, x_2, k) \frac{\overleftrightarrow{\partial}}{\partial Q_i^\mu} \mathcal{W}_j(x_1, x_2, k)\,,$$

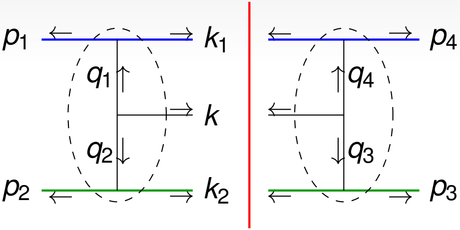
The linear and angular momentum

- Compute from the eikonal operator the emitted energy and momentum

$$\boldsymbol{P}^{\mu} = \int_k \tilde{\mathcal{A}}^{(5)} k^{\mu} \tilde{\mathcal{A}}^{(5)*}$$

- Conveniently rewritten in terms of the Fourier transform of the three-particle cut:

$$\boldsymbol{P}^{\mu} = \mathrm{FT} \int d(\mathrm{LIPS})\, k^{\mu}$$



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with the Lorentz invariant phase space measure

$$d(LIPS) = \frac{d^D k}{(2\pi)^D} 2\pi\theta(k^0)\delta(k^2) \frac{d^D q_1}{(2\pi)^D} 2\pi\delta(2p_1 q_1) 2\pi\delta(2p_2(q_1 + k))$$

- By reinterpreting the δ-functions in the LIPS as cut propagators, one can use reverse unitarity to get

$$P^{\mu}_{\text{rad}} = \frac{G^3 m_1^2 m_2^2}{b^3} \left(\check{v}_1^{\mu} + \check{v}_2^{\mu}\right) \mathcal{E}(\sigma) \; ; \; \check{v}^{\mu}_{1,2} = \frac{\sigma v^{\mu}_{2,1} - v_{1,2}}{\sigma^2 - 1} \; ; \; p_i = -m_i v_i$$

- where

$$\frac{\mathcal{E}(\sigma)}{\pi} = f_1(\sigma) + f_2(\sigma) \log \frac{\sigma + 1}{2} + f_3(\sigma) \frac{\sigma \;\text{arccosh}\, \sigma}{2\sqrt{\sigma^2 - 1}} \,,$$

$$f_1(\sigma) = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2 - 1)^{3/2}}$$

$$f_2(\sigma) = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2 - 1}} \,,$$

$$f_3(\sigma) = \frac{\left(2\sigma^2 - 3\right)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8\left(\sigma^2 - 1\right)^{3/2}} \,.$$

[Herrmann, Parra-Martinez, Ruf and Zeng, 2101.07255]

- Check that this emission of energy and momentum is matched by the corresponding radiative losses of energy-momentum of the colliding objects by using the saddle point conditions

$$\boldsymbol{Q}_{(1,2)\mu} = \frac{1}{2} \int_k \left[-i \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial x^{\mu}_{(1,2)}} \tilde{\mathcal{A}}^{(5)*} + i \tilde{\mathcal{A}}^{(5)} \frac{\partial \tilde{\mathcal{A}}^{(5)*}}{\partial x^{\mu}_{(1,2)}} \right]$$

- Using again reverse unitarity we get

$$\boldsymbol{Q}_1^{\mu} = -\frac{G^3 m_1^2 m_2^2}{b^3} \check{v}_2^{\mu} \, \mathcal{E}(\sigma) \;\; ; \;\; \boldsymbol{Q}_2^{\mu} = -\frac{G^3 m_1^2 m_2^2}{b^3} \check{v}_1^{\mu} \, \mathcal{E}(\sigma)$$

- The radiative part of energy and momentum is then conserved:

$$\boldsymbol{P}^{\mu} + \boldsymbol{Q}_1^{\mu} + \boldsymbol{Q}_2^{\mu} = 0$$

- The saddle point equation contains two additional contributions for the two particles that cancel among themselves:

$$\mathcal{Q}_1^{\mu} + \mathcal{Q}_2^{\mu} = 0$$

- They are obtained in different way in the two cases.

- If static modes are not included they are obtained from

$$\mathcal{Q}_1^\mu = \frac{\partial 2\,\mathrm{Re}\,\delta_s^{RR}(b)}{\partial b_\mu} = -\frac{b^\mu}{b} p\Theta_s^{RR} = -\frac{G}{2} Q_{1PM}^2 \frac{b^\mu}{b^2}\mathcal{I}(\sigma)$$

where

$$\frac{1}{2}\mathcal{I}(\sigma) = \frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma(2\sigma^2-3)}{(\sigma^2-1)^{3/2}}\cosh^{-1}(\sigma)$$

- With static modes, from one of saddle point conditions and from the relation between b and b_J one gets

$$b^\mu = x^\mu\left(1 + \frac{GQ(b=x)}{2x}\right) \;\; ; \;\; Q = Q_{1PM} + Q_{2PM}$$

- Then from another saddle point equation one gets:

$$Q^\mu = \frac{\partial 2\tilde{\delta}(x + \frac{GQ}{2}\mathcal{I}(\sigma))}{\partial x^\mu} = \frac{\partial 2\tilde{\delta}(x)}{\partial x^\mu} + \frac{GQ}{2}\mathcal{I}(\sigma)\frac{\partial^2 2\tilde{\delta}(x)}{\partial x \partial x^\mu}$$

- It implies

$$\mathcal{Q}_1^\mu = \frac{G}{4b}\mathcal{I}(\sigma) b^\mu \frac{\partial Q^2}{\partial b} = \frac{1}{2}\frac{\partial Q^2}{\partial b^\mu}\frac{G}{2}\mathcal{I}(\sigma) \to -\frac{G}{2}Q_{1PM}^2\frac{b^\mu}{b^2}\mathcal{I}(\sigma)$$

- For the angular momentum we have two terms:

$$J^{\alpha\beta} = \boldsymbol{J}^{\alpha\beta} + \mathcal{J}^{\alpha\beta}.$$

- The second is the contribution of the static modes that starts at 2PM and that we have already computed.
- The first is a genuine radiative term that starts at 3PM and is obtained by replacing the "soft factor" $F^{\mu\nu}$ with the gravitational waveform $\tilde{A}^{(5)\mu\nu}$.
- It is given by:

$$\boldsymbol{J}_{\alpha\beta} = \boldsymbol{J}^{(o)}_{\alpha\beta} + \boldsymbol{J}^{(s)}_{\alpha\beta} \, ; \; \boldsymbol{J}^{(o)}_{\alpha\beta} = -i\int_k k_{[\alpha}\frac{\partial\tilde{A}^{(5)}}{\partial k^{\beta]}}\tilde{A}^{(5)*} \, ; \; \boldsymbol{J}^{(s)}_{\alpha\beta} = i\int_k 2\tilde{A}^{(5)\mu}_{[\alpha}\tilde{A}^{(5)*}_{\beta]\mu}$$

- It is computed with reverse unitarity getting:

$$\boldsymbol{J}^{\alpha\beta} \simeq \frac{G^3 m_1^2 m_2^2}{b^3}\,\mathcal{F}\left(b^{[\alpha}\check{v}_1^{\beta]} - b^{[\alpha}\check{v}_2^{\beta]}\right).$$

where

$$\mathcal{F} = \frac{\mathcal{E}(\sigma-1) - 2\mathcal{C}\sqrt{\sigma^2-1}}{2(\sigma+1)}$$

- and

$$\frac{\mathcal{C}}{\pi} = g_1 + g_2 \log\frac{\sigma+1}{2} + g_3 \frac{\sigma\ \text{arccosh}\,\sigma}{2\sqrt{\sigma^2-1}} \,,$$

$$g_1 = \frac{105\sigma^7 - 411\sigma^6 + 240\sigma^5 + 537\sigma^4 - 683\sigma^3 + 111\sigma^2 + 386\sigma - 2}{24(\sigma^2-1)^2}$$

$$g_2 = \frac{35\sigma^5 - 90\sigma^4 - 70\sigma^3 + 16\sigma^2 + 155\sigma - 62}{4(\sigma^2-1)} \,,$$

$$g_3 = -\frac{(2\sigma^2-3)\left(35\sigma^5 - 60\sigma^4 - 70\sigma^3 + 72\sigma^2 + 19\sigma - 12\right)}{4\left(\sigma^2-1\right)^2}$$

- The previous result is valid in the frame where $b_1 + b_2 = 0$
- We can go to another frame by the transformation:

$$b_i^\mu \to b_i^\mu + a^\mu \quad ; \quad \boldsymbol{J}^{\alpha\beta} \to \boldsymbol{J}^{\alpha\beta} + a^{[\alpha}\boldsymbol{P}^{\beta]}$$

that follows from

$$b_{1,2}^\mu \to b_{1,2}^\mu + a^\mu \Longrightarrow \tilde{A}_5^{\mu\nu} \to e^{-ik\cdot a}\tilde{A}_5^{\mu\nu}$$

- By choosing

$$a = \frac{E_2 - E_1}{2(E_1 + E_2)}$$

we go to the center of energy frame ($E_1 b_1^\mu + E_2 b_2^\mu = 0$) and in this frame we agree with the corresponding radiative term of [Manohar, Ridgway and Shen, 2203.04283].

- Compute now the angular momentum lost by each particle given, for particle 1, by

$$\Delta L_{(1)\alpha\beta} = \langle\psi|S^\dagger L_{(1)\alpha\beta} S|\psi\rangle - \langle\psi|L_{(1)\alpha\beta}|\psi\rangle$$

$$L_{(1)\alpha\beta} = -i\int_k a_1^\dagger(k_1) k_{1[\alpha}\frac{\partial a_1^\dagger}{\partial k_1^{\beta]}}$$

$$|\psi\rangle = \int_{-p_1}\int_{-p_2} \Phi(-p_1)\Phi(-p_2) e^{ip_1 b_1 + ip_2 b_2}|-p_1, -p_2\rangle$$

$$S|\psi\rangle = \int_{p_3}\int_{p_4} |p_3, p_4\rangle \int \frac{d^D Q}{(2\pi)^D}\int d^D x$$

$$\times\, e^{iQ(b_1-b_2)} e^{2i\delta(x)} e^{ixQ}\Phi(p_4 - Q)\Phi(Q + p_3)$$

- After some calculation we get:

$$\Delta L_1^{\alpha\beta} = x_{1[\alpha|} Q_{1|\beta]} + p_{4[\alpha]} \frac{\partial 2\tilde{\delta}(b)}{\partial p_4^{\beta]}} - i \int_k \tilde{\mathcal{A}}^{(5)} p_{4[\alpha} \frac{\partial}{p_4^{\beta]}} \tilde{\mathcal{A}}^{(5)} - i \int_k F^* O_{(1)\alpha\beta} F$$

where

$$O_{(1)\alpha\beta} F^{\mu\nu}(p_3, p_4; p_1 = Q_1 - p_4, p_2 = Q_2 - p_3)$$
$$= p_{4[\alpha} \frac{\partial F^{\mu\nu}}{\partial p_4^{\beta]}} + p_{1[\alpha} \frac{\partial F^{\mu\nu}}{\partial p_1^{\beta]}}$$

- We kept the second term of $O_{(1)}$ only in the last term because it gives higher powers of G in the other terms.

- It consists of three terms:

$$\Delta L_1^{\alpha\beta} = \Delta L_{(1c)}^{\alpha\beta} + \Delta \boldsymbol{L}_1^{\alpha\beta} + \Delta \mathcal{L}_1^{\alpha\beta}$$

  and similarly for particle 2.
- The radiative term is

$$\Delta \boldsymbol{L}_i^{\alpha\beta} = \mathrm{Im}\, \boldsymbol{J}_i^{\alpha\beta} + b_i^{[\alpha} \boldsymbol{Q}_i^{\beta]} \ \ ; \ \ \boldsymbol{J}_{i\alpha\beta} = \int_k p_{i[\alpha} \frac{\partial \tilde{\mathcal{A}}^{(5)}}{\partial p_i^{\beta]}} \tilde{\mathcal{A}}^{(5)*}$$

  where $\boldsymbol{Q}_i^\alpha$ is the radiative contribution to the impulse.
- We use again reverse unitarity getting:

$$\Delta \boldsymbol{L}_1^{\alpha\beta} \simeq \frac{G^3 m_1^2 m_2^2}{b^3} \left[+ \frac{\mathcal{E}_+ b^{[\alpha} u_1^{\beta]}}{\sigma - 1} - \frac{1}{2} \mathcal{E}\, b^{[\alpha} \check{u}_2^{\beta]} \right]$$

$$\Delta \boldsymbol{L}_2^{\alpha\beta} \simeq \frac{G^3 m_1^2 m_2^2}{b^3} \left[- \frac{\mathcal{E}_+ b^{[\alpha} u_2^{\beta]}}{\sigma - 1} + \frac{1}{2} \mathcal{E}\, b^{[\alpha} \check{u}_1^{\beta]} \right]$$

- The balance equation for the radiative modes is satisfied:

$$\boldsymbol{J}^{\alpha\beta} + \Delta \boldsymbol{L}_1^{\alpha\beta} + \Delta \boldsymbol{L}_2^{\alpha\beta} = 0$$

- The contribution of the static modes is given by:

$$\Delta\mathcal{L}_1^{\alpha\beta} = -i\int_k F^*\left(p_{4[\alpha}\frac{\overleftrightarrow{\partial}}{\partial p_4^{\beta]}} + p_{1[\alpha}\frac{\overleftrightarrow{\partial}}{\partial p_1^{\beta]}}\right)F + b_1^{[\alpha}\mathcal{Q}_1^{\beta]}$$

where $\mathcal{Q}_{1\alpha}$ is the static contribution to the impulse.

- It can be shown that it is equal to

$$\Delta\mathcal{L}_1^{\alpha\beta} = J_{(1)\alpha\beta} + J_{(4)\alpha\beta} + b_1^{[\alpha}\mathcal{Q}_1^{\beta]}$$

where

$$2\eta_m J_{(m)}^{\alpha\beta} = \sum_{\eta_n=-\eta_m} c_{nm}\, p_n^{[\alpha} p_m^{\beta]} - \sum_{\substack{\eta_n=\eta_m \\ n\neq m}} d_{nm}\, p_n^{[\alpha} p_m^{\beta]}$$

- In conclusion, we get

$$\Delta\mathcal{L}_1^{\alpha\beta} = J_1^{\alpha\beta} + J_4^{\alpha\beta} + b_1^{[\alpha}\mathcal{Q}_1^{\beta]}\,, \qquad \Delta\mathcal{L}_2^{\alpha\beta} = J_2^{\alpha\beta} + J_3^{\alpha\beta} + b_2^{[\alpha}\mathcal{Q}_2^{\beta]}$$

where

$$\mathcal{Q}_1^{\alpha} = -\mathcal{Q}_2^{\alpha} = -\frac{GQ_{1\mathrm{PM}}^2 b^{\alpha}}{2b^2}\mathcal{I}(\sigma), \qquad Q_{1\mathrm{PM}} = \frac{2Gm_1m_2(2\sigma^2-1)}{b\sqrt{\sigma^2-1}}$$

is the 3PM radiation-reaction contribution to the impulse

- It can be shown that

$$J_1^{\alpha\beta} + J_2^{\alpha\beta} + J_3^{\alpha\beta} + J_4^{\alpha\beta} = -\mathcal{J}^{\alpha\beta}\,.$$

- Moreover, $b_1^{[\alpha}\mathcal{Q}_1^{\beta]} + b_2^{[\alpha}\mathcal{Q}_2^{\beta]} = (b_1-b_2)^{[\alpha}\mathcal{Q}_1^{\beta]} = b_J^{[\alpha}\mathcal{Q}_1^{\beta]} = 0$ (up to $\mathcal{O}(G^4)$) which vanishes by antisymmetry.
- In conclusion, also the static part of the angular momentum is conserved

$$\mathcal{J}^{\alpha\beta} + \Delta\mathcal{L}_1^{\alpha\beta} + \Delta\mathcal{L}_2^{\alpha\beta} = 0$$

- Finally, we can also compute the conservative part of the angular momentum:

$$\Delta L_{(1c)\alpha\beta} = b_{1\alpha} Q_{\beta]} + p_{4[\alpha} \frac{\partial 2\delta(b)}{\partial p_{4^{\beta]}}} \;\; ; \;\; \Delta L_{(2c)\alpha\beta} = b_{2\alpha} Q_{\beta]} + p_{3[\alpha} \frac{\partial 2\delta(b)}{\partial p_{3^{\beta]}}}$$

- They satisfy:

$$\Delta L_{(1c)\alpha\beta} + \Delta L_{(2c)\alpha\beta} = 0$$

- As expected, no mechanical momentum is lost by the two-body system in the conservative approximation.

Waveform

- The waveform observed at distance r from the source is given by

$$\widehat{W}^{\mu\nu}(k) = \frac{2G}{r}\frac{\mathcal{W}^{\mu\nu}(k)}{\sqrt{8\pi G}}$$

in terms of the FT transform of the tree-level 5-point amplitude:

$$\mathcal{W}^{\mu\nu}(k) = \frac{1}{4m_1 m_2}\int \frac{d^4 q_1}{(2\pi)^4}\, e^{ib_1\cdot q_1 + ib_2\cdot q_2} 2\pi\delta(v_1\cdot q_1)$$
$$\times 2\pi\delta(v_2\cdot q_2)\mathcal{A}_0^{(5)\mu\nu}(q_1, q_2, k)$$

where $q_1 + q_2 + k = 0$.

- We introduce the explicit parametrisation for k

$$k^\mu = \omega(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and of the polarisation vectors

$$\tilde{e}^\mu_\theta = -\frac{1}{\sin\theta}(\cos\theta, 0, 0, 1)\,, \qquad e^\mu_\phi = (0, -\sin\phi, \cos\phi, 0),$$

satisfying $k\tilde{e}_\theta = ke_\phi = e_\phi\tilde{e}_\theta = 0$.

- In terms of them we can construct the following transverse-traceless polarization tensors

$$\varepsilon_{\times}^{\mu\nu} = \frac{1}{2}(\tilde{e}_{\theta}^{\mu} e_{\phi}^{\nu} + \tilde{e}_{\theta}^{\nu} e_{\phi}^{\mu}) , \qquad \varepsilon_{+}^{\mu\nu} = \frac{1}{2}(\tilde{e}_{\theta}^{\mu} \tilde{e}_{\theta}^{\mu} - e_{\phi}^{\mu} e_{\phi}^{\nu})$$

and define

$$\widehat{W}_{\times}(k) = \varepsilon_{\times\mu\nu} \widehat{W}_0^{\mu\nu}(k) , \qquad \widehat{W}_{+}(k) = \varepsilon_{+\mu\nu} \widehat{W}_0^{\mu\nu}(k)$$

- Separate the two contributions according to

$$\widehat{W}_{\times/+}(k) = \widehat{W}_{12,\times/+}(k) + \widehat{W}_{\text{irr},\times/+}(k) ,$$

- For the $\times$ polarisation we get

$$\widehat{W}_{12,\times}(k) = -\frac{4iG^2 m_1 m_2 c_0}{br(\sigma^2-1)}\, b\cdot e_{\phi} \times \left(e^{-ib_1\cdot k} K_1(\Omega_1)\, v_1\cdot \tilde{e}_{\theta} - e^{-ib_2\cdot k} K_1(\Omega_2)\, v_2\cdot \tilde{e}_{\theta} \right)$$

where

$$\Omega_{1,2} = \frac{\omega_{1,2} b}{\sqrt{\sigma^2-1}} \; ; \; \omega_{1,2} = -k\cdot v_{1,2} \; ; \; c_0 = 2\sigma^2 - 1 (2\sigma^2) \; (GR, \mathcal{N}=8)$$

- and

$$\widehat{W}_{\mathrm{irr},\times}(k) = \frac{4iG^2 m_1 m_2}{r\sqrt{\sigma^2-1}} \left(\frac{c_0 \omega_1 \omega_2}{\sqrt{\mathcal{P}}} - 2\sigma\sqrt{\mathcal{P}} \right) \times b \cdot e_\phi \int_0^1 e^{-ib(x)\cdot k} K_0(\Omega(x)) dx$$

  where

$$\Omega(x) == \sqrt{\Omega_1^2 x^2 + 2\Omega_1\Omega_2\sigma xy + \Omega_2^2 y^2} \quad ; \quad b^\mu(x) = b_1^\mu x + b_2^\mu (1-x)$$

  and

$$\mathcal{P} = -\omega_1^2 + 2\omega_1\omega_2\sigma - \omega_2^2 = \omega^2(\sigma^2-1)\sin^2\theta$$

- K_0 and K_1 are two Bessel functions.
- The waveform was originally computed by Kovacs and Thorne (1978). Recently it has been computed using scattering amplitudes by [Jakobsen, Mogull,Plefka and Steinhoff, 2101.12688] in time domain and by [Riva and Vernizzi, 2102.08339] in frequency domain.

- For the $+$ polarization, we find

$$\widehat{W}_{12,+}(k) = \frac{2G^2 m_1 m_2}{r\omega_1\omega_2\left(\sigma^2-1\right)}\Big[i\frac{b\cdot k}{b}\,c_0$$
$$\times\left(e^{-ib_2\cdot k}K_1\left(\Omega_2\right)\left(v_2\cdot\tilde{e}_\theta\right)^2\omega_1 - e^{-ib_1\cdot k}K_1\left(\Omega_1\right)\left(v_1\cdot\tilde{e}_\theta\right)^2\omega_2\right)$$
$$+\frac{e^{-ib_1\cdot k}K_0(\Omega_1)\,v_1\cdot\tilde{e}_\theta\omega_2 - e^{-ib_2\cdot k}K_0(\Omega_2)v_2\cdot\tilde{e}_\theta\omega_1}{\sqrt{\sigma^2-1}\sqrt{\mathcal{P}}}$$
$$\times\left(\left(\sigma^2-1\right)\left(4\mathcal{P}\sigma - c_0\omega_1\omega_2\right) - c_0\mathcal{P}\sigma\right)\Big]$$

and finally

$$\widehat{W}_{\mathrm{irr},+}(k) = \frac{2G^2 m_1 m_2}{r\sqrt{\sigma^2-1}}\int_0^1 dx\, e^{-ib(x)\cdot k}$$
$$\times\Big[\frac{(b\cdot e_\phi)^2}{b^2}c_0 K_1(\Omega(x))\Omega(x) - c_0 K_0(\Omega(x))$$
$$+\frac{b^2 K_1(\Omega(x))}{\Omega(x)\mathcal{P}}\left(c_0\omega_1^2\omega_2^2 + 2\mathcal{P}^2 - 4\sigma\omega_1\omega_2\mathcal{P}\right)\Big]$$

- They are functions of $\sigma, b_1, b_2, \omega_1, \omega_2$ and of r, ω, θ, ϕ.

- To leading order in the soft limit (term of order $\frac{1}{\omega}$), we find

$$\widehat{W}^{\mu\nu}_{L.}(k) = \frac{4G^2 i m_1 m_2 c_0}{b^2 r\sqrt{\sigma^2-1}\,\omega_1^2\omega_2^2}$$
$$\times \Big(v_2{}^{\mu} v_2{}^{\nu}\omega_1^2 (b\cdot k) - (b\cdot k) v_1{}^{\mu} v_1{}^{\nu}\omega_2^2 - v_1{}^{(\mu} b^{\nu)}\omega_2^2\omega_1 + v_2{}^{(\mu} b^{\nu)}\omega_2\omega_1^2 \Big)$$

  which is in agreement with the PM limit of Weinberg's soft theorem translated to b-space.
- At the next to the leading order we get:

$$e^{\mu\nu}_{S.L.}(k) = \frac{4G^2 m_1 m_2\left(2\sigma^2-3\right)\sigma\log(\omega b)}{r\left(\sigma^2-1\right)^{3/2}\omega_1\omega_2}$$
$$\times\Big(v_1{}^{\mu} v_1{}^{\nu}\omega_2^2 - v_1{}^{(\mu} v_2{}^{\nu)}\omega_1\omega_2 + \omega_1^2 v_2^{\mu} v_2^{\nu}\Big)$$

  where $e^{\mu\nu} = \widehat{W}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\alpha\beta}\widehat{W}^{\alpha\beta}$, in agreement with [Sahoo and Sen, 1808.03288] and [Addazi, Bianchi and Veneziano, 1901.10986]
- In our case the log term comes automatically from the calculation.

The impulse at 4PM

- From the eikonal operator we can get also the impulse at 4PM.
- Using:

$$\sigma_{34} = \sigma_{12} - \frac{(p_1 + p_2)P + \frac{1}{2}P^2}{m_1 m_2} = \sigma_{12} - \frac{E P^0_{rad}}{m_1 m_2} + \mathcal{O}(G^6)$$

in the eikonal written as follows

$$2\delta_s(b) = \frac{1}{2}[\mathrm{Re}\, 2\delta(s_{12}, b) + \mathrm{Re}\, 2\delta(s_{34}, b)]$$

we get a contribution to the impulse of order G^4

$$Q^{(1)}_\mu = \frac{E P^0_{rad}}{2 m_1 m_2} \frac{\partial}{\partial \sigma} \left(\frac{\partial 2\delta(b)}{\partial x^\mu} \right) = \frac{E P^0_{rad}}{2 m_1 m_2} \frac{\partial}{\partial \sigma} \left(Q_{1PM} \frac{b_\mu}{b} \right)$$
$$= \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{b^4} \mathcal{E}(\sigma) \frac{b_\mu}{b} \frac{\partial}{\partial \sigma} \left(\frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}} \right)$$

where we have used

$$P^0_{rad} = \frac{G^3 m_1 m_2 (m_1 + m_2)\mathcal{E}}{E b^3} \; ; \quad Q_{1PM} = \frac{2 G m_1 m_2 (2\sigma^2 - 1)}{b\sqrt{\sigma^2 - 1}}$$

- From reverse unitarity one gets:

$$Q^\mu (x-b)_\mu = QL \;\; ; \;\; L = \frac{(m_1+m_2)G^3 m_1 m_2}{2(\sigma^2-1)b^2}(\sigma\mathcal{E} - \sqrt{\sigma^2-1}\mathcal{C})$$

- Since $Q^\mu = -\frac{b^\mu}{b} Q$ the previous condition implies:

$$b \cdot x = b^2 - bL \Longrightarrow x^\mu = b^\mu (1 - \frac{L}{b}) \Longrightarrow x = b - L$$

- Then one gets a new RR term from:

$$Q_\mu = \frac{\partial(2\tilde{\delta}(b))}{\partial x^\mu} = \frac{\partial(2\tilde{\delta}(x))}{\partial x^\mu} + L\frac{\partial^2 2\tilde{\delta}(x)}{\partial x \partial x^\mu}$$

- Going back to b it is given by

$$Q^{(2)}_{\mu\,RR} = L\frac{\partial}{\partial b}\left(-\frac{b^\mu}{b}Q\right) = \frac{G^4 m_1^2 m_2^2 (m_1+m_2)(2\sigma^2-1)}{(\sigma^2-1)^{3/2} b^4}$$
$$\times \frac{b^\mu}{b}(\sigma\mathcal{E} - \sqrt{\sigma^2-1}\mathcal{C}) \;\; ; \;\; \frac{\partial Q}{\partial b} = -\frac{Q_{1PM}}{b}$$

- Finally from the following equation that we have already derived:

$$Q^{(3)}_{\mu\, RR} = \frac{G}{4b}\mathcal{I}(\sigma) b^{\mu} \frac{\partial Q^2}{\partial b} = \frac{1}{2}\frac{\partial Q^2}{\partial b^{\alpha}}\frac{G}{2}\mathcal{I}(\sigma) \;\; ; \;\; Q = Q_{1PM} + Q_{2PM}$$

we get

$$Q^{(3)}_{\mu\, RR} = -\left(\frac{G^4 m_1^2 m_2^2 (m_1+m_2)}{b^4}\right)\frac{3\pi(2\sigma^2-1)(5\sigma^2-1)}{\sigma^2-1}\frac{3\mathcal{I}(\sigma)}{4}\frac{b_\mu}{b}$$

- In conclusion, we get:

$$Q^{4PM}_{\mu\, RR} = Q^{(1)}_{\mu\, RR} + Q^{(2)}_{\mu\, RR} + Q^{(3)}_{\mu\, RR} = \frac{G^4 m_1^2 m_2^2 (m_1+m_2)}{b^4} \times \left[\mathcal{E}\frac{\partial}{\partial\sigma}\left(\frac{2\sigma^2-1}{\sqrt{\sigma^2-1}}\right) + \frac{2\sigma^2-1}{(\sigma^2-1)^{3/2}}(\sigma\mathcal{E} - \sqrt{\sigma^2-1}\mathcal{C}) - \frac{3\pi(2\sigma^2-1)(5\sigma^2-1)}{\sigma^2-1}\frac{3\mathcal{I}(\sigma)}{4}\right]\frac{b_\mu}{b}$$

- If we want it along b_J^μ instead of b^μ we need to add a term that is orthogonal to b^μ but will contribute along b_J^μ.
- Such additional term is constructed starting from the radiative momentum:

$$\boldsymbol{Q}_1^\mu = -\frac{G^3 m_1^2 m_2^2}{b^3} \frac{(\sigma u_1 - u_2)^\mu}{\sigma^2 - 1} \mathcal{E}(\sigma)$$

that implies $(-u_1^\mu = (\frac{E_1}{m_1}, 0, 0, \frac{p}{m_1})$ and $-u_2^\mu = (\frac{E_2}{m_2}, 0, 0, -\frac{p}{m_2}))$

$$\boldsymbol{Q}_1^{\mu=3} = \frac{G^3 m_1 m_2}{b^3} p(\sigma m_2 + m_1)\mathcal{E}(\sigma)$$

- From it we can compute $(\sin\frac{\Theta_s}{2} = \frac{G(2\sigma^2-1)E}{b(\sigma^2-1)})$

$$\sin\frac{\Theta_s}{2} Q_1^3 = \frac{G^3 m_1^2 m_2^2 (m_1 + m_2)}{b^3} \frac{2\sigma^2 - 1}{(\sigma^2 - 1)^{3/2}} \mathcal{E}(\sigma) \frac{m_1 + m_2\sigma}{m_1 + m_2}$$

- Finally we get

$$Q^{\mu}_{1\,RR} = \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{b^4} \times \Bigg[\mathcal{E} \left(\frac{\sigma(6\sigma^2 - 5)}{(\sigma^2 - 1)^{3/2}} - \frac{m_1}{m_1 + m_2} \frac{2\sigma^2 - 1}{(\sigma + 1)(\sigma^2 - 1)^{1/2}} \right) - \frac{3\pi(2\sigma^2 - 1)(5\sigma^2 - 1)}{\sigma^2 - 1} \frac{3\mathcal{I}(\sigma)}{4} - \frac{2\sigma^2 - 1}{(\sigma^2 - 1)} \mathcal{C}(\sigma) \Bigg] \frac{b_J^{\mu}}{b_J}$$

  that agrees with [Bini, Damour and Geralico, 2107.08896], with [Manohar, Ridgway and Shen, 2203.04283], with [Bini, Damour, Geralico, 2210.07165], with the first line of Eq. (15) of [Dlapa, Kälin, Liu, Neef and Porto, 2210.05541] and with Eqs. (4.15) and (4.22) of [Damgaard, Hansen, Planté and Vanhove, 2307.04746].
- Actually, the last two papers have computed the elastic amplitude at 4PM including both potential and radiation.

Extension to the case with spin

- Both the Schwarzschild and Kerr solutions involve the full non-linear structure of GR.
- Construct a linearised version of the Kerr black hole by keeping in the GR Lagrangian only the kinetic term of the gravitational field and a term that describes its interaction with the energy-momentum tensor of the spinning matter.
- Then, from it, one can extract the three-point amplitude involving two massive particles with spin and a graviton:

$$\tau^{\mu\nu}(p, p', k; a) = i\kappa \Big[\cosh(a \cdot k) 2\bar{p}^\mu \bar{p}^\nu + i \frac{\sinh(a \cdot k)}{a \cdot k} \times \Big(\bar{p}^\mu \epsilon^\nu{}_{\rho\alpha\beta} a^\alpha k^\beta \bar{p}^\rho + \bar{p}^\nu \epsilon^\mu{}_{\rho\alpha\beta} a^\alpha k^\beta \bar{p}^\rho \Big) \Big] \;\; ; \;\; \bar{p} = \frac{1}{2}(p'^\mu - p^\mu)$$

[J. Vines, 1709.06016]

- The vertex involves the spin vector a^μ of the massive object.

- This is related to the spin tensor $S^{\mu\nu}$ through the following relations:

$$S^\mu = \frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\bar{p}_\nu S_{\rho\sigma}, \qquad a^\mu = \frac{S^\mu}{m} \qquad a\cdot\bar{p} = 0$$

- From it we can compute the four-point vertex by sewing together two three-point vertices and a de Donder propagator:

$$i\mathcal{A}_0 = \tau^{\mu\nu}(p_1, p_4, -q; a_1)G_{\mu\nu,\rho\sigma}(q)\tau^{\rho\sigma}(p_2, p_3, q; a_2)$$

- One obtains

$$\mathcal{A}_0 = \frac{2\kappa^2 m_1^2 m_2^2 \sigma^2}{q^2} \Big[(1+v^2)\cosh(i(\hat{p}\times\vec{a})\cdot\vec{q}\,) + 2v\sinh(i(\hat{p}\times\vec{a})\cdot\vec{q}\,) \Big]$$

neglecting analytic terms in q^2.
A. Guevara, A. Ochirov and J. Vines, 1812.06895, 1906.10071
Y.F. Bautista and A. Guevara, 1903.12419

- We can go to impact parameter space and compute the eikonal:

$$2\delta_0 = \frac{\kappa^2 m_1 m_2 \sigma}{4v} \frac{1}{4\pi^{1-\epsilon}} \sum_{\eta=\pm 1} (1+\eta v)^2 \frac{\Gamma(-\epsilon)}{\left(|\vec{b}+\eta\vec{c}\,|^2\right)^{-\epsilon}}$$

where $\vec{c} = \hat{p}\times\vec{a}$.

- The impulse is given by

$$-\vec{Q} = -\frac{\partial 2\delta_0}{\partial \vec{b}} = \frac{\kappa^2 m_1 m_2 \sigma}{2v} \frac{1}{4\pi} \sum_{\eta=\pm 1} (1+\eta v)^2 \frac{\vec{b}+\eta\vec{c}}{|\vec{b}+\eta\vec{c}\,|^2}$$

- We see that the entire spin dependence is encoded in the shift $\vec{b}\to\vec{b}\pm\vec{c}$, which is reminiscent of the Newman-Janis shift, relating Kerr to Schwarzschild black holes.

- These results are valid for generic spin orientations.
- Let us now consider the case in which both spins are parallel (or anti-parallel) to the orbital angular momentum in the centre of mass frame.
- We take

$$\vec{b} = (b, 0, 0) \;\; ; \;\; \vec{p} = (0, 0, p) \;\; ; \;\; \vec{L} = \vec{b} \times \vec{p} = (0, -pb, 0)$$

$$\vec{a} = (0, \mp a, 0) \;\; ; \;\; \vec{c} = \hat{p} \times \vec{a} = (\pm a, 0, 0) \;\; : \;\; \vec{b} + \eta \vec{c} = (b \pm \eta a, 0, 0)$$

- In this case we get

$$2\delta_0 = -\frac{\kappa^2 m_1 m_2 \sigma}{2v} \frac{1}{4\pi} \left[(1 + v^2) \log(b^2 - a^2) + 2v \log \frac{b \pm a}{b \mp a} \right] + \mathcal{O}(\epsilon)$$

- The impulse is given by

$$-Q^\mu = Q \frac{b^\mu}{b} = -\frac{\partial 2\delta_0}{\partial b^\mu} , \qquad Q = \frac{\kappa^2 m_1 m_2 \sigma}{v} \frac{1}{4\pi b} \frac{(1 + v^2) \mp \frac{2va}{b}}{1 - \frac{a^2}{b^2}}$$

where $-$ $(+)$ for spins parallel (anti-parallel) to the orbital angular momentum.

- We do not have the complete five-point amplitude with spin to use in the three-particle cut to compute $Im 2\delta_2$.
- But, in order to compute the divergent contribution we need only its soft limit.
- The leading soft term of the 5-point amplitude is given by

$$\mathcal{A}_5^{\mu\nu}(q,k) = \kappa \sum_{i=1}^{4} \frac{p_i^\mu p_i^\nu}{k \cdot p_i} \mathcal{A}_4^{\text{tree}}(q,\sigma) + \mathcal{O}(k^0)$$

- Proceeding as in the case without spin we get

$$\mathrm{Im}\, 2\delta_2(\sigma, b) \simeq -\frac{1}{2\epsilon}\frac{\pi}{2\hbar(2\pi)^3}\left(\frac{\kappa^3 m_1 m_2 \sigma^2}{8\pi\sqrt{\sigma^2-1}}\right)^2 \left(\sum_{\pm}\frac{(1\pm v)^2 \mathbf{b}_\pm}{\mathbf{b}_\pm^2}\right)^2 \mathcal{I}(\sigma)$$

where $\vec{b}_\pm \equiv \mathbf{b} \pm \hat{\mathbf{p}} \times \mathbf{a}$ and

$$\mathcal{I}(\sigma) = 2\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)\right]$$

- Introducing the spatial vector

$$\sum_{\pm}\frac{(1\pm v)^2 \mathbf{b}_\pm}{\mathbf{b}_\pm^2} \equiv \frac{2(2\sigma^2-1)}{\sigma^2 b}\mathbf{f}(a, b, \sigma)$$

- we finally get

$$\mathrm{Im}\, 2\delta_2(\sigma, b) \simeq -\frac{1}{2\epsilon}\frac{G^3(2m_1 m_2(2\sigma^2-1))^2}{2\pi\hbar(\sigma^2-1)b^2}\mathcal{I}(\sigma)\mathbf{f}^2(a, b, \sigma)$$

- When the spins are anti-parallel to the orbital angular momentum the vector **f** is

$$\mathbf{f}(a,b,\sigma) \equiv f(a,b,\sigma)\frac{\mathbf{b}}{b}, \qquad f(a,b,\sigma) = \frac{1 + \frac{2\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}}{1 - \left(\frac{a}{b}\right)^2}$$

- Otherwise **f** has non-vanishing components also along $\hat{\mathbf{p}} \times \mathbf{a}$

- The ZFL of the spectrum of emitted energy:

$$\frac{dE^{\rm rad}}{d\omega}\bigg|_{\omega\to 0} = \frac{4G^3 m_1^2 m_2^2 (2\sigma^2-1)^2}{\pi b^2(\sigma^2-1)}\mathcal{I}(\sigma)\mathbf{f}^2(a,b,\sigma)$$

- As before, using analyticity, one gets the radiative contribution to the real part of the eikonal:

$$\begin{aligned}\mathrm{Re}\,2\delta_2^{\rm rr}(\sigma,b) &= \frac{G\beta^2(\sigma)}{4\hbar(\sigma^2-1)b^2}\mathcal{I}(\sigma)\mathbf{f}^2(a,b,\sigma)\\ &= \mathrm{Re}\,2\delta_2^{\rm rr}(\sigma,b)\Big|_{\mathbf{a}=0}\mathbf{f}^2(a,b,\sigma)\end{aligned}$$

- and the radiative part of 3PM deflection angle:

$$\theta_3^{\rm rad}(\sigma, b) = \frac{G^3(2m_1 m_2(2\sigma^2-1))^2}{2(\sigma^2-1)pb^3}\mathcal{I}(\sigma)$$

$$\times \frac{\left(1+\frac{2\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}\right)\left[1+\frac{4\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}+\left(\frac{a}{b}\right)^2\right]}{\left[1-\left(\frac{a}{b}\right)^2\right]^3}$$

- For spin 1 we get

$$\theta_3^{\rm rad}(\sigma, b) = \frac{G^3(2m_1m_2(2\sigma^2-1))^2}{2(\sigma^2-1)pb^3}\mathcal{I}(\sigma)$$
$$\times\left(1 + \frac{6\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b} + 4\frac{6\sigma^4-6\sigma^2+1}{(2\sigma^2-1)^2}\frac{a^2}{b^2}\right)$$

that agrees with [Jakobsen and Mogull, 2201.07778].
- When added to the conservative part one gets a perfectly well defined deflection angle at high energy.

- Finally, from the Bini-Damour relation:

$$\theta^{\rm rad} = -\frac{1}{2}\frac{\partial\theta^{\rm cons}}{\partial E}E^{\rm rad} - \frac{1}{2}\frac{\partial\theta^{\rm cons}}{\partial J}J^{\rm lost}$$

and the conservative deflection angle $\theta_1^{\rm cons}$

$$\theta_1^{\rm cons}(\sigma, b) = \frac{2m_1 m_2 G(2\sigma^2-1)}{pb\sqrt{\sigma^2-1}} f(a, b, \sigma)$$

$$= \frac{2m_1 m_2 G(2\sigma^2-1)}{J\sqrt{\sigma^2-1}}\left(\frac{1+\frac{2\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{pa}{J}}{1-\left(\frac{pa}{J}\right)^2}\right)$$

we can extract the loss of angular momentum.

- We find the 2PM loss of angular momentum $(Q = p\Theta_s, \Theta_s = \frac{2m_1 m_2 G(2\sigma^2-1) f(a,b,\sigma)}{J\sqrt{\sigma^2-1}})$

$$\mathbf{J}_2^{\text{lost}}(\sigma, b) = J\frac{2m_1 m_2 G^2(2\sigma^2 - 1)}{b^2\sqrt{\sigma^2 - 1}}\mathcal{I}(\sigma)\hat{\mathbf{p}} \times \left(f(a, b, \sigma)\frac{\mathbf{b}}{b}\right)$$
$$= J\frac{2m_1 m_2 G^2(2\sigma^2 - 1)}{b^2\sqrt{\sigma^2 - 1}}\mathcal{I}(\sigma) f(a, b, \sigma)\mathbf{e}_2$$
$$= \frac{p}{Q}\lim_{\epsilon\to 0}\left[-4\pi\epsilon \,\mathrm{Im}\, 2\delta\right] f(a, b, \sigma)\mathbf{e}_2$$

  in agreement with the angular momentum computed by [C.Heissenberg, R.Russo, PDV, 2203.11915] with $f = 1$.
- Note that the angular momentum is lost only along the $\mathbf{e}_2$ direction, perpendicular to the scattering plane.
- This is due to the fact that in the aligned-spin case the scattering dynamics is planar, just as in the spinless scenario.

- A natural generalisation to non-aligned spin is

$$\mathbf{J}_2^{\rm lost}(\sigma, b) = J\frac{2m_1 m_2 G^2(2\sigma^2 - 1)}{b^2\sqrt{\sigma^2 - 1}}\mathcal{I}(\sigma)\hat{\mathbf{p}} \times \mathbf{f}(a, b, \sigma)$$

- In this case **f** does not lie entirely along **b**, but has also one non-vanishing component along $\hat{\mathbf{p}} \times \mathbf{a}$.
- J^{lost} for spin one agrees with the expression found by [G.Jakobsen and G.Mogull, 2201.07778]
- It agrees also with [C.Heissenberg, R.Russo, PDV, 2203.11915] for non-aligned spin.

Conclusions and Outlook

- We have constructed an eikonal operator that reproduces all data up to 3PM.
- With some small modifications we reproduce the part of the impulse along b_J at 4PM.
- Since, in gravity theories, the graviton is the massless particle with the highest spin we expect universality at high energy.
- For instance, what one gets from GR and from $\mathcal{N} = 8$ massive supergravity should be the same at high energy.
- To achieve this we must go over the bound of D'Eath, Kovacs and Thorne characterised by

$$\max\left\{\frac{m_1}{m_2}\sigma\,\Theta_s^2,\ \frac{m_2}{m_1}\sigma\,\Theta_s^2\right\} \gtrsim 1$$

- In this regime we cannot neglect Q^2 with respect to m_i^2.

- The PM expansion is not valid anymore and we get a non-perturbative answer
- This may also be the solution of the energy-crisis's problem:

$$\frac{E^{rad}_{3PM}}{E} \sim \Theta_s^3 \sqrt{\sigma} \;\; ; \;\; \frac{E^{rad}_{4PM}}{E} \sim \Theta_s^4 \log(\sigma)$$

  and of the divergence of the deflection angle at high energy at 4PM :

$$\chi^{(4)} \sim \Theta_s^4 \sqrt{\sigma}$$

  [Dlapa, Kälin, Liu, Neef and Porto, 2210.05541].
- Or may be not and then one needs to find another mechanism for explaining the divergence at high energy of the deflection angle.
- More data from NR at higher and higher energy would be very useful to confirm or exclude the trend shown in [Rettegno, Pratten, Thomas, Schmidt and Damour, 2307.06999].