Universality at high energy in black-hole scattering

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Foreword

This talk is based on recent work at two loops in GR and massive $N = 8$ supergravity together with

C. Heissenberg, R. Russo and G. Veneziano, arXiv:2008.12743, arXiv:2101.05772 and arXiv:2104.03256

on universality at high energy for inelastic processes together with

C. Heissenberg, R. Russo and G. Veneziano, arXiv:2204.02378

and on the radiation reaction with any spin together with

F. Alessio, arXiv:2203.13272

on angular momentum together with

C. Heissenberg and R. Russo, arXiv:2203.11915

and on more recent work with

C. Heissenberg, R. Russo and G. Veneziano, to appear

Plan of the talk

Introduction

- In gravity the massless particle with the highest spin is the graviton and a theory with a massless particle with spin 2 is consistent only if it is invariant under any change of coordinates.
- Since, at high energy, the massless particle with the highest spin dominates, we expect to find, in this limit, a universal behaviour of the various observables.
- This should be valid for both elastic and inelastic processes with extra gravitons.
- In this seminar we will analyse both elastic and inelastic processes and will show how universality is obtained.
- We will limit ourselves to the case in which the graviton is soft.
- We will see that, in the case of the inelastic processes, universality is also recovered at high energy, but in a very not trivial way.
- At the end we will also briefly discuss the generic case.

A bit of history

- In an impressive calculation more than three years ago Z. Bern et al, 1901.04424 computed the conservative part of the elastic scattering (involving two scalar particles with mass m_1 and m_2 at two-loop order (3PM)).
- They extracted the deflection angle that turned out to be divergent at high energy ($s = -(p_1 + p_2)^2 \to \infty$).
- This was in contradiction with the results of ACV90 based on unitarity, analyticity and crossing symmetry that instead gave a finite result at high energy for the scattering of massless scalars.
- Some people thought that one should find a universal result at high energy because the masses are negligible in this limit.
- Others thought instead that one could not compare the two cases.

- After infinite discussions on what was the origin of this problem, only an explicit calculation in massive $\mathcal{N} = 8$ supergravity convinced everybody that the problem disappears if one adds to the conservative piece, computed by [J. Parra-Martinez, M. Ruf and M. Zeng, 2005.04236], also the contribution of radiation reaction.
- This was done by approximating the integrals in the soft rather than in the potential region [C. Heissenberg, R. Russo, G. Veneziano, PDV, 2008.12743].
- But then how to compute this extra piece in GR if the total classical integrand of the two-loop amplitude was not known?
- From the loss of angular momentum T. Damour computed the radiation reaction contribution to the deflection angle in GR that, added to the conservative part, eliminated the problem with ACV90 also in GR, T. Damour, 2010.01641.
- And this without knowing the complete classical part of the two-loop amplitude!

- Here I will present an alternative approach, based on unitarity and real analyticity, that allows to compute the radiation reaction contribution to the deflection angle again without knowing the complete classical two-loop amplitude.
- It is still not clear from a physical point of view why the two previous approaches give the same result.
- On the other hand, at this point, there is no doubt that they correctly complete the conservative contribution of the classical amplitude, as shown in recent beautiful papers by
  N.E.J. Bjerrum-Bohr, P.H. Damgaard, L. Planté and P. Vanhove, 2104.04510, 2105.05218.
  N.E.J. Bjerrum-Bohr, L. Planté and P. Vanhove, 2111.02976
  A. Brandhuber, G. Chen, G. Travaglini and C. Wen, 2108.04216.
  E. Herrmann, J. Parra-Martinez, M. Ruf, M. Zeng, 2104.03957.
- They managed to extract from the quantum amplitude the complete classical integrand (including both the conservative part and the part due to radiation reaction).
- They confirmed the previous results with a direct calculation.

- We will start describing our way of computing the radiation reaction effects in GR.
- Our approach is based on the calculation of the 3-particle cut from the unitarity relation.
- This 3-particle cut becomes relevant only at 3PM and, having an intermediate graviton exchanged, it naturally provides the radiation reaction contribution.
- Performing the calculation in parallel for $\mathcal{N} = 8$ massive supergravity and GR, one obtains the same result at high energy: universality at high energy.
- We will then consider inelastic processes with emission of very soft gravitons and we will compute the waveforms and the ZFL of the spectrum of emitted energy.
- We will show that also in these cases one gets a universal behaviour at high energy.
- but, this time, in a very non-trivial way.

- Let me mention here few references on inelastic processes.
- From few years ago there are the three papers
  Gruzinov and Veneziano, 1409.4555 .
  Ciafaloni, Coradeschi, Colferai and Veneziano, 1512.00281.
  Ciafaloni, Colferai and Veneziano, 1812.08137.
  The first one using GR and the other two using amplitudes.
- More recently
  Jakobsen, Mogull, Plefka and Steinhoff, 2101.12688.
  Mougiakakos, Riva and Vernizzi, 2102.08339.
  Cristofoli, Gonzo, Kosower and O'Connell, 2107.10193.
  Riva and Vernizzi, 2110.10140.
  Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola and White, 2112.07556.
  Britto, Gonzo and Jehu, 2112.07036.

The three-particle cut from unitarity

- The 3-particle cut can be extracted from the unitarity relation:

$$2[\text{Im}\, A_2]_{3pc} = \int \frac{d^{D-1}k_1}{(2\pi)^{D-1}2k_1^0}\frac{d^{D-1}k_2}{(2\pi)^{D-1}2k_2^0}\frac{d^{D-1}k}{(2\pi)^{D-1}2k^0}$$

$$\times A_5^{MN}(P_1, P_2, K_1, K_2, k)\left[\sum_i \epsilon_{MN}^{(i)}\epsilon_{RS}^{(i)}\right]$$

$$\times A_5^{RS}(P_4, P_3, -K_1, -K_2, -k)(2\pi)^D\delta^{(D)}(p_1+p_2+k_1+k_2+k)$$

- In $N = 8$ supergravity the indices are 10-dim

$$\sum_i \epsilon_{MN}^{(i)}\epsilon_{RS}^{(i)} = \eta_{MR}\eta_{NS}$$

while in GR they are 4-dim

$$\sum_i \epsilon_{\mu\nu}^{(i)}\epsilon_{\rho\sigma}^{(i)} = \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}$$

- No need to symmetrise in $(MN)(RS)$ because amplitudes are symmetric.

- The 5-point classical amplitude is given by

$$A_5^{MN} = (8\pi G)^{\frac{3}{2}} \Bigg\{ \frac{8\left(P_1kP_2^M - P_2kP_1^M\right)\left(P_1kP_2^N - P_2kP_1^N\right)}{q_1^2 q_2^2}$$

$$+8P_1P_2\left[\frac{P_1^M P_1^N \frac{kP_2}{kP_1} - P_1^{(M}P_2^{N)}}{q_2^2} + \frac{P_2^M P_2^N \frac{kP_1}{kP_2} - P_1^{(M}P_2^{N)}}{q_1^2}\right.$$

$$\left. -2\frac{P_1kP_2^{(M}q_1^{N)} - P_2kP_1^{(M}q_1^{N)}}{q_1^2 q_2^2}\right]$$

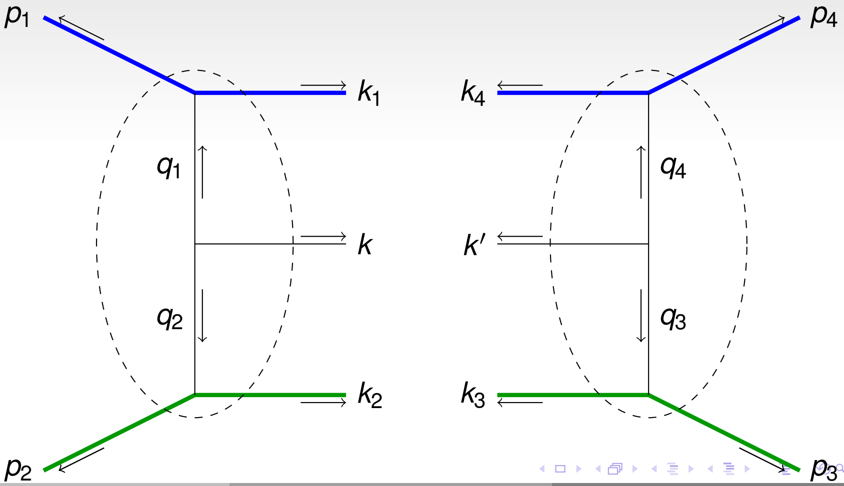
$$+\beta\left[-\frac{P_1^M P_1^N (kq_1)}{(P_1k)^2 q_2^2} - \frac{P_2^M P_2^N (kq_2)}{(P_2k)^2 q_1^2}\right.$$

$$\left.\left. +2\left(\frac{P_1^{(M}q_1^{N)}}{(P_1k)q_2^2} - \frac{P_2^{(M}q_1^{N)}}{(P_2k)q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2}\right)\right]\right\} ; \; k_M A_5^{MN} = k_N A_5^{MN} = 0$$

W. Goldberger and A. Ridgway, 1611.03493
A. Luna, I. Nicholson, D. ’O Connell and C. White, 1711.03901
G. Mogull, J. Plefka and J. Steinhoff, 2010.02865.

- Separate long. and trans. directs wrt the class. dir. of propagation:

$$p_1 = \left(-E_1, \frac{\mathbf{q}}{2}, -\bar{p}\right) \qquad p_4 = \left(E_1, \frac{\mathbf{q}}{2}, \bar{p}\right)$$
$$p_2 = \left(-E_2, -\frac{\mathbf{q}}{2}, \bar{p}\right) \qquad p_3 = \left(E_2, -\frac{\mathbf{q}}{2}, -\bar{p}\right)$$



- In $N = 8$ it is convenient to choose the following 10-dim kinematics:

$$P_1 = (p_1; 0, 0, 0, 0, 0, m_1) \qquad P_1^2 = 0$$
$$P_2 = (p_2; 0, 0, 0, 0, m_2, 0) \qquad P_2^2 = 0$$
$$K_1 = (k_1; 0, 0, 0, 0, 0, -m_1) \qquad K_1^2 = 0$$
$$K_2 = (k_2; 0, 0, 0, 0, -m_2, 0) \qquad K_2^2 = 0$$

while in GR all momenta are 4-dim:

$$P_1 = (p_1; 0, 0, 0, 0, 0, 0) \qquad p_1^2 = -m_1^2$$
$$P_2 = (p_2; 0, 0, 0, 0, 0, 0) \qquad p_2^2 = -m_2^2$$
$$K_1 = (k_1; 0, 0, 0, 0, 0, 0) \qquad k_1^2 = -m_1^2$$
$$K_2 = (k_2; 0, 0, 0, 0, 0, 0) \qquad k_2^2 = -m_2^2$$

and

$$\beta^{\mathcal{N}=8} = 4m_1^2 m_2^2 \sigma^2 \quad \beta^{GR} = 4m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{D-2}\right) \quad \sigma = -\frac{p_1 p_2}{m_1 m_2}$$

- Using the momentum conservation δ-functions we can perform the integral over the longitudinal directions of k_1 and k_2.
- The resulting Jacobian cancels the factors $k^0_{1,2}$ and produces an extra factor of $|k_1^0 k_2^L - k_2^0 k_1^L|^{-1}$.
- We are then left with the integrals over q_1 and q_2 only along the remaining $D-2$ transverse directions:

$$[\mathrm{Im}\, 2A_2]_{3pc} = \int \frac{d^{D-1}k}{(2\pi)^{D-1} 2k^0} \int \frac{d^{D-2}q_1}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_2}{(2\pi)^{D-2}}$$
$$\times \frac{(2\pi)^{D-2}\delta^{(D-2)}(k+q_1+q_2)}{4|k_1^0 k_2^L - k_2^0 k_1^L|} A_5^{MN}(P_1, P_2, K_1, K_2, k)$$
$$\times \left[\sum_i \epsilon^{(i)}_{MN} \epsilon^{(i)}_{RS} \right] A_5^{RS}(P_4, P_3, -K_1, -K_2, -k)$$

where we changed variables of integration from $k_{1,2}$ to $q_{1,2}$

$$q_{1,2} = p_{1,2} + k_{1,2} = (q^0_{1,2}, \mathbf{q}_{1,2}, q^L_{1,2}) \,, \qquad k = (k^0, \vec{k}) = (k^0, \mathbf{k}, k^L)$$

- In the classical limit we can safely approximate

$$k_1^L \simeq \bar{p} \simeq p\,;\; k_2^L \simeq -\bar{p} \simeq -p\,;\; k_1^0 \simeq E_1\;;\; k_2^0 \simeq E_2$$
$$\Longrightarrow 4|k_1^0 k_2^L - k_2^0 k_1^L| \simeq 4Ep\;\;;\;\; E = E_1 + E_2$$

- In order to treat the two 5-point amplitudes more symmetrically we can introduce q_3 and q_4 such that

$$q_1 + q_4 = P_1 + P_4 = q\;\;;\;\; q_2 + q_3 = P_2 + P_3 = -q$$

by introducing the two δ-functions:

$$1 = \int d^{D-2}q_4\, \delta^{(D-2)}(q_1 + q_4 - q) \int d^{D-2}q_3\, \delta^{(D-2)}(q_3 + q_2 + q)$$

- Then going to impact parameter space we get

$$2\,\mathrm{Im}\,2\delta_2(b,s) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{ib\cdot q}\, \frac{[\mathrm{Im}\,2A_2]_{3pc}}{4Ep} =$$

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}2k^0} \sum_i$$

$$\left[\int \frac{d^{D-2}q_1\, d^{D-2}q_2}{(2\pi)^{D-2}} \delta^{(D-2)}(q_1+q_2+k) \frac{e^{i\frac{b}{2}(q_1-q_2)}}{4Ep} \right.$$

$$\left. \times A_5^{MN}(P_1,P_2,K_1,K_2,k)\epsilon^{(i)}_{MN} \right]$$

$$\times \left[\int \frac{d^{D-2}q_3\, d^{D-2}q_4}{(2\pi)^{D-2}} \delta^{(D-2)}(q_3+q_4-k) \frac{e^{i\frac{b}{2}(q_4-q_3)}}{4Ep} \right.$$

$$\left. \times A_5^{RS}(P_4,P_3,-K_1,-K_2,-k)\epsilon^{(i)}_{RS} \right]$$

where we have used $q = \frac{1}{2}(q_1 - q_2 + q_4 - q_3)$.

- The previous expression is very powerful because it allows to compute $Im(2\delta_2)$ directly from unitarity without needing to know the complete two-loop amplitude.
- Writing the previous expression in a more compact form we get

$$2\,\mathrm{Im}\,2\delta_2(b,s) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2\omega} \sum_i |\tilde{A}_{5i}(b,\vec{k})|^2$$

just in terms of the classical tree five-point amplitude in impact parameter space.
- In GR sum over i means a sum over the two graviton polarisations.
- In $\mathcal{N}=8$ massive sugra is a sum over all massless degrees of freedom (graviton, dilaton...).
- Inserting in the previous relation the double-Regge limit of the 5-point amplitude, ACV90 computed $Im(2\delta_2)$ in the massless case that turned out to be divergent as $\log s$.
- Then, using analyticity and crossing symmetry, ACV90 managed to deduce from it also $Re(2\delta_2)$ that gave a finite deflection angle at high energy.

- We consider instead the massive case and keep only the leading divergent soft term for the momentum of the graviton $k \to 0$.
- This leading soft term completely fix the infrared divergent part of $Im(2\delta_2)$ that is the quantity we want to compute.
- In the soft graviton limit the five-point amplitude drastically simplifies

$$A_5^{\mu\nu} \simeq \kappa$$
$$\times \left[\left(\frac{\bar{p}_1^\mu \bar{p}_1^\nu}{(\bar{p}_1 k)^2} - \frac{\bar{p}_2^\mu \bar{p}_2^\nu}{(\bar{p}_2 k)^2} \right) (qk) - \frac{\bar{p}_1^\mu q^\nu + \bar{p}_1^\nu q^\mu}{(\bar{p}_1 k)} + \frac{\bar{p}_2^\mu q^\nu + \bar{p}_2^\nu q^\mu}{(\bar{p}_2 k)} \right] A_0$$

  in terms of a product of a soft factor times the four-point amplitude without the graviton. Keep only linear term in q in classical limit.
- Inserting this amplitude in previous relation we get for the graviton:

$$(\mathrm{Im}\, 2\delta_2)_{gr}(\sigma, b)$$
$$\simeq -\frac{1}{2\epsilon} \frac{G^3 \beta^2(\sigma)}{\pi b^2 (\sigma^2 - 1)^2} \left[\frac{8 - 5\sigma^2}{3} - \frac{\sigma(3 - 2\sigma^2)}{(\sigma^2 - 1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right]$$

- Then, using arguments based on real analyticity, we argued that the radiation reactions terms should appear in the following combination:

$$\left[1 + \frac{i}{\pi}\left(-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right)\right] Re(2\delta_2^{(rr)})$$

- The part in the round bracket comes from the integral over the frequency of the graviton given by

$$\int_0^{\overline{\omega b}} \frac{d\omega}{\omega}(\omega b)^{-2\epsilon} = -\frac{(\overline{\omega b})^{-2\epsilon}}{2\epsilon} = -\frac{1}{2\epsilon} + \log(\overline{\omega b})$$

- Then we argue that $\overline{\omega b} = (\sigma^2 - 1)$.
- Then real analyticity implies the connection with the real part

$$\log(1 - \sigma^2) = \log(\sigma^2 - 1) - i\pi$$

- In this way we extracted $Re(2\delta_2^{(rr)})$ from the divergent part of $Im(2\delta_2)$, finding in GR agreement with T. Damour, 2010.01641.
- For the complete amplitude we have to use again the technique of differential equations and master integrals.

- and we get

$$
\begin{aligned}
\text{Im}\, 2\delta_2^{(gr)} &= \frac{2m_1^2 m_2^2 G^3}{\pi b^2} \frac{(2\sigma^2-1)^2}{(\sigma^2-1)^2} \\
&\times \Bigg\{ -\frac{1}{\epsilon} \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
&+ \Big(\log(4(\sigma^2-1)) - 3\log(\pi b^2 e^{\gamma_E}) \Big) \\
&\times \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
&+ (\cosh^{-1}(\sigma))^2 \left[\frac{\sigma\left(3-2\sigma^2\right)}{(\sigma^2-1)^{\frac{1}{2}}} - 2\frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{\left(2\sigma^2-1\right)^2} \right] \\
&+ \cosh^{-1}(\sigma) \left[\frac{\sigma\left(88\sigma^6-240\sigma^4+240\sigma^2-97\right)}{3\left(2\sigma^2-1\right)^2 (\sigma^2-1)^{\frac{1}{2}}} \right] \\
&+ \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \text{Li}_2(1-z^2) + \frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9\left(2\sigma^2-1\right)^2} \Bigg\}
\end{aligned}
$$

- The divergent term reproduces the one obtained using the leading soft term of the amplitude or using the IR exponentiation in momentum space C. Heissenberg, arXiv:2105.04594.
- The divergent term and the term proportional to $\log(\sigma^2 - 1)$ are related precisely as argued above.
- It behaves as $\log s$ at high energy as predicted in ACV90.
- The complete $Re(2\delta_2)$ is then given by

$$\mathrm{Re}\, 2\delta_2^{(gr)} = \frac{4G^3 m_1^2 m_2^2}{b^2} \Bigg\{ \frac{(2\sigma^2-1)^2(8-5\sigma^2)}{6(\sigma^2-1)^2} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} + \frac{s(12\sigma^4-10\sigma^2+1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} + \cosh^{-1}\sigma \times \left[\frac{\sigma(2\sigma^2-1)^2(2\sigma^2-3)}{2(\sigma^2-1)^{\frac{5}{2}}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1} \right] \Bigg\}$$

- At high energy we get the same behaviour for GR and $\mathcal{N}=8$ supergravity: universality.

- If we don't integrate over the momentum of the graviton we get the differential spectrum of the number of emitted gravitons according

$$dN_{\rm gr} = \sum_i \left| \tilde{A}_{5,{\rm gr,i}}(b,\vec{k}) \right|^2 \frac{d^3k}{\hbar(2\pi)^3 2\omega}$$

  that, because of a factor $\frac{1}{\hbar}$, is divergent in the classical limit.
- By multiplying it with $\hbar\omega$ we get the differential spectrum of the energy:

$$dE_{gr} = \hbar\omega dN_{gr} = \frac{1}{2}\sum_i \left| \tilde{A}_{5,{\rm gr,i}}(b,\vec{k}) \right|^2 \frac{d^3k}{(2\pi)^3}$$

  that is a classical quantity.
- Integrating over the angles we get the spectrum $\frac{dE}{d\omega}(\omega)$ of emitted energy. For $\omega = 0$ we get the ZFL.
- Integrating over the momentum of the graviton we get the total energy emitted.
- Using the complete classical amplitude we reproduced the results of E. Herrmann, J. Parra-Martinez, M. Ruf, M. Zeng, 2104.03957.

The elastic eikonal: a brief reminder

- The leading eikonal is obtained from the tree-level elastic amplitude:

$$\mathcal{A}_0(\sigma, q^2) = \frac{8\pi G}{q^2}\left[4m_1^2 m_2^2\left(\sigma^2 - \frac{\zeta}{D-2}\right)\right] + \dots$$

$$2\delta_0(\sigma, b) = \widetilde{\mathcal{A}}_0(\sigma, b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}_0(\sigma, q^2)}{4Ep} e^{i\frac{bq}{\hbar}}$$

- One gets

$$2\delta_0 = \frac{2G\, m_1 m_2 \left(\sigma^2 - \frac{\zeta}{D-2}\right) \Gamma\left(\frac{D-4}{2}\right)}{\hbar\sqrt{\sigma^2 - 1}(\pi b^2)^{\frac{D-4}{2}}} \ ; \ \sigma = -\frac{p_1 p_2}{m_1 m_2}$$

$\zeta = 1$ for GR and $\zeta = 0$ for $\mathcal{N} = 8$ supergravity.

- The process above involves the exchange of a single quantum.
- After the eikonal resummation the leading contribution to the S-matrix is captured by the phase $e^{2i\delta_0}$, which effectively resums infinitely many exchanges.

- This can be seen by rewriting the resumed amplitude in momentum space:

$$S^{(M)}(\sigma, Q) \simeq \int d^{D-2}b\, e^{-i\frac{bQ}{\hbar}}\, e^{2i\delta_0(\sigma,b)}$$

- The Fourier transform above is dominated by the saddle point

$$Q_s^\mu = \hbar\frac{\partial(2\delta_0)}{\partial b^\mu}; \quad N_s \simeq \frac{|Q_s|}{|q|} \simeq \frac{4Gm_1 m_2\left(\sigma^2 - \frac{\zeta}{D-2}\right)\Gamma\left(\frac{D-2}{2}\right)}{\hbar\sqrt{\sigma^2-1}\,\pi^{\frac{D-4}{2}}\, b^{D-4}}$$

  Q_s represents the momentum exchanged in the classical deflection.

- N_s is the number of soft particles exchanged during the scattering obtained dividing Q_s by the typical momentum of each soft particle $q \sim \frac{\hbar}{b}$.
- N_s is large and becomes infinite in the strict classical limit.

- The classical deflection angle Θ_s is derived from the momentum $|Q_s|$ and at 1PM order we have

$$p\,\Theta_s \simeq |Q_s| \simeq \frac{4Gm_1 m_2 \left(\sigma^2 - \frac{\zeta}{D-2}\right) \Gamma\left(\frac{D-2}{2}\right)}{\sqrt{\sigma^2 - 1}\, \pi^{\frac{D-4}{2}}\, b^{D-3}}$$

- Straightforward to formally generalise this discussion beyond the case of the 1PM elastic eikonal.
- One just needs to use the full eikonal and write the long-range elastic S-matrix as follows

$$S^{(M)}(\sigma, Q) = \int d^{D-2}b\, e^{-i\frac{bQ}{\hbar}}\, (1 + 2i\Delta(\sigma, b))\, e^{2i\delta(\sigma,b)}$$

- Again the classical deflection angle Θ_s is derived from the momentum $|Q_s|$ by a saddle point now related to δ instead of δ_0

$$Q_s^\mu = \hbar \frac{\partial(2\,\mathrm{Re}\,\delta)}{\partial b_\mu}\,, \qquad \sin\frac{\Theta_s}{2} = \frac{|Q_s|}{2p}$$

The inelastic case: the eikonal operator

- The S-matrix element for the emission of N soft gravitons factorises as the matrix element $S^{(M)}(\sigma, Q)$ for the background elastic process and N universal factors $f_j(k)$

$$S^{(M)}_{s.r.,N} = \prod_{r=1}^{N} f_{j_r}(k_r)\, S^{(M)}(\sigma, Q)\ ;\ f_j(k) = \varepsilon_j^{*\mu\nu}(k) F_{\mu\nu}(k)$$

$$F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

- Of course an analogous formula holds for soft absorptions, with $f_j(k)$ replaced by $-f_j^*(k)$.
- We want to write an eikonal operator that reproduces the previous equation.

- We introduce the creation and annihilation operators for the gravitons satisfying the following commutation relation:

$$[a_i(k), a_j^\dagger(k')] = \delta(\vec{k}, \vec{k}\,')\delta_{ij}$$

and

$$\delta(\vec{k}, \vec{k}') = 2\hbar\omega(2\pi)^{D-1}\delta^{D-1}(\vec{k} - \vec{k}')\ ;\ \int_{\vec{k}} \equiv \int_0^{\omega_*} \frac{d^{D-1}\vec{k}}{2\omega(2\pi)^{D-1}}$$

- We restrict ourselves to soft gravitons: $\omega = |\vec{k}| \leq \omega_*$.
- ω_* is a frequency scale below which the soft approximation is valid ($\frac{\omega^* b}{v} < 1$). v is the relative velocity given by $\sigma = \frac{1}{\sqrt{1-v^2}}$.
- Following the approach of Bloch-Nordsieck, we can write the S-matrix for the emissions of soft gravitons as a product of two terms: one describing the emission of gravitons and the other the elastic scattering amplitude.

- We introduce the operator

$$e^{2i\hat{\delta}_{s.r.}} = \exp\left(\frac{1}{\hbar}\int_{\vec{k}}\sum_j \left[f_j\,(k)\,a_j^\dagger(k) - f_j^*(k)\,a_j\,(k)\right]\right)$$

- Then the S-matrix for the emission of soft gravitons is given by

$$S_{s.r.}^{(M)} = e^{2i\hat{\delta}_{s.r.}}\,\frac{S^{(M)}(\sigma, Q)}{\langle 0|e^{2i\hat{\delta}_{s.r.}}|0\rangle}$$

- The amplitude for the emission of N soft graviton is obtained from the following matrix element:

$$S_{s.r.,N}^{(M)} = \langle 0|a_{j_1}(k_1)\cdots a_{j_N}(k_N)S_{s.r.}^{(M)}|0\rangle$$

- We can go to impact parameter space getting

$$\tilde{S}_{s.r.}(\sigma, b; a, a^\dagger) = \exp\left(\frac{1}{\hbar}\int_{\vec{k}}\sum_j\left[f_j(k)a_j^\dagger(k) - f_j^*(k)a_j(k)\right]\right) \times [1 + 2i\Delta(\sigma, b)]\, e^{2i\,\mathrm{Re}\,\delta(\sigma,b)}$$

- Because of the factor $\langle 0|e^{2i\hat{\delta}_{s.r.}}|0\rangle$ in the denominator one gets only the real part of the eikonal in the exponent.
- Remember

$$f_j(k) = \varepsilon_j^{*\mu\nu}(k)F_{\mu\nu}(k) \;\; ; \;\; F^{\mu\nu}(k) = \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

- Remember that $f_j(k)$ depends on the momenta of the massive particles that are given by:

$$p_1^\mu = -\overline{m}_1 u_1^\mu + \frac{Q^\mu}{2}\ ; \qquad p_2^\mu = -\overline{m}_2 u_2^\mu - \frac{Q^\mu}{2}$$
$$p_4^\mu = \overline{m}_1 u_1^\mu + \frac{Q^\mu}{2}\ ; \qquad p_3^\mu = \overline{m}_2 u_2^\mu - \frac{Q^\mu}{2}$$

  where $\overline{m}_i^2 = m_i^2 + \frac{Q^2}{4}$ from mass-shell conditions.
- By taking the Fourier transform we use the following relation:

$$Q^\mu \to -i\hbar \frac{\partial}{\partial b_\mu} \to \hbar \, \frac{\partial\, 2\,\mathrm{Re}\,\delta}{\partial b_\mu} = \hat{b}^\mu\, 2p \sin\frac{\Theta_s}{2}$$

  where $\hat{b}^\mu = b^\mu/|b|$.
- $\hbar\partial_b\, \mathrm{Re}\, 2\delta \sim \mathcal{O}(\hbar^0)$, while if we act on $\mathrm{Re}\, 2\delta$ more than once with $\hbar\partial_b$, we would only produce terms of higher order in $\hbar$.

- Equation that clarifies the step above:

$$\varphi(Q)\, e^{i\,\mathrm{Re}\,2\delta} \to \sum_n c_n(-i\hbar\partial_b)^{2n} e^{i\,\mathrm{Re}\,2\delta}$$
$$= \sum_n c_n \left[\hbar \frac{\partial\,\mathrm{Re}\,2\delta}{\partial b}\right]^{2n} e^{i\,\mathrm{Re}\,2\delta} + \mathcal{O}(\hbar)$$
$$= \varphi\left(\hbar\,\frac{\partial\,\mathrm{Re}\,2\delta}{\partial b}\right) e^{i\,\mathrm{Re}\,2\delta} + \mathcal{O}(\hbar)\,,$$

- Then in $f_j(k)$ we should use the following momenta for the external hard particles ($\hat{b}^\mu = \frac{b^\mu}{b}$):

$$p_1^\mu = -\,\overline{m}_1\, u_1^\mu + \hat{b}^\mu p \sin\frac{\Theta_s}{2} \;\;;\;\; p_2^\mu = -\overline{m}_2 u_2^\mu - \hat{b}^\mu p \sin\frac{\Theta_s}{2}$$
$$p_4^\mu = \overline{m}_1\, u_1^\mu + \hat{b}^\mu p \sin\frac{\Theta_s}{2} \;\;;\;\; p_3^\mu = \overline{m}_2 u_2^\mu - \hat{b}^\mu p \sin\frac{\Theta_s}{2}$$

that are the initial and final momenta in the classical elastic scattering.

- In a PM expansion one keeps only the leading term in the expansion of the terms involving Θ_s since Θ_s is proportional to G.
- This is equivalent to expand for small Q in the operator part of the eikonal.
- This expansion is not justified in all kinematic regimes, as we will see in the discussion of the waveforms and of the ZFL below.
- The eikonal operator is unitary since only the real part of the elastic δ enters in this equation.
- The imaginary part of δ is obtained by putting all creation operators on the left of the annihilation operators.
- Apply the eikonal operator to discuss the contribution of low-energy gravitons to several observables: waveforms and the energy emission spectrum.
- Given an observable $\mathcal{O}$ the general strategy is to take its expectation value according to

$$\langle \mathcal{O} \rangle = \langle 0 | S^{\dagger}_{s.r.} \mathcal{O} \, S_{s.r.} | 0 \rangle$$

Waveforms

- The classical field is obtained by inserting in the expectation value the free gravitational field

$$H_{\mu\nu} = \int_{\vec{k}} \left[\epsilon^*_{\mu\nu;i} a_i(k) e^{ikx} + \epsilon_{\mu\nu;i} a_i^\dagger(k) e^{-ikx} \right]$$

- We have to compute:

$$h_{\mu\nu}(x) = \int_{\vec{k}} \langle 0| S^\dagger_{s.r.} \left[\epsilon^*_{\mu\nu;i} a_i(k) e^{ikx} + \epsilon_{\mu\nu;i} a_i^\dagger(k) e^{-ikx} \right] S_{s.r.} |0\rangle$$

- We get

$$h_{\mu\nu}(x) = \int_{\vec{k}} e^{-2iRe\delta} \left[\epsilon^*_{\mu\nu;i} f_i(k) e^{ikx} + \epsilon_{\mu\nu;i} f_i^*(k) e^{-ikx} \right] e^{2iRe\delta}$$

- We consider the asymptotic limit for the gravitational field, where $x^\mu = (x^0, \vec{x}) = (u + r, r\hat{x})$ and the detector's distance is taken to infinity, $r \to \infty$, for fixed retarded time u and angles $\hat{x}$:

$$h_i \sim \frac{1}{4i\pi r} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f_i(\omega, \omega\hat{x})\, e^{-i\omega u}$$

- Staying in frequency domain we get for the two leading soft waveforms $(\times, +)$

$$W_{\times,+} = \frac{\kappa}{4\pi r} \epsilon^{\times,+}_{\mu\nu} \sum_n \frac{\kappa\, p_n^\mu p_n^\nu}{p_n \cdot k}$$

where

$$\varepsilon^{\mu\nu}_\times = \epsilon^\mu_\phi \epsilon^\nu_\theta \ , \quad \varepsilon^{\mu\nu}_+ = \frac{1}{2}(\epsilon^\mu_\theta \epsilon^\nu_\theta - \epsilon^\mu_\phi \epsilon^\nu_\phi)$$

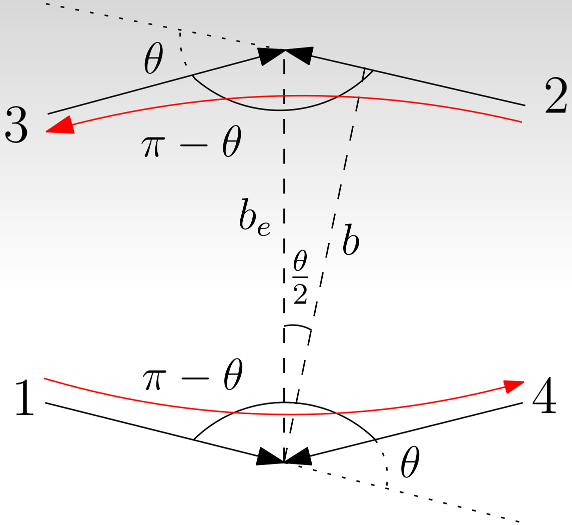
with

$$\epsilon^\mu_\phi = (0, -\sin\phi, \cos\phi, 0)\ ; \epsilon^\mu_\theta = (0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$k^\mu = (\omega, \mathbf{k}, k^L) = \omega n^\mu \ , \quad n^\mu = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

- We specialize our expressions using the following kinematics

$$p_1 = \left(-E_1, \hat{b}_e\, p \sin\frac{\Theta_s}{2}, -p\cos\frac{\Theta_s}{2}\right)$$
$$p_2 = \left(-E_2, -\hat{b}_e\, p \sin\frac{\Theta_s}{2}, +p\cos\frac{\Theta_s}{2}\right)$$
$$p_4 = \left(+E_1, \hat{b}_e\, p \sin\frac{\Theta_s}{2}, +p\cos\frac{\Theta_s}{2}\right)$$
$$p_3 = \left(+E_2, -\hat{b}_e\, p \sin\frac{\Theta_s}{2}, -p\cos\frac{\Theta_s}{2}\right)$$

- Added subindex e to specify that is the $b_{eikonal}$.
- They are obtained from the previous ones + mass-shell conditions.



θ
3
2
$\pi - \theta$
b_e
b
$\frac{\theta}{2}$
1
$\pi - \theta$
4
θ

- We get

$$W_\times^{LS}(k) = \frac{2G}{r}\frac{p}{\omega}\left(-\sin\phi\sin\frac{\Theta_s}{2}\right)$$

$$\left[\frac{\cos\theta\cos\phi\sin\frac{\Theta_s}{2}+\sin\theta\cos\frac{\Theta_s}{2}}{\frac{E_1}{p}+\sin\frac{\Theta_s}{2}\sin\theta\cos\phi-\cos\frac{\Theta_s}{2}\cos\theta}\right.$$

$$+\frac{\cos\theta\cos\phi\sin\frac{\Theta_s}{2}+\sin\theta\cos\frac{\Theta_s}{2}}{\frac{E_2}{p}-(\sin\frac{\Theta_s}{2}\sin\theta\cos\phi-\cos\frac{\Theta_s}{2}\cos\theta)}$$

$$-\frac{\cos\theta\cos\phi\sin\frac{\Theta_s}{2}-\sin\theta\cos\frac{\Theta_s}{2}}{\frac{E_1}{p}-\sin\frac{\Theta_s}{2}\sin\theta\cos\phi-\cos\frac{\Theta_s}{2}\cos\theta}$$

$$\left.-\frac{\cos\theta\cos\phi\sin\frac{\Theta_s}{2}-\sin\theta\cos\frac{\Theta_s}{2}}{\frac{E_2}{p}+\sin\frac{\Theta_s}{2}\sin\theta\cos\phi+\cos\frac{\Theta_s}{2}\cos\theta}\right]$$

- LS stands for leading soft.

- $$W_+^{LS} = \frac{G}{r}\frac{p}{\omega}\Bigg[\frac{\left(\sin\frac{\Theta_s}{2}\cos\phi\cos\theta + \sin\theta\cos\frac{\Theta_s}{2}\right)^2 - \sin^2\phi\sin^2\frac{\Theta_s}{2}}{\frac{E_1}{p} + \sin\frac{\Theta_s}{2}\sin\theta\cos\phi - \cos\frac{\Theta_s}{2}\cos\theta}$$
$$+ \frac{\left(\sin\frac{\Theta_s}{2}\cos\phi\cos\theta + \sin\theta\cos\frac{\Theta_s}{2}\right)^2 - \sin^2\phi\sin^2\frac{\Theta_s}{2}}{\frac{E_2}{p} - \sin\frac{\Theta_s}{2}\sin\theta\cos\phi + \cos\frac{\Theta_s}{2}\cos\theta}$$
$$+ \frac{\left(\sin\frac{\Theta_s}{2}\cos\phi\cos\theta - \sin\theta\cos\frac{\Theta_s}{2}\right)^2 - \sin^2\phi\sin^2\frac{\Theta_s}{2}}{-\frac{E_1}{p} + \sin\frac{\Theta_s}{2}\sin\theta\cos\phi + \cos\frac{\Theta_s}{2}\cos\theta}$$
$$+ \frac{\left(-\sin\frac{\Theta_s}{2}\cos\phi\cos\theta + \sin\theta\cos\frac{\Theta_s}{2}\right)^2 - \sin^2\phi\sin^2\frac{\Theta_s}{2}}{-\frac{E_2}{p} - \sin\frac{\Theta_s}{2}\sin\theta\cos\phi - \cos\frac{\Theta_s}{2}\cos\theta}\Bigg]$$
- They have a complicated dependence on Newton constant G through Θ_s .
- The usual way of performing the PM expansion on the waveforms is to assume that Θ_s is small with respect to any other kinematic ratio and then Taylor expand the results for $\Theta_s \ll 1$.

- At leading PM order we get

$$W_{\times}^{LS}(k) \simeq -\frac{2G}{r}\frac{p\Theta_s}{\omega}\sin\phi\sin\theta\left[\frac{1}{\frac{E_1}{p}-\cos\theta}+\frac{1}{\frac{E_2}{p}+\cos\theta}\right]$$

$$W_{+}^{LS}(k) \simeq \frac{2G}{r}\frac{\Theta_s}{\omega}\cos\phi\sin\theta\Bigg\{-\frac{p}{2}\sin^2\theta$$

$$\times\left[\frac{1}{(\frac{E_1}{p}-\cos\theta)^2}-\frac{1}{(\frac{E_2}{p}+\cos\theta)^2}\right]$$

$$+\frac{\cos\theta\sqrt{s}}{(\frac{E_1}{p}-\cos\theta)(\frac{E_2}{p}+\cos\theta)}\Bigg\}$$

- The results above diverge when Θ_s is small but fixed and we send m_i to zero (or equivalently we take $\sigma \to \infty$) and we also take the $\theta \to 0$ limit.
- For $W_{\times}^{LS}$ this is already visible at the leading $\mathcal{O}(\Theta_s)$ PM-order.
- In the limit $E_i/p \to 1$, the denominators vanish quadratically as $\theta \to 0$ or $\theta \to \pi$ while the prefactor vanishes just linearly.

- However, when $\theta \simeq \Theta_s \ll 1$ the corrections to the expressions in the denominators become important and in particular they compete with the leading term $E_i/p - 1$ when

$$\max\left\{ \frac{m_1}{m_2}\sigma\,\Theta_s^2,\ \frac{m_2}{m_1}\sigma\,\Theta_s^2 \right\} \gtrsim 1$$

- The bound above has been discussed in papers by D'Eath, Kovacs and Thorne
- The dominant PM contributions to the ZFL come from the polarisation $W_\times^{LS}$ and is of order Θ_s, while W_+^{LS} is of order Θ_s^3.
- The denominator for Θ_s small and $\theta \sim \Theta_s$ is equal:

$$E_1 - p\cos\theta + \frac{p\Theta_s^2}{2} \sim p - p\cos\theta + \frac{m_1^2}{2p} + \frac{p\Theta_s^2}{2} \sim \frac{m_1^2}{2p} + \frac{p\Theta_s^2}{2}$$

- If $p^2\Theta_s^2 > m_1^2$ we cannot neglect the last term in the denominator.
- The previous equation implies:

$$p^2\Theta_s^2 > m_1^2 \Longrightarrow \frac{m_2}{m_1}\sigma\Theta_s^2 > 1$$

- That something should be done comes also from the fact that the ratio

$$\frac{E^{rad}}{\sqrt{s}} = \Theta_s^3 \sqrt{\sigma} \ ,$$

at high enough energy, becomes bigger that 1.

- In the massless limit we have $E_i/p \to 1$, $p \to \sqrt{s}/2$ and the waveforms takes a simpler form

$$W_{\times}^{LS}(k) = -\frac{2G}{r}\frac{\sqrt{s}}{\omega}\sin\phi\sin\frac{\Theta_s}{2} \times \sum_{\alpha=\pm 1}\left[\frac{\alpha\cos\theta\cos\phi\sin\frac{\Theta_s}{2} + \sin\theta\cos\frac{\Theta_s}{2}}{1-(\sin\frac{\Theta_s}{2}\sin\theta\cos\phi - \alpha\cos\frac{\Theta_s}{2}\cos\theta)^2}\right]$$

$$W_{+}^{LS}(k) = -\frac{2G}{r}\frac{\sqrt{s}}{\omega}\sum_{\alpha=\pm 1}\left[\frac{\alpha\sin^2\phi\sin^2\frac{\Theta_s}{2}}{1-(\sin\frac{\Theta_s}{2}\sin\theta\cos\phi - \alpha\cos\frac{\Theta_s}{2}\cos\theta)^2}\right]$$

Infrared divergences

- Using the BCH formula

$$e^{va^\dagger - v^* a} = e^{va^\dagger} e^{-v^* a} e^{-\frac{1}{2}|v|^2[a,a^\dagger]}$$

- to compute

$$\langle 0|S_{s.r.}|0\rangle = \exp\left[-\frac{1}{2}\int_{\vec{k}} F^*_{\mu\nu}(k)\Pi^{\mu\nu,\rho\sigma}F_{\rho\sigma}(k)\right] e^{2i\,\mathrm{Re}\,\delta(\sigma,b)}$$

where $\Pi^{\mu\nu,\rho\sigma}$ is the usual transverse-traceless projector and

$$F^{\mu\nu} = \sum_n \frac{\kappa p_n^\mu p_n^\nu}{p_n k}$$

- Since $F_{\mu\nu}$ is transverse we can use

$$F^*_{\mu\nu}\Pi^{\mu\nu,\rho\sigma}F_{\rho\sigma} = F^*_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \frac{1}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right)F_{\rho\sigma}$$

- We get

$$\langle 0|S_{s.r.}|0\rangle = \exp\left[-\frac{1}{2}\int_{\vec{k}}\sum_{n,m}\frac{\kappa^2}{\hbar}\frac{(p_n p_m)^2 - \frac{m_m^2 m_n^2}{D-2}}{(p_n k)(p_m k)}\right] e^{2i\,\mathrm{Re}\,\delta(\sigma,b)}$$

- The first exponential is a damping factor that can be interpreted as an imaginary contribution to the classical eikonal.
- In this way we have

$$\langle 0|S_{s.r.}|0\rangle = e^{2i\delta(\sigma,b)}$$

where

$$\mathrm{Im}\,2\delta(\sigma,b) = \frac{1}{2}\int_{\vec{k}}\sum_{n,m}\frac{\kappa^2}{\hbar}\frac{(p_n p_m)^2 - \frac{m_m^2 m_n^2}{D-2}}{(p_n k)(p_m k)}$$

- The dependence on the impact parameter b is implicit through the identification $Q^\mu \to \hat{b}^\mu 2p\sin\frac{\Theta_s}{2}$.

- To the leading order in ϵ we get

$$\operatorname{Im} 2\delta(\sigma, b) = -\frac{\kappa^2 \omega_*^{-2\epsilon}}{(4\pi)^2 \epsilon} \sum_{n,m} m_n m_m \left(\sigma_{nm}^2 - \frac{1}{2} \right) F_{nm}$$

where

$$F_{nm} = \frac{\eta_n \eta_m \operatorname{arccosh} \sigma_{nm}}{\sqrt{\sigma_{nm}^2 - 1}} , \qquad \sigma_{nm} = -\eta_n \eta_m \frac{p_n \cdot p_m}{m_n m_m}$$

$\eta_n = 1$ for outgoing and $\eta_n = -1$ for ingoing momentum.

- The dependence on the kinematics of elastic process is exact.
- Moreover, as expected, $\operatorname{Im} 2\delta > 0$ for $\epsilon < 0$, which grants the convergence of the integral.
- This means that $e^{-\operatorname{Im} 2\delta}$ is indeed an exponentional suppression.

- An explicit expression for $\text{Im}\, 2\delta$ for gravitational $2 \to 2$ scattering is obtained by using $\sigma_{nn} = 1$ and $F_{nn} = 1$, while for $n \neq m$ we have $\sigma_{nm} = \sigma_{mn}$ and

$$\sigma_{12} = \sigma_{34} = \sigma \,, \quad \sigma_{13} = \sigma_{24} = \sigma_Q \,, \quad \sigma_{14} = 1 + \frac{Q^2}{2m_1^2} \,;\, \sigma_{23} = 1 + \frac{Q^2}{2m_2^2}$$

- We introduce the shorthand notation $\sigma_Q = \sigma - \frac{Q^2}{2m_1 m_2}$

- We get ($|Q| = 2p \sin \frac{\Theta_s}{2}$)

$$\mathrm{Im}\, 2\delta(\sigma, b) = -\frac{G}{\pi\epsilon}\Bigg[2m_1 m_2 \left(\sigma^2 - \tfrac{1}{2}\right) \frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2 - 1}}$$
$$- 2m_1 m_2 \left(\sigma_Q^2 - \tfrac{1}{2}\right) \frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2 - 1}}$$
$$+ \frac{m_1^2}{2} - m_1^2 \left(\left(1 + \tfrac{Q^2}{2m_1^2}\right)^2 - \tfrac{1}{2}\right) \frac{\operatorname{arccosh}\left(1 + \frac{Q^2}{2m_1^2}\right)}{\sqrt{\left(1 + \frac{Q^2}{2m_1^2}\right)^2 - 1}} \tag{1}$$
$$+ \frac{m_2^2}{2} - m_2^2 \left(\left(1 + \frac{Q^2}{2m_2^2}\right)^2 - \tfrac{1}{2}\right) \frac{\operatorname{arccosh}\left(1 + \frac{Q^2}{2m_2^2}\right)}{\sqrt{\left(1 + \frac{Q^2}{2m_2^2}\right)^2 - 1}} \Bigg]$$

- The standard relativistic regime requires that

$$Q^2 \sim (p\Theta_s)^2 \ll 2m_i^2$$

- In this case only the first two lines contribute and one can extract the 3PM contribution:

$$\text{Im}\, 2\delta(\sigma, b) \simeq -\frac{GQ^2}{2\pi\epsilon}\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{(2\sigma^2-3)\sigma\cosh^{-1}\sigma}{(\sigma^2-1)^{3/2}}\right]$$

- In this limit we reproduce the result obtained from the 3-particle cut.
- In the case of $\mathcal{N} = 8$ massive supergravity one gets:

$$\text{Im}\, 2\delta(\sigma, b) \simeq -\frac{GQ^2}{\pi\epsilon}\left[\frac{\sigma^2}{\sigma^2-1} + \frac{\sigma(\sigma^2-1)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)\right]$$

- It looks universal at high energy, but the factor $\log\frac{s}{m_1 m_2}$ is singular for zero masses.
- As discussed above the PM approximation can break down even when Θ_s is small if the energy is high enough.

- This happens when

$$\frac{Q}{\sqrt{2}m_i} \gtrsim 1 \quad \Rightarrow \quad \frac{\sqrt{2}p}{m_i} \sin \frac{\Theta_s}{2} \gtrsim 1$$

- In this regime, one cannot expand the last two lines of Eq. (1) for small $\Theta_s \sim G$.
- It is particularly interesting to focus on the extreme ultrarelativistic regime, or equivalently the massless limit, where $2p \to \sqrt{s}$ and $m_1, m_2 \ll Q = \sqrt{s} \sin \frac{\Theta_s}{2}$.
- Then one gets:

$$\text{Im}\, 2\delta(\sigma, b) \simeq -\frac{G}{\pi\epsilon} \left[s \log \frac{s}{s - Q^2} + Q^2 \log \frac{s - Q^2}{Q^2} \right]_{Q=\sqrt{s}\sin\frac{\Theta_s}{2}} ,$$

- It can be put in the form:

$$\text{Im}\, 2\delta(\sigma, b) \simeq \frac{Gs}{\pi\epsilon} \left[\cos^2 \frac{\Theta_s}{2} \log \cos^2 \frac{\Theta_s}{2} + \sin^2 \frac{\Theta_s}{2} \log \sin^2 \frac{\Theta_s}{2} \right] .$$

- Exactly the same result also describes the extreme ultrarelativistic regime of $\mathcal{N} = 8$ supergravity: universality at high energy.

Number of emitted quanta

- We can insert 1 and compute the probability of emission of N gravitons

$$\mathcal{P}_N = \langle 1 \rangle_N \,, \qquad \langle 1 \rangle = \sum_{N=0}^{\infty} \mathcal{P}_N = 1 \,.$$

- We get

$$\mathcal{P}_N = \frac{1}{N!} \sum_{j_1,\ldots,j_N} \int_{\vec{k}_1} \cdots \int_{\vec{k}_N} \langle 0|S^\dagger_{s.r.} a^\dagger_{j_1} \cdots a^\dagger_{j_N}|0\rangle\langle 0|a_{j_1} \cdots a_{j_N} S_{s.r.}|0\rangle \,.$$

-

$$\mathcal{P}_N = \frac{1}{N!}\langle 0|S^\dagger_{s.r.}|0\rangle \left[\int_{\vec{k}} F^*_{\mu\nu}(k)\Pi^{\mu\nu,\rho\sigma} F_{\rho\sigma}(k)\right]^N \langle 0|S_{s.r.}|0\rangle \,.$$

- Recognizing the same integral that appeared before, up to a crucial factor of -2, and using the fact that $\langle 0|S_{s.r.}|0\rangle = e^{2i\delta}$, we thus have

$$\mathcal{P}_N = \frac{1}{N!} \left[2 \; \mathrm{Im}\, 2\delta\right]^N e^{-2\,\mathrm{Im}\, 2\delta}$$

- In this way we obtain that the probability for the emission of N gravitons follows the Poisson distribution

$$\mathcal{P}_N = \frac{(\bar{n})^N}{N!} e^{-\bar{n}} \qquad \bar{n} = 2 \ \text{Im}\, 2\delta(\sigma, b)$$

- where $\bar{n}$ is the average number of emitted gravitons

$$\sum_{N=0}^{\infty} N \mathcal{P}_N = \bar{n}$$

Energy spectrum

- The ZFL of the energy spectrum can be easily obtained from $\mathrm{Im}(2\delta_2)$ by the following observation.
- In the case of $\mathrm{Im}\, 2\delta_2$ we need to compute the following integral:

$$\frac{1}{2}\int_0^{\omega_*} \frac{d|\vec{k}\,|}{|\vec{k}\,|^{1+2\epsilon}} = -\frac{\omega_*^{-2\epsilon}}{4\epsilon}$$

- In the case of the ZFL we have to multiply with $\hbar\omega$ and we get a convergent integral.
- This implies

$$\lim_{\omega\to 0} \frac{dE^{\mathrm{rad}}}{d\omega} = \lim_{\epsilon\to 0} \left[-4\epsilon\, \mathrm{Im}\, 2\delta(\sigma, b)\right]$$

- In the standard PM regime one gets:

$$\lim_{\omega\to 0} \frac{dE_{\rm rad}}{d\omega} \simeq \frac{2G}{\pi}(p\,\Theta_s)^2\left[\frac{8-5\sigma^2}{3(\sigma^2-1)}+\frac{(2\sigma^2-3)\sigma\,\mathrm{arccosh}\,\sigma}{(\sigma^2-1)^{3/2}}\right]$$

- We find again logarithmic divergent terms if we send $\sigma\to\infty$ keeping Θ_s fixed.
- To restore universality we must use the following expression:

$$\mathrm{Im}\, 2\delta(\sigma, b) \simeq -\frac{G}{\pi\epsilon}\left[s\log\frac{s}{s-Q^2}+Q^2\log\frac{s-Q^2}{Q^2}\right]_{Q=\sqrt{s}\sin\frac{\Theta_s}{2}}$$

- We get

$$\frac{dE^{\rm rad}}{d\omega}(\omega\to 0)\simeq -\frac{4Gs}{\pi}\left[\cos^2\frac{\Theta_s}{2}\log\cos^2\frac{\Theta_s}{2}+\sin^2\frac{\Theta_s}{2}\log\sin^2\frac{\Theta_s}{2}\right]$$

that agrees with the leading soft limit of
Sahoo and Sen, 2105.08739

- At leading order for $\Theta_s << 1$ we get

$$\frac{dE^{\text{rad}}}{d\omega}(\omega \to 0) \simeq \frac{Gs\Theta_s^2}{\pi}\left[1 + \log\frac{4}{\Theta_s^2}\right]$$

- It reproduces the result obtained by
  Gruzinov and Veneziano, 1409.4555
  within a classical GR approach.
- The same result has been obtained by
  Ciafaloni, Colferai and Veneziano, 1812.08137
  from a scattering amplitude perspective.
- Needless to say that the same result holds also per $\mathcal{N} = 8$ supergravity: universality at high energy.

Extension to the case of spin

- In the case with spin, connected to the Kerr black-hole, the four-point amplitude to use is:

$$\mathcal{A}_4^{tree}(q,\sigma) = \frac{8\pi G m_1^2 m_2^2}{q^2}\sigma^2 \sum_{\pm}(1\pm v)^2 \exp\left(\pm i\mathbf{q}\cdot(\hat{\mathbf{p}}\times\mathbf{a})\right)$$

  A. Guevara, A. Ochirov and J. Vines, 1812.06895, 1906.10071
  Y.F. Bautista and A. Guevara, 1903.12419

- We work in the center-of-mass frame, defined by

$$p_1 = (E_1, \mathbf{p}), \qquad p_2 = (E_2, -\mathbf{p}), \qquad |\mathbf{p}| = p\ ;\ \hat{\mathbf{p}} = \frac{\mathbf{p}}{p}$$

- The spin is introduced through the quantities:

$$a_i^\mu = \frac{S_i^\mu}{m_i}\ \ ,\ \ i = 1,2\ \ ;\ \ a = a_1 + a_2$$

- We take

$$\mathbf{p} = pe_3\ \ ;\ \ \mathbf{b} = be_1\ \ ;\ \ a = ae_2$$

  if the spins are aligned.

- The leading soft term of the 5-point amplitude is given by

$$\mathcal{A}_5^{\mu\nu}(q,k) = \kappa \sum_{i=1}^{4} \frac{p_i^\mu p_i^\nu}{k \cdot p_i} \mathcal{A}_4^{\mathrm{tree}}(q,\sigma) + \mathcal{O}(k^0)$$

- We can go to impact parameter space:

$$\tilde{\mathcal{A}}_5^{\mu\nu}(b,k) \simeq \frac{i\kappa^3}{8\pi} \frac{m_1 m_2 \sigma^2}{\sqrt{\sigma^2-1}} \sum_{\pm} \frac{(1\pm v)^2}{\mathbf{b}_\pm^2} \Bigg(\frac{\bar{p}_1^\mu \bar{p}_1^\nu}{(\bar{p}_1 \cdot k)^2} (\mathbf{k}\cdot\mathbf{b}_\pm) - \frac{\bar{p}_1^\mu b_\pm^\nu + \bar{p}_1^\nu b_\pm^\mu}{\bar{p}_1 \cdot k} - 1 \leftrightarrow 2 \Bigg)$$

where $b_\pm^\mu \equiv (0, \mathbf{b} \pm \hat{\mathbf{p}} \times \mathbf{a})$.

- We are now ready to compute the infrared divergent part of $\mathrm{Im}\, 2\delta_2$

$$\mathrm{Im}\, 2\delta_2(b,\sigma) = \frac{1}{2} \int \frac{d^{3-2\epsilon}\mathbf{k}}{2\,|\mathbf{k}|\,(2\pi)^{3-2\epsilon}} \tilde{\mathcal{A}}_5^{\mu\nu}(b,k) P_{\mu\nu;\rho\sigma} \tilde{\mathcal{A}}_5^{*\rho\sigma}(b,k)$$

- We get

$$\mathrm{Im}\,2\delta_2(\sigma,b) \simeq -\frac{1}{2\epsilon}\frac{\pi}{2\hbar(2\pi)^3}\left(\frac{\kappa^3 m_1 m_2 \sigma^2}{8\pi\sqrt{\sigma^2-1}}\right)^2\left(\sum_{\pm}\frac{(1\pm v)^2\mathbf{b}_\pm}{\mathbf{b}_\pm^2}\right)^2\mathcal{I}(\sigma)$$

where

$$\mathcal{I}(\sigma) = 2\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)\right]$$

- Introducing the spatial vector

$$\sum_{\pm}\frac{(1\pm v)^2\mathbf{b}_\pm}{\mathbf{b}_\pm^2} \equiv \frac{2(2\sigma^2-1)}{\sigma^2 b}\mathbf{f}(a,b,\sigma)$$

- we finally get

$$\mathrm{Im}\,2\delta_2(\sigma,b) \simeq -\frac{1}{2\epsilon}\frac{G\beta^2(\sigma)}{2\pi\hbar(\sigma^2-1)b^2}\mathcal{I}(\sigma)\mathbf{f}^2(a,b,\sigma).$$

- When the spins are parallel to the orbital angular momentum the vector **f** is

$$\mathbf{f}(a,b,\sigma) \equiv f(a,b,\sigma)\frac{\mathbf{b}}{b}, \qquad f(a,b,\sigma) = \frac{1 + \frac{2\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}}{1 - \left(\frac{a}{b}\right)^2}$$

- Otherwise **f** has non-vanishing components also along $\hat{\mathbf{p}} \times \mathbf{a}$

- Simple rule to go from $a = 0$ to $a \neq 0$:

$$\left(\frac{2(2\sigma^2 - 1)}{\sigma^2 b^2}\mathbf{b}\right)^2 = \left(\frac{(1+v)^2\mathbf{b}}{\mathbf{b}^2} + \frac{(1-v)^2\bar{\mathbf{b}}}{\bar{\mathbf{b}}^2}\right)^2$$

where we introduced an auxiliary vector $\bar{\mathbf{b}}$ such that $\bar{\mathbf{b}} \overset{\mathbf{a}=0}{=} \mathbf{b}$.

- Then to get the result for $\mathbf{a} \neq 0$ we just need to do the replacements

$$\mathbf{b} \xrightarrow{\mathbf{a}\neq 0} \mathbf{b}_+, \qquad \bar{\mathbf{b}} \xrightarrow{\mathbf{a}\neq 0} \mathbf{b}_-.$$

- With this substitution one goes from Schwarzschild to Kerr reminiscent of the Newman-Janis shift.
- The ZFL of the spectrum of emitted energy:

$$\left.\frac{dE^{\mathrm{rad}}}{d\omega}\right|_{\omega\to 0} = \frac{4G^3 m_1^2 m_2^2 (2\sigma^2 - 1)^2}{\pi b^2(\sigma^2 - 1)}\mathcal{I}(\sigma)\mathbf{f}^2(a, b, \sigma)$$

- One gets the radiative contribution to the real part of the eikonal:

$$\mathrm{Re}\, 2\delta_2^{\mathrm{rr}}(\sigma, b) = \frac{G\beta^2(\sigma)}{4\hbar(\sigma^2-1)b^2}\mathcal{I}(\sigma)\mathbf{f}^2(a, b, \sigma)$$
$$= \mathrm{Re}\, 2\delta_2^{\mathrm{rr}}(\sigma, b)\Big|_{\mathbf{a}=0}\mathbf{f}^2(a, b, \sigma)$$

- and the radiative part of 3PM deflection angle:

$$\theta_3^{\mathrm{rad}}(\sigma, b) = \frac{G\beta^2(\sigma)}{2(\sigma^2-1)pb^3}\mathcal{I}(\sigma)$$
$$\times \frac{\left(1+\frac{2\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}\right)\left[1+\frac{4\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}+\left(\frac{a}{b}\right)^2\right]}{\left[1-\left(\frac{a}{b}\right)^2\right]^3}$$

- For spin 1 we get

$$\theta_3^{\rm rad}(\sigma, b) = \frac{G\beta^2(\sigma)}{2(\sigma^2-1)pb^3}\mathcal{I}(\sigma)$$
$$\times\left(1+\frac{6\sigma\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{a}{b}+4\frac{6\sigma^4-6\sigma^2+1}{(2\sigma^2-1)^2}\frac{a^2}{b^2}\right)$$

that agrees with [Jakobsen and Mogull, 2201.07778].
- When added to the conservative part one gets a perfectly well defined deflection angle at high energy.
- Assuming the same cancellation for arbitrary spin, from the high energy behaviour of the real part of the two-loop eikonal given by

$$\mathrm{Re}\, 2\delta_2^{\rm rr}(\sigma, b) \sim \frac{16G^3 m_1^2 m_2^2}{(1-\frac{a}{b})^2}\sigma^2\log\sigma \qquad \sigma\to\infty$$

we can deduce the high energy limit of the conservative part of the real part of the two-loop eikonal

$$\mathrm{Re}\, 2\delta_2^{\rm cons}(\sigma, b) \sim -\frac{16G^3 m_1^2 m_2^2}{(1-\frac{a}{b})^2}\sigma^2\log\sigma \qquad \sigma\to\infty$$

- This implies that the high energy limit of the conservative scattering angle must be equal to

$$\theta_3^{\rm cons}(\sigma, b) \simeq -\frac{32G^3 E m_1 m_2}{(1-\frac{a}{b})^3}\sigma\log\sigma \qquad \sigma\to\infty$$

- Finally, from the Bini-Damour relation:

$$\theta^{\rm rad} = -\frac{1}{2}\frac{\partial\theta^{\rm cons}}{\partial E}E^{\rm rad} - \frac{1}{2}\frac{\partial\theta^{\rm cons}}{\partial J}J^{\rm lost}$$

and the conservative deflection angle $\theta_1^{\rm cons}$

$$\theta_1^{\rm cons}(\sigma, b) = \frac{\beta(\sigma)}{pb\sqrt{\sigma^2-1}}f(a,b,\sigma) = \frac{\beta(\sigma)}{J\sqrt{\sigma^2-1}}\left(\frac{1+\frac{2\sqrt{\sigma^2-1}}{2\sigma^2-1}\frac{pa}{J}}{1-\left(\frac{pa}{J}\right)^2}\right)$$

we can extract the loss of angular momentum.

- We find the 2PM loss of angular momentum $(Q = p\Theta_s, \Theta_s = \frac{\beta(\sigma) f(a,b,\sigma)}{J\sqrt{\sigma^2-1}})$

$$\mathbf{J}_2^{\text{lost}}(\sigma, b) = J\frac{G\beta(\sigma)}{b^2\sqrt{\sigma^2-1}}\mathcal{I}(\sigma)\hat{\mathbf{p}} \times \left(f(a,b,\sigma)\frac{\mathbf{b}}{b}\right)$$

$$= J\frac{G\beta(\sigma)}{b^2\sqrt{\sigma^2-1}}\mathcal{I}(\sigma) f(a,b,\sigma)\mathbf{e}_2 = \frac{p}{Q}\lim_{\epsilon\to 0}\left[-4\pi\epsilon \operatorname{Im} 2\delta\right] f(a,b,\sigma)\mathbf{e}_2$$

  in agreement with the angular momentum computed by [C.Heissenberg, R.Russo, PDV, 2203.11915] with $f = 1$.
- Note that the angular momentum is lost only along the $\mathbf{e}_2$ direction, perpendicular to the scattering plane.
- This is due to the fact that in the aligned-spin case the scattering dynamics is planar, just as in the spinless scenario.

- A natural generalisation to non-aligned spin is

$$\mathbf{J}_2^{\text{lost}}(\sigma, b) = J \frac{G\beta(\sigma)}{b^2\sqrt{\sigma^2 - 1}} \mathcal{I}(\sigma) \hat{\mathbf{p}} \times \mathbf{f}(a, b, \sigma)$$

- In this case **f** does not lie entirely along **b**, but has also one non-vanishing component along $\hat{\mathbf{p}} \times \mathbf{a}$.
- J^{lost} for spin one agrees with the expression found by [G.Jakobsen and G.Mogull, 2201.07778]
- It agrees also with [C.Heissenberg, R.Russo, PDV, 2203.11915] for non-aligned spin.

Beyond the soft limit but below the KT bound

- Most relevant paper is the one by [Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola and White, 2112.07556].
- In the elastic scattering write the momenta

$$p_4 = -\bar{p}_1 + \frac{Q}{2} \; ; \; p_3 = -\bar{p}_2 - \frac{Q}{2} \; ; \; \bar{p}_{1,2} Q = 0$$

$$p_1 = \bar{p}_1 + \frac{Q}{2} \; ; \; p_2 = \bar{p}_2 - \frac{Q}{2}$$

- In KMOC one starts from an in state:

$$|in\rangle = \int \frac{d^D p_1}{(2\pi)^D} \left[(2\pi)\delta(p_1^2 + m_1^1)\theta(p_1^0) \right] \int \frac{d^D p_2}{(2\pi)^D} \times \left[(2\pi)\delta(p_2^2 + m_2^1)\theta(p_2^0) \right] \Phi(-p_1)\Phi(-p_2) e^{ip_1 b_1 + ip_2 b_2} | -p_1, -p_2\rangle$$

- that can be rewritten as follows:

$$\prod_{i=1,2} \left[\int \frac{d^D p_i}{(2\pi)^D} \right] (2\pi)^2 \delta(2Q\bar{p}_1)\delta(2Q\bar{p}_2)\Phi(-p_1)\Phi(-p_2) e^{-ip_1 b_1 - ip_2 b_2} \times | -p_1, -p_2\rangle$$

- Then one introduces an out state

$$S|in\rangle = |out\rangle = |in\rangle + iT|in\rangle$$

- where

$$S|in\rangle = \int \prod_{i=1}^{4} \left(\frac{d^{D-1} p_i}{2E_i (2\pi)^{D-1}} \right) \Phi(-p_1)\Phi(-p_2)$$

$$\times\, e^{ip_1 b_1 + ip_2 b_2} |p_3, p_4\rangle\langle p_3, p_4|S| - p_1, -p_2\rangle$$

- with

$$\langle p_3, p_4|S| - p_1, -p_2\rangle = \int \frac{d^D Q}{(2\pi)^D}$$

$$\times (2\pi)^D \delta^D(p_1 + p_4 - Q)(2\pi)^D \delta^D(p_2 + p_3 + Q) iA(s_{12}, Q^2)$$

where $s_{12} = -(p_1 + p_2)^2$.

- We can then write

$$i(2\pi)\delta(2\bar{p}_2 Q)(2\pi)\delta(2\bar{p}_1 Q)A(s_{12}, Q^2) = \int d^D b_e e^{2i\delta(\tilde{b}_e)} e^{-ib_e Q}$$

- and then we get

$$S|in\rangle = \left(\prod_{i=3}^{4} \frac{d^{D-1}p_i}{(2\pi)^{D-1}2E_i}\right) |p_3, p_4\rangle e^{-ib_1 p_4} e^{-ib_2 p_3} \int \frac{d^D Q}{(2\pi)^D} \int d^D b_e$$
$$\times e^{iQ(b_1 - b_2)} e^{2i\delta(\tilde{b}_e)} e^{-ib_e Q} \Phi(p_4 - Q)\Phi(Q + p_3)$$

- Perform the integrals by saddle point.

- Kinematic in the inelastic case:

$$p_4 = -\bar{p}_1 + \frac{Q}{2} - \frac{K}{2} \;\; ; \;\; p_3 = -\bar{p}_2 - \frac{Q}{2} - \frac{K}{2}$$
$$p_1 = \bar{p}_1 + \frac{Q}{2} \;\; ; \;\; p_2 = \bar{p}_2 - \frac{Q}{2}$$

- Inelastic case up to two loops

$$S|in\rangle = \int \frac{d^{D-1}p_3}{(2\pi)^{D-1}2E_3} \int \frac{d^{D-1}p_4}{(2\pi)^{D-1}2E_4} e^{-ib_1 p_4} e^{-ib_2 p_3} \int d^D\lambda \int d^D b_e$$
$$\int \frac{d^D Q}{(2\pi)^D} \int \frac{d^D K}{(2\pi)^D} \; e^{i\lambda K} e^{iQ(b_1-b_2)} e^{-ib_e Q} e^{-\frac{i}{2}K(b_1+b_2)} e^{2i\,\mathrm{Re}\,\delta(\tilde{b}_e)}$$
$$\exp\left[\int \frac{d^D k}{2\omega(2\pi)^{D-1}} \sum_{j=X,+} \left(\tilde{A}_j(k,b_e)a_j(k)e^{i\lambda k} - \tilde{A}_j(k,b_e)a_j^\dagger(k)e^{-i\lambda k}\right)\right]$$
$$\Phi\left(p_4 - Q + \frac{K}{2}\right)\Phi\left(Q + p_3 + \frac{K}{2}\right)|p_3, p_4, 0\rangle$$

where now A_j is the 5-point amplitude in impact parameter space.

- $\mathrm{Re}\,\delta(\tilde{b}_e)$ is up to two loops.
- k is the momentum of a single graviton emitted while K is the sum of the momenta of all graviton emitted.
- This is enforced by the integral over λ.
- By choosing $b_1 = b_1'$ and $b_2 = b_2'$ the norm of the in state is equal to

$$\langle in' | in \rangle = \int \frac{d^{D-1} p_1}{2E_1 (2\pi)^{D-1}} \int \frac{d^{D-1} p_2}{2E_2 (2\pi)^{D-1}} |\Phi(-p_1)|^2 |\Phi(-p_2)|^2$$

- where the following equation has been used:

$$\langle -p_1', -p_2' | -p_1, -p_2 \rangle = (2\pi)^{D-1} 2E_1 \delta^{D-1}(p_1 - p_1')(2\pi)^{D-1} \times 2E_2 \delta^{D-1}(p_2 - p_2')$$

- One can also compute the norm of the out state
- Unitarity of the S-matrix requires the two norms to be equal.
- Solve all integrals by saddle point.

- From the λ saddle we get:

$$K^\mu = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2\omega} k^\mu \sum_{j=+,X} \tilde{A}_j^*(k, b_e) A_j(k, b_e)$$

- From the b_e saddle we get

$$Q^\mu = \frac{\partial 2\,\mathrm{Re}\,\delta^{cons}(|\tilde{b}_e|)}{\partial |\tilde{b}_e|} \frac{\tilde{b}_e^\mu}{|\tilde{b}_e|} - \frac{i}{2} \int \frac{d^{D-1}k}{(2\pi)^{D-1}2\omega}$$
$$\times \sum_{j=+,X} \left(\tilde{A}_j^*(k, b_e) \frac{\partial}{\partial b_{e\mu}} \tilde{A}_j(k, b_e) - \frac{\partial}{\partial b_{e\mu}} \tilde{A}_j^*(k, b_e) \tilde{A}_j(k, b_e) \right)$$

- and from the Q saddle we get ($b_1 - b_2 \equiv -b$)

$$-b^\mu + (b_e)^\mu = \frac{\partial 2\,\mathrm{Re}\,\delta(|\tilde{b}_e|)}{\partial |\tilde{b}_e|} \frac{\partial |\tilde{b}_e|}{\partial Q_\mu}$$

- Since

$$\frac{\partial}{\partial b_{e\mu}} = \frac{1}{2}\left(\frac{\partial}{\partial b_{1\mu}} - \frac{\partial}{\partial b_{2\mu}}\right)$$

one can compute Q_{1rad} and Q_{2rad} getting:

$$Q^{\mu}_{1rad} = \frac{G^3 m_1^2 m_2^2}{\tilde{b}_e^3}\left[\frac{\tilde{b}_e^{\mu}}{|\tilde{b}_e|}\frac{4\mathcal{E}^{LS}(\sigma)}{\sqrt{\sigma^2-1}} - \frac{\sigma u_1^{\mu} - u_2^{\mu}}{\sigma^2-1}\mathcal{E}(\sigma)\right]$$

and

$$Q^{\mu}_{2rad} = \frac{G^3 m_1^2 m_2^2}{\tilde{b}_e^3}\left[-\frac{\tilde{b}_e^{\mu}}{|\tilde{b}_e|}\frac{4\mathcal{E}^{LS}(\sigma)}{\sqrt{\sigma^2-1}} - \frac{\sigma u_2^{\mu} - u_1^{\mu}}{\sigma^2-1}\mathcal{E}(\sigma)\right]$$

- The first terms come from zero frequency gravitons, while the second terms are obtained from reverse unitarity.

- They imply:

$$Q^{\mu}_{rad} = \frac{1}{2}(Q_1 - Q_2)^{\mu}_{rad} = \frac{G^3 m_1^2 m_2^2}{\tilde{b}_e^3}\left[\frac{\tilde{b}^{\mu}_e}{|\tilde{b}_e|}\frac{4\mathcal{E}^{LS}(\sigma)}{\sqrt{\sigma^2-1}} - \frac{u_1^{\mu} - u_2^{\mu}}{2(\sigma-1)}\mathcal{E}(\sigma)\right]$$

- $\mathcal{E}^{LS}$ is given by

$$\mathcal{E}^{LS}(\sigma) = f_1^{LS}(\sigma) + f_3^{LS}(\sigma)\frac{\sigma\cosh^{-1}(\sigma)}{2\sqrt{\sigma^2-1}}$$

where

$$f_1^{LS}(\sigma) = -\frac{(2\sigma^2-1)^2(5\sigma^2-8)}{3(\sigma^2-1)^{\frac{3}{2}}}\ ;\ f_3^{LS}(\sigma) = \frac{2(2\sigma^2-1)^2(2\sigma^2-3)}{(\sigma^2-1)^{\frac{3}{2}}}$$

- This is the quantity computed before:

$$\hat{Q}_{rad} = -\frac{\partial 2\,\mathrm{Re}\,\delta_2^{RR}}{\partial b} = \frac{2}{b}\lim_{\epsilon\to 0}\left[-\pi\epsilon Im 2\delta_2\right]$$

- It is equal to

$$\hat{Q}_{rad} = \frac{GQ_{1PM}^2}{\tilde{b}_e}\left[\frac{8-5\sigma^2}{3(\sigma^2-1)} + \frac{\sigma(2\sigma^2-3)\cosh^{-1}(\sigma)}{(\sigma^2-1)^{\frac{3}{2}}}\right]$$

where

$$Q_{1PM} = \frac{2Gm_1 m_2(2\sigma^2-1)}{b\sqrt{\sigma^2-1}}$$

- With reverse unitarity one can also compute:

$$K^\mu = \int \frac{d^{D-1}k}{(2\pi)^{D-1}2\omega} k^\mu \sum_{j=+,X} \tilde{A}_j^*(k, b_e) A_j(k, b_e)$$

$$= \frac{G^3 m_1^2 m_2^2}{\tilde{b}_e^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma)$$

- One gets:

$$Q_{1rad}^\mu + Q_{2rad}^\mu + K^\mu = 0$$

- We can now use the eikonal operator to compute observables.
- The first observable is the change of momenta $\Delta p_1 = p_1 + p_4$ and $\Delta p_2 = p_2 + p_3$ that are given by

$$\langle (\Delta p_1)^\mu \rangle \equiv \langle in' | S^\dagger (\Delta p_1)^\mu S | in \rangle = Q^\mu - \frac{K^\mu}{2} = \frac{\tilde{b}_e^\mu}{|\tilde{b}_e|} \frac{\partial 2\text{Re}\delta^{cons}(|\tilde{b}_e|)}{\partial |\tilde{b}_e|}$$

$$+ \frac{G^3 m_1^2 m_2^2}{b^3} \left[\frac{4\mathcal{E}^{LS}}{\sqrt{\sigma^2 - 1}} \frac{\tilde{b}_e^\mu}{|\tilde{b}_e|} - \frac{u_1^\mu \sigma - u_2^\mu}{\sigma^2 - 1} \mathcal{E}(\sigma) \right]$$

and

$$\langle (\Delta p_2)^\mu \rangle \equiv \langle in' | S^\dagger (\Delta p_2)^\mu S | in \rangle = -Q^\mu - \frac{K^\mu}{2} = -\frac{\tilde{b}_e^\mu}{|\tilde{b}_e|} \frac{\partial 2\text{Re}\delta^{cons}(|\tilde{b}_e|)}{\partial |\tilde{b}_e|}$$

$$+ \frac{G^3 m_1^2 m_2^2}{b^3} \left[- \frac{4\mathcal{E}^{LS}}{\sqrt{\sigma^2 - 1}} \frac{\tilde{b}_e^\mu}{|\tilde{b}_e|} - \frac{u_2^\mu \sigma - u_1^\mu}{\sigma^2 - 1} \mathcal{E}(\sigma) \right]$$

- They imply

$$(\Delta p_1)^\mu + (\Delta p_2)^\mu + K^\mu = o$$

Conclusions and Outlook

- In this talk we focused on the soft eikonal operator describing the emission of low frequency gravitons from the $2 \to 2$ scattering of massive scalar particles.
- By combining Weinberg's exponentiation in momentum space and the eikonal exponentiation in impact parameter space we obtain explicit formulae for the waveforms and the energy spectrum in the zero frequency limit.
- The main feature is that these observables are smooth as the energy of the collision increases and display a qualitative change in their behaviour when one goes above the threshold:

$$\max\left\{\frac{m_1}{m_2}\sigma\,\Theta_s^2,\ \frac{m_2}{m_1}\sigma\,\Theta_s^2\right\} \gtrsim 1$$

- In the extreme ultrarelativistic regime in which *both* ratios go to infinity (e.g. in the massless case), the universality of gravitational scattering, already found for the elastic case, is restored also for radiative observables (at least in the ZFL regime).

- Also the ZFL of the energy emitted reproduces a universal result.
- If we perform the high energy limit below the bound one gets logarithmic divergent non-universal results.
- Above the bound the logarithmic increase with the energy is substituted by a non-analytic dependence on the scattering angle (and thus on the Newton constant).
- Here this has been shown to happen only in the soft limit $\frac{\omega b}{v} < 1$.
- For $\sigma \to \infty$ and still below the threshold above one gets that ratio of the total energy radiated and the initial energy $\frac{E^{rad}}{\sqrt{s}} \sim \Theta_s^3 \sqrt{\sigma}$, while the prediction for the massless case (above the threshold discussed before) is $\frac{E^{rad}}{\sqrt{s}} = \frac{\Theta_s^2}{2\pi} \log \frac{1}{\Theta_s^2}$ [Gruzinov and Veneziano, 1409.4555].

- Extend the eikonal operator beyond the soft frequency limit.
- The first step in this direction comes from the paper by Cristofoli, Gonzo, Moynihan, O'Connell, Ross, Sergola and White, 2112.07556
  which focuses in the regime below the threshold above.
- We have proposed an explicit form of the eikonal operator that reproduces all known results up to two loops.
- We have used it to compute observables as the impulse and the angular momentum.