Happy anniversary to the Niels Bohr Institute!

Niels Bohr in the Institute for Advanced Study archives

(thanks to Caitlin Rizzo)

[From the Shelby White and Leon Levy Archives Center at the Institute for Advanced Study]

Four Nobel laureates:
Bohr, Franck, Einstein, Rabi

[From the Shelby White and Leon Levy Archives Center at the Institute for Advanced Study]

Unitarity Cuts of the Worldsheet

Sebastian Mizera (IAS)

based on work with Lorenz Eberhardt

Veneziano amplitude

Polarization dependence $t_8 = s\, p_1 \cdot \epsilon_2\, p_2 \cdot \epsilon_1\, \epsilon_3 \cdot \epsilon_4 + \ldots$

$$\mathcal{A}^{\text{planar}}_{\text{tree}}(s,t) = -t_8 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)}{\Gamma(1 - \alpha' s - \alpha' t)}$$

Center of mass energy

Momentum transfer

Inverse string tension

No counterpart is known at loop level

But why is the Veneziano amplitude so much better than

$$\mathcal{A}^{\text{planar}}_{\text{tree}}(s,t) = \frac{t_8}{t}\int_0^1 z^{-\alpha' s-1}(1-z)^{-\alpha' t}\mathrm{d}z \quad \textbf{?}$$

Doesn't converge in the physical kinematics, e.g., $s>0,\ t,u<0$

$\implies$ Have to define it via analytic continuation

A sign of a more general problem

Textbook definition of string amplitudes

$$\mathcal{A}_{g,n}(p_1, p_2, \ldots, p_n) \stackrel{?}{=} \int_{\mathcal{M}_{g,n}} \text{(correlation function)}$$

or $\Gamma \subset \mathcal{M}_{g,n}$ ← Moduli space of genus-g Riemann surfaces with n punctures

isn't entirely correct, e.g., not consistent with unitarity
(the integration contour isn't known)

The underlying problem is that we formulate string amplitudes on a *Euclidean* worldsheet, but the target space is *Lorentzian*

Why hasn't it been a problem before?

Most computations done:

- At tree level
  (meromorphic functions)
- At loop level in the $\alpha' \to 0$ expansion
  (branch cuts fixed by matching with QFT)

[enormous literature: Green, Schwarz, Gross, Veneziano, Di Vecchia, Koba, Nielsen, D'Hoker, Phong, Bern, Dixon, Polyakov, Kosower, Vanhove, Schlotterer, Mafra, Stieberger, Brown, Broedel, Hohenegger, Kleinschmidt, Gerken, Roiban, Lipstein, Mason, Monteiro, ...]

where we can get away without being careful about the integration contour

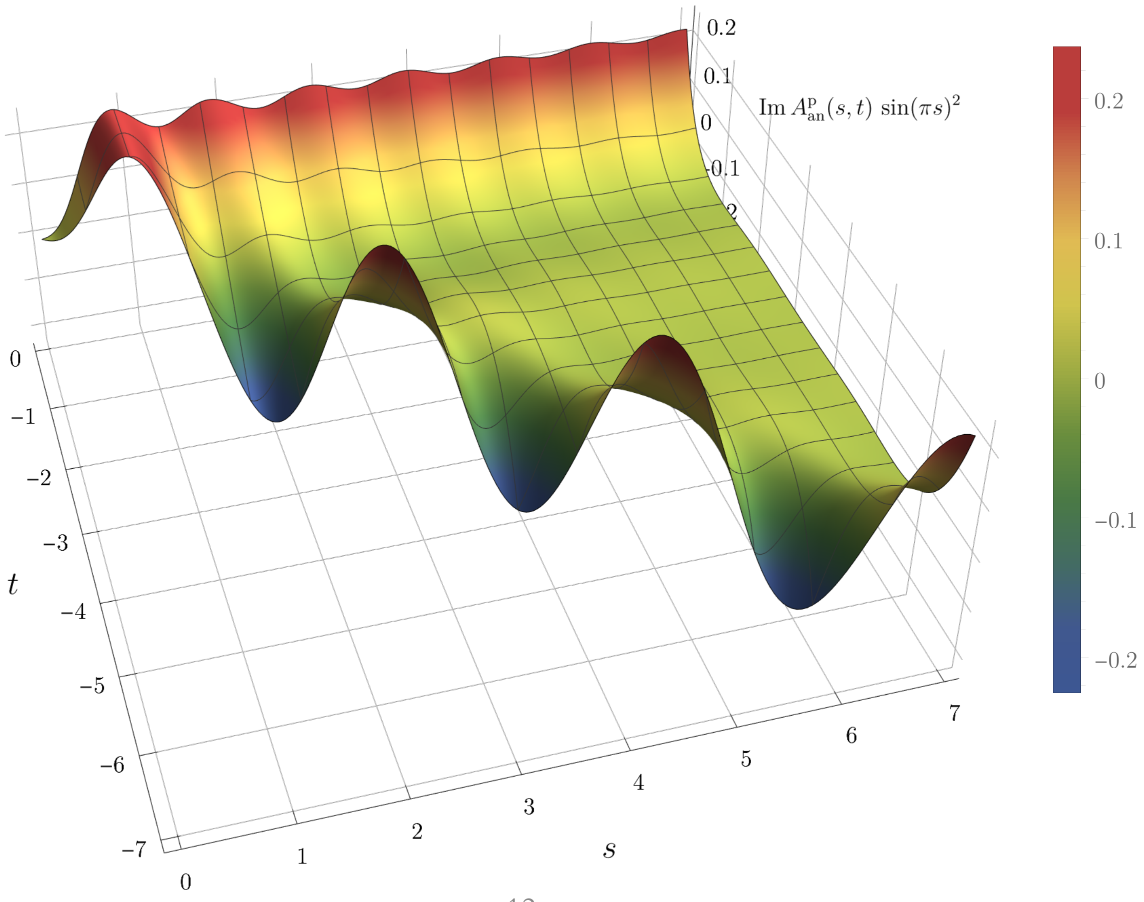
So what does it mean to "compute" an amplitude?

Pragmatic answer:

Be able to efficiently evaluate it numerically

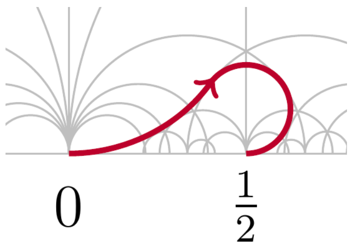
(e.g., known hypergeometric functions, fast convergent integrals, infinite sums, …)

In this talk we'll do it for the imaginary parts of genus-one amplitudes

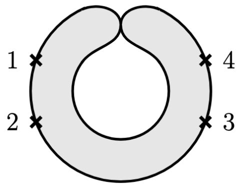


Outline of the talk

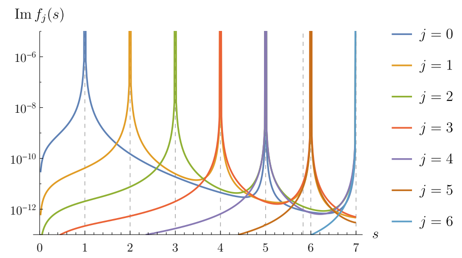
1) Continuation from Euclidean to Lorentzian



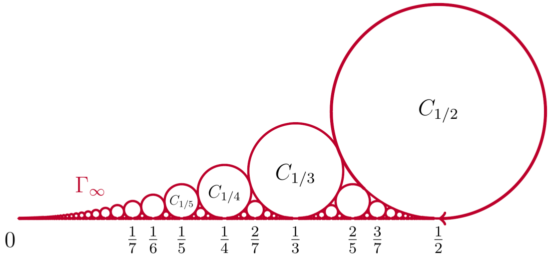
2) Unitarity cuts of the worldsheet



3) Physical properties of the imaginary parts

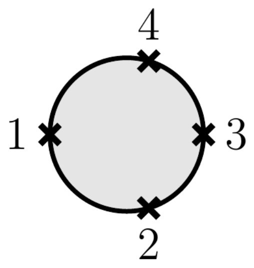


4) Glimpse of the real part (if there's time)



Let’s start at tree level

($\alpha' = 1$ from now on)



$$= \frac{t_8}{t} \int_0^1 z^{-s-1}(1-z)^{-t}\, \mathrm{d}z$$

s-channel poles come from $z \approx 0$, so set $z = e^{-\tau}$ and take $\tau \to \infty$

$$t_8 \int_0^{\infty} e^{\tau s}\left(\# + \# e^{-\tau} + \# e^{-2\tau} + \ldots\right) \mathrm{d}\tau$$

$-\frac{1}{s}$	$-\frac{\#}{s-1}$	$-\frac{\#}{s-2}$
massless	level-1	level-2

Important distinction

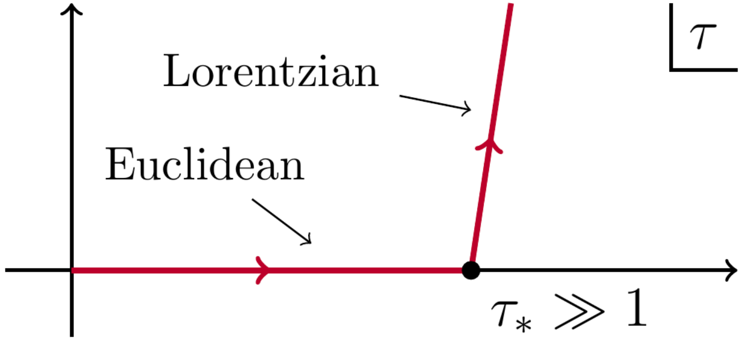
$$\frac{-1}{s-m^2} = \int_0^\infty \mathrm{d}\tau_{\mathrm{E}}\, e^{\tau_{\mathrm{E}}(s-m^2)}$$

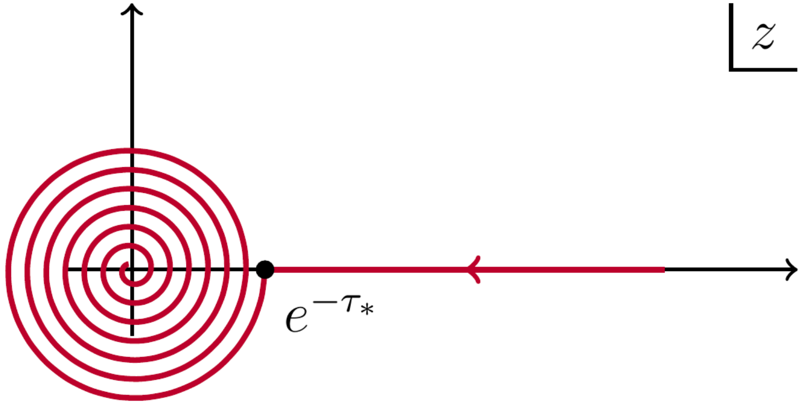
Euclidean proper time

$$\frac{i}{s-m^2} = \lim_{\varepsilon\to 0^+} \int_0^\infty \mathrm{d}\tau_{\mathrm{L}}\, e^{i\tau_{\mathrm{L}}(s-m^2+i\varepsilon)}$$

Lorentzian proper time

This tells us about the correct integration contour





We can resum $\circlearrowright + e^{-2\pi i s} \circlearrowright + e^{-4\pi i s} \circlearrowright + \ldots = \dfrac{1}{1 - e^{-2\pi i s}} \circlearrowright$

infinite number of string resonances

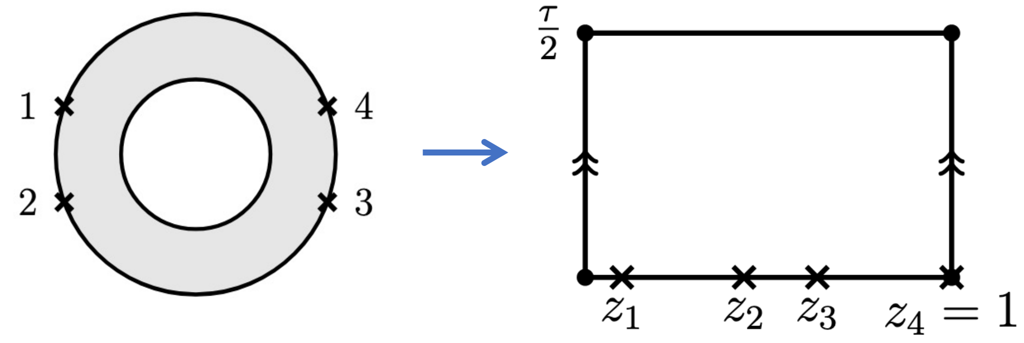
Strategy for finding the contour at higher genus

- Identify local variables $q \sim e^{-(\text{Schwinger parameter})}$
- Continue to Lorentzian signature locally in the moduli space
- Glue everything together

[Witten '13]

Genus-one amplitudes

In this talk we focus on the planar annulus contribution



Modular parameter

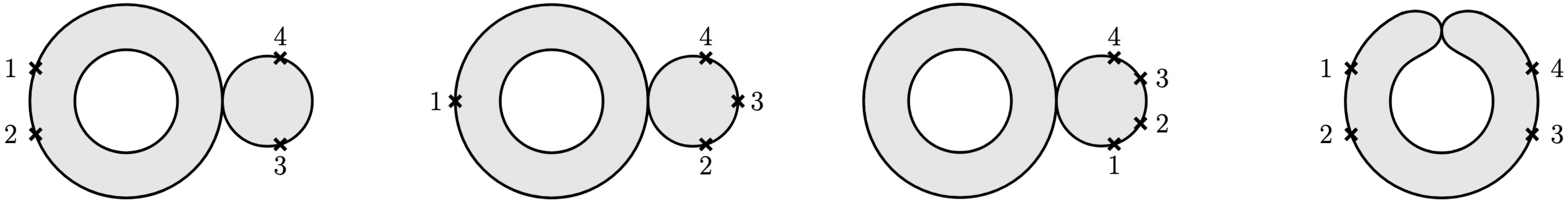
$z_{ij} = z_i - z_j$

$$\mathcal{A}^{\text{planar}}_{\text{annulus}} \stackrel{?}{=} -i\, t_8 \int_0^{i\infty} \mathrm{d}\tau \int_{0<z_1<z_2<z_3<1} \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \left(\frac{\vartheta_1(z_{21},\tau)\vartheta_1(z_{43},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-s} \left(\frac{\vartheta_1(z_{32},\tau)\vartheta_1(z_{41},\tau)}{\vartheta_1(z_{31},\tau)\vartheta_1(z_{42},\tau)} \right)^{-t}$$

Jacobi theta function

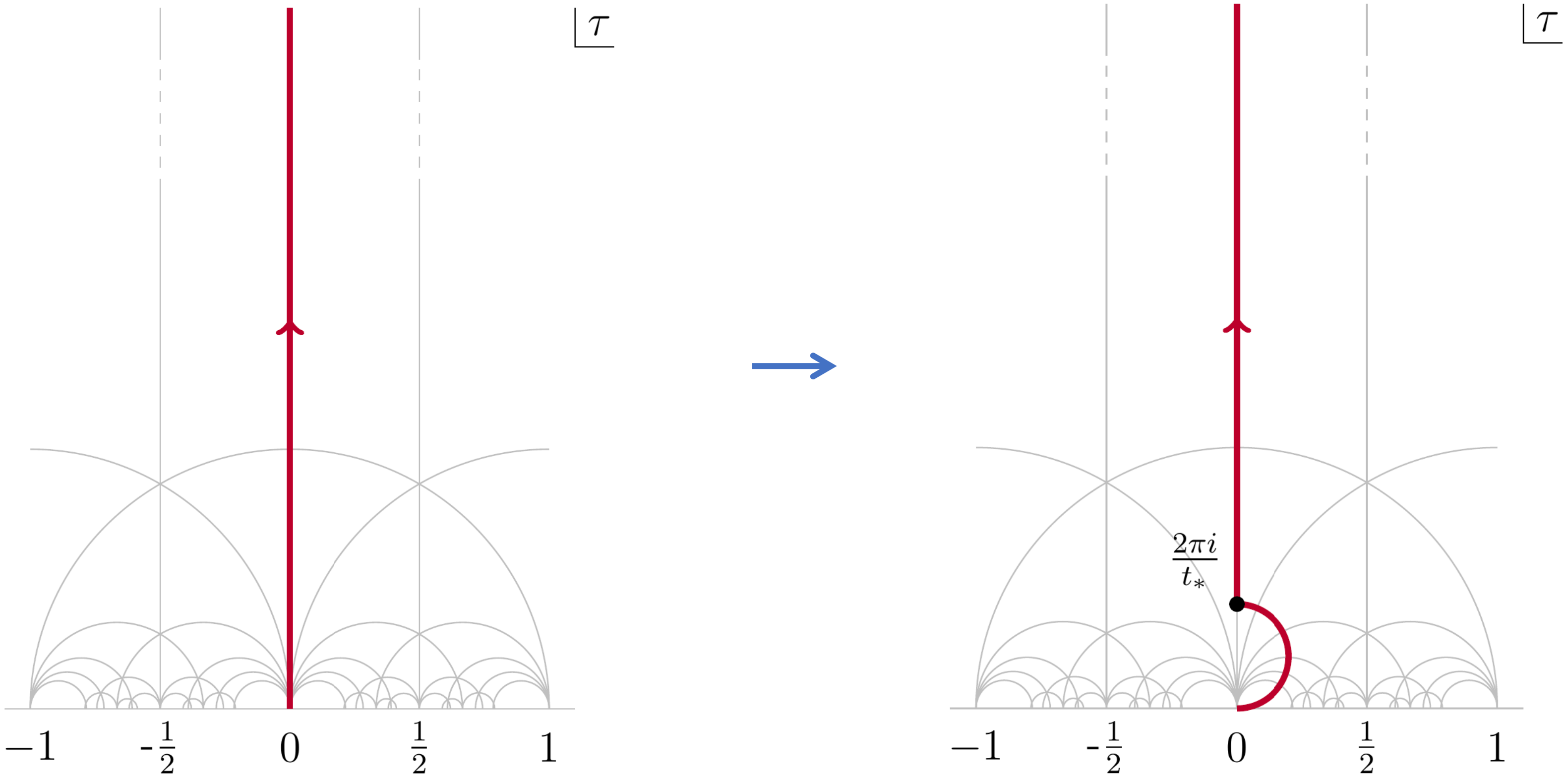
$$\vartheta_1(z,\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{2\pi i (n-\frac{1}{2})z + \pi i (n-\frac{1}{2})^2 \tau}$$

Various degenerations need the Witten $i\varepsilon$



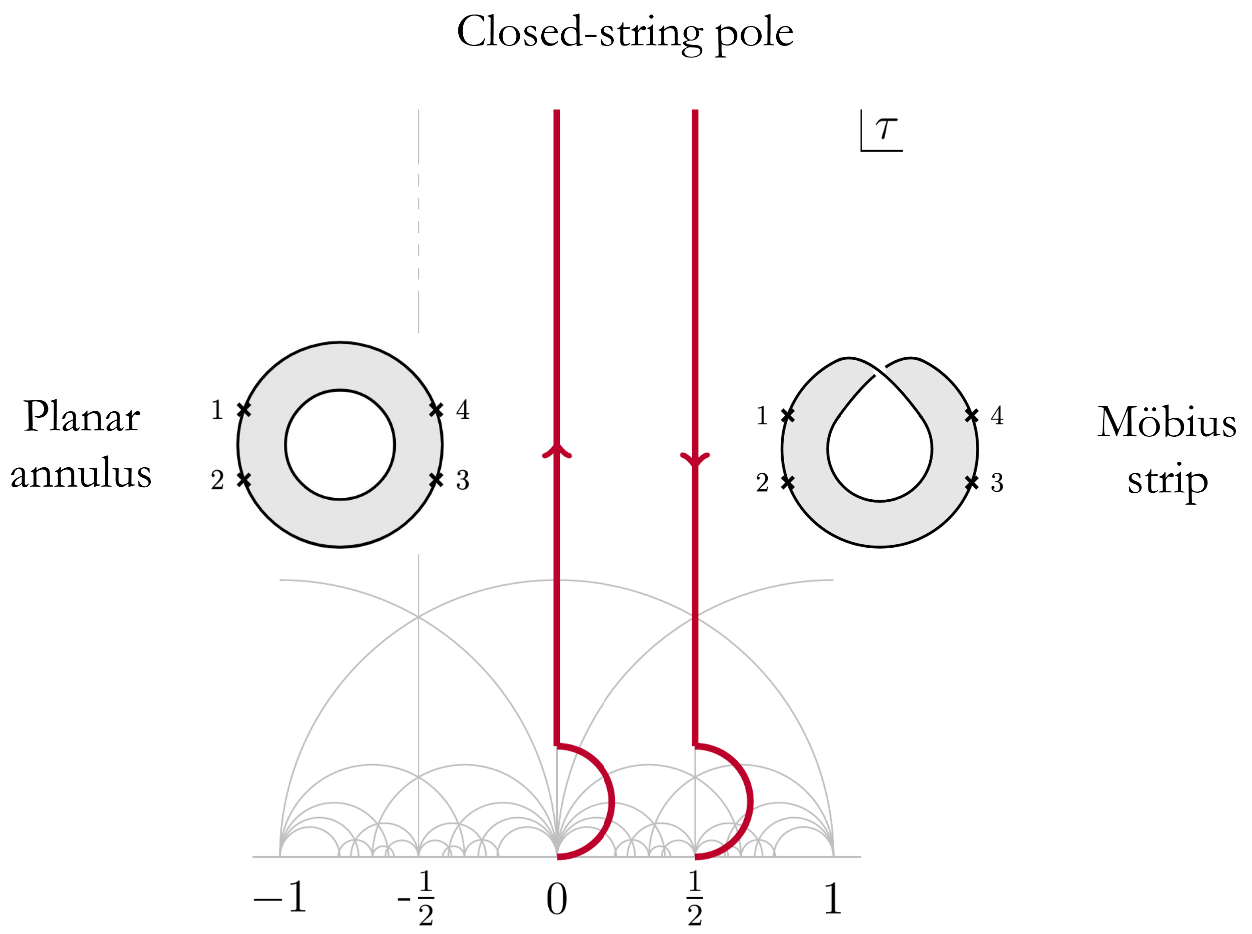
Massive pole exchange	Wave-function renormalization	Tadpole	Non-separating degeneration
$q = z_{43}$	$q = z_{42}$	$q = z_{41}$	$q = e^{-\frac{2\pi i}{\tau}}$
			Unitarity cuts

Let’s focus on the contour in the fundamental domain, $\tau = \dfrac{2\pi i}{t_* + it}$



Approach the essential singularity from the right

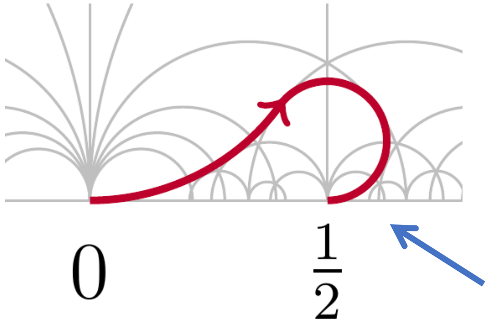
Adding the other planar contribution: Möbius strip



Our proposal for the correct integration contour

(similar for other topologies)

Precise shape
doesn't matter

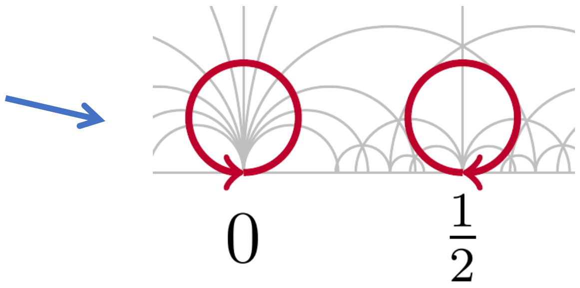


Approach 0 and ½
from the right

We'll come back to it at the end of the talk

For the imaginary part we only need

Size of the circles doesn't matter



They'll give as unitarity cuts of the planar annulus and the Möbius strip

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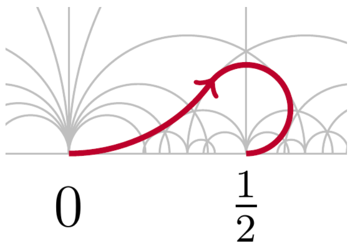

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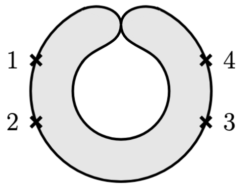

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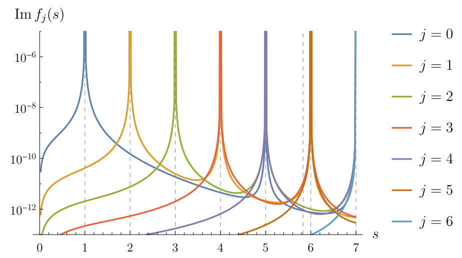

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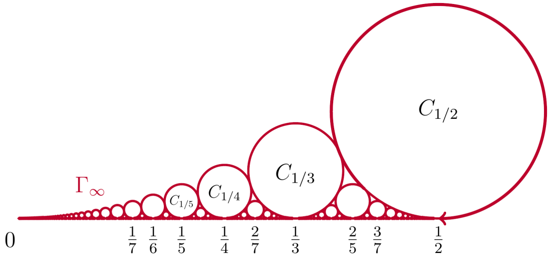


(Holomorphic) unitarity cuts

$$\begin{aligned}\operatorname{Im} T &= \tfrac{1}{2} T T^\dagger \\ &= \underbrace{\tfrac{1}{2} T^2 - \tfrac{i}{2} T^3 + \mathcal{O}(T^4)}\end{aligned}$$

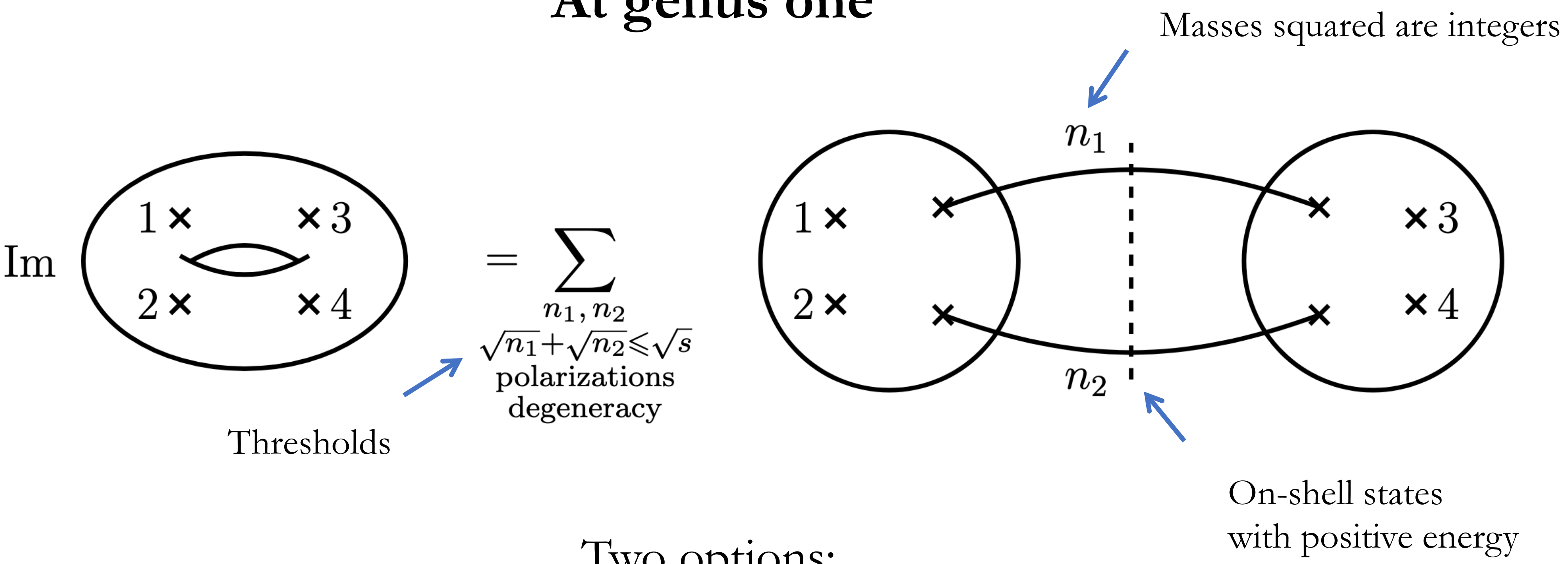
No complex conjugate

[Hannesdottir, SM ‘22]

Just as in quantum field theory, much easier to prove the holomorphic version using contour deformations

[talk by Hannesdottir]

At genus one



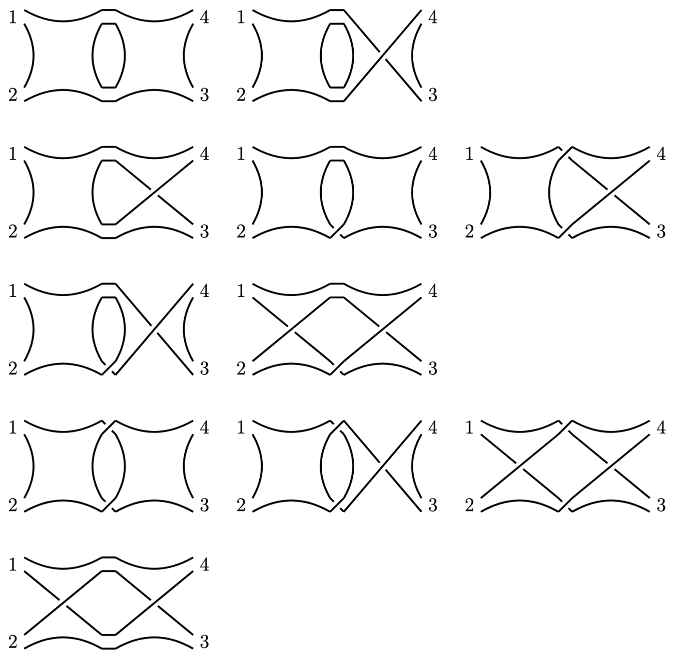
Two options:

- Do unitarity cuts "by hand" just as in field theory
- Let the worldsheet do it for us

First do it by hand

(not feasible beyond the massless cut)

- Color sums



- Polarization sums

$$\mathcal{P} = \sum_{\text{pol}} \Big[\underbrace{t_8^b(1256)\, t_8^b(34\overline{56})}_{\text{gluons}} - \underbrace{t_8^f(1256)\, t_8^f(34\overline{56})}_{\text{gluinos}} \Big] = \frac{s^2}{2} t_8$$

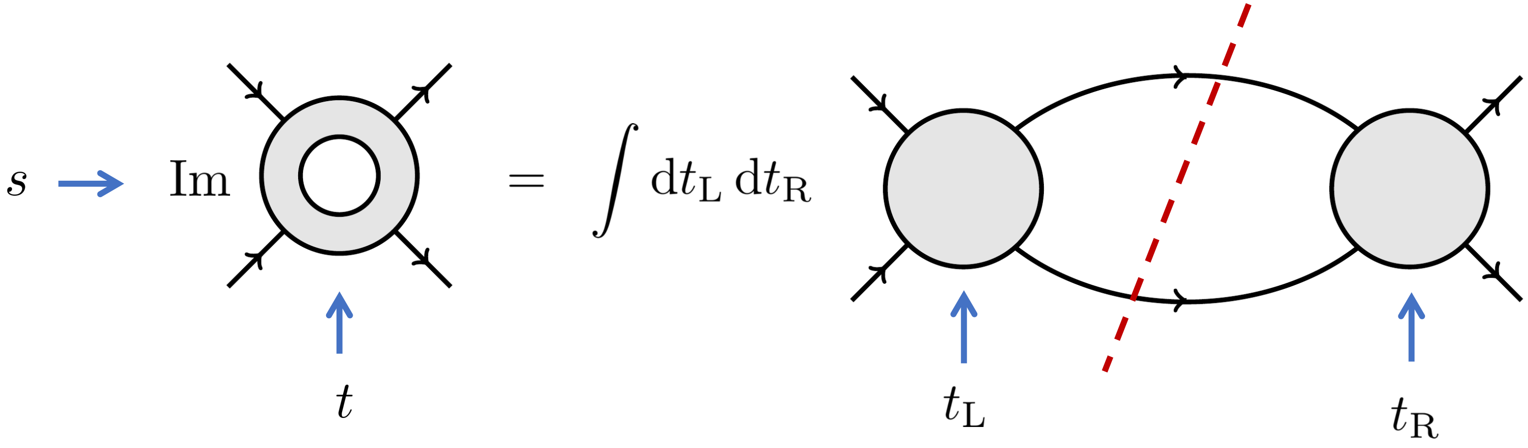
- Loop integration

(Baikov representation)

$$\int \mathrm{d}^{\mathrm{D}}\ell \, \delta^+[\ell^2]\delta^+[(p_{12}-\ell)^2](\cdots)$$

$$\propto \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}} P^{\frac{\mathrm{D}-5}{2}}(\cdots)$$

After the dust settles



$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}\Big|_{s<1} = \frac{N\pi}{60\sqrt{stu}} \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\ P(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{L}})}{\Gamma(1-s-t_{\mathrm{L}})} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{R}})}{\Gamma(1-s-t_{\mathrm{R}})}$$

On-shell phase space

$$P(t_{\mathrm{L}}, t_{\mathrm{R}}) = -\frac{s(t^2 + t_{\mathrm{L}}^2 + t_{\mathrm{R}}^2 - 2tt_{\mathrm{L}} - 2tt_{\mathrm{R}} - 2t_{\mathrm{L}}t_{\mathrm{R}}) - 4tt_{\mathrm{L}}t_{\mathrm{R}}}{4tu}$$

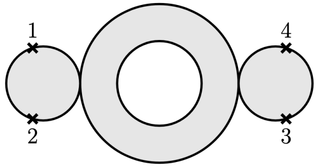
General form after including massive exchanges

New thresholds opening up

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi N}{60} \frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\big[s - (\sqrt{n_1}+\sqrt{n_2})^2\big] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}}$$

$$\times Q_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}}) \frac{\Gamma(-t_{\mathrm{L}})\Gamma(-t_{\mathrm{R}})}{\Gamma(n_1+n_2+1-s-t_{\mathrm{L}})\Gamma(n_1+n_2+1-s-t_{\mathrm{R}})}$$

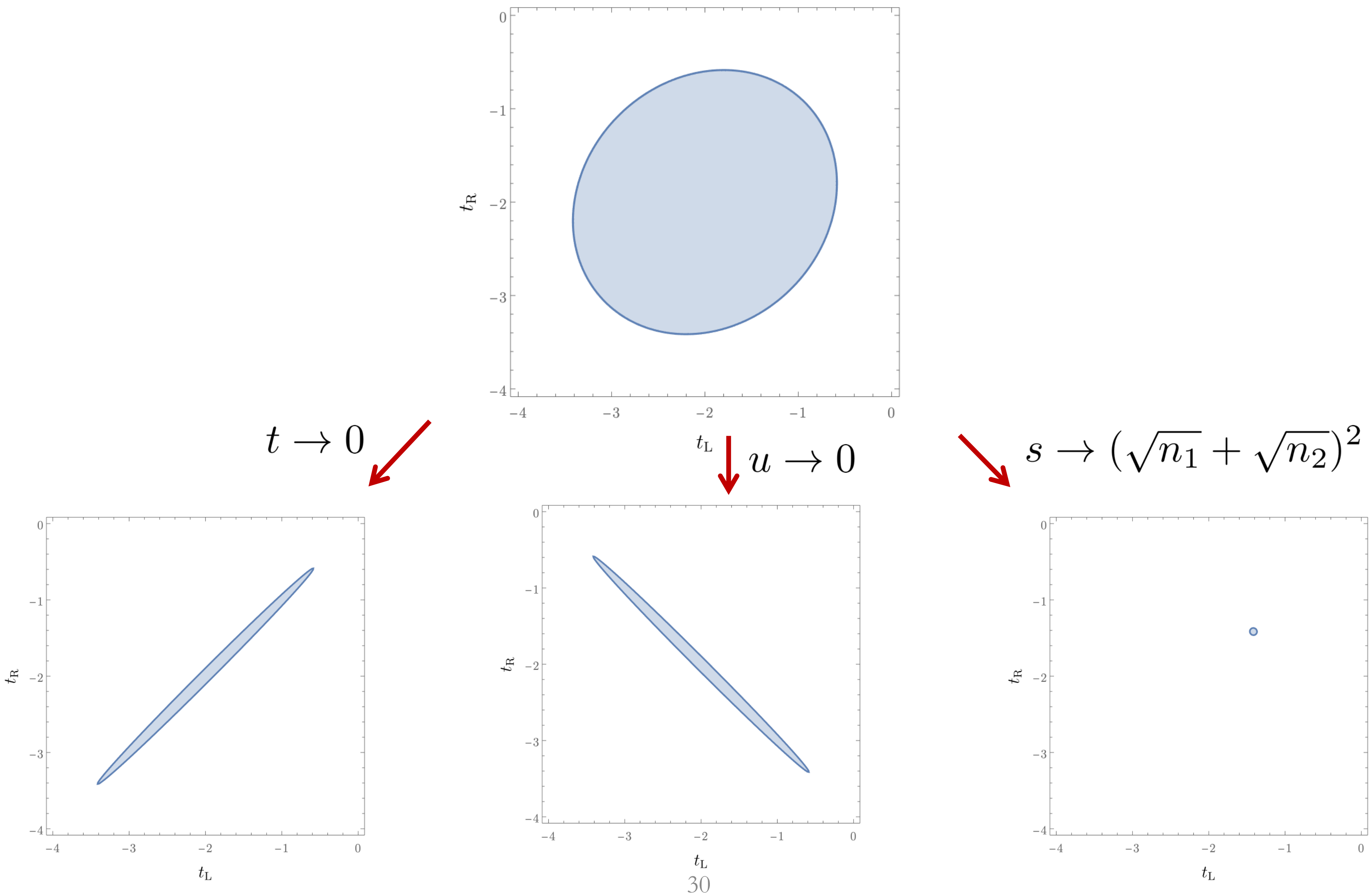
Double poles at every positive integer



Need a computation to determine the integrand, e.g., $Q_{0,0} = 1$

with $P_{n_1,n_2} = -\frac{1}{4stu} \det \begin{bmatrix} 0 & s & u & n_2-s-t_{\mathrm{L}} \\ s & 0 & t & t_{\mathrm{L}}-n_1 \\ u & t & 0 & n_1-t_{\mathrm{R}} \\ n_2-s-t_{\mathrm{L}} & t_{\mathrm{L}}-n_1 & n_1-t_{\mathrm{R}} & 2n_1 \end{bmatrix}$

On-shell phase space

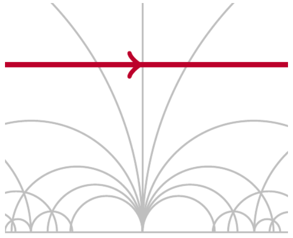


Unitarity cuts of the worldsheet

0

$$\tau \to -\frac{1}{\tau}$$

$$q \to e^{2\pi i \tau}$$



0

q can be arbitrarily small

After the modular transformation:

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = -\frac{N}{64} \int_{\longrightarrow} \frac{\mathrm{d}\tau}{\tau^2} \int \mathrm{d}z_1\, \mathrm{d}z_2\, \mathrm{d}z_3 \; \underbrace{q^{s z_{41} z_{32} - t z_{21} z_{43}} \left(\frac{\vartheta_1(z_{21}\tau, \tau)\vartheta_1(z_{43}\tau, \tau)}{\vartheta_1(z_{31}\tau, \tau)\vartheta_1(z_{42}\tau, \tau)} \right)^{-s} \left(\frac{\vartheta_1(z_{41}\tau, \tau)\vartheta_1(z_{32}\tau, \tau)}{\vartheta_1(z_{31}\tau, \tau)\vartheta_1(z_{42}\tau, \tau)} \right)^{-t}}$$

$\sim q^{\mathrm{Trop}(s,t,z_i)}$ as $q \to 0$

Tropical analysis

The integrand goes as q^{Trop} so only terms with $\text{Trop} < 0$ can contribute

It tells us how many terms in the q-expansion we need to keep, e.g.,

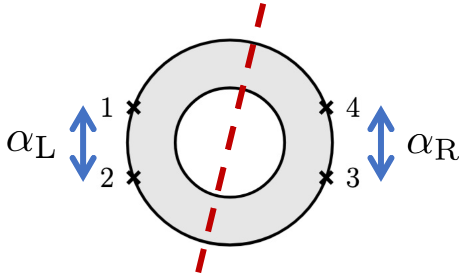
$$\vartheta_1(z\tau, \tau) = i q^{\frac{1}{8}} \left(q^{-\frac{z}{2}} - q^{\frac{z}{2}} - q^{1-\frac{3z}{2}} \right) (1 + \mathcal{O}(q)) \qquad z \in [0, 1]$$

always dominates — needed near $z \approx 0$ — needed near $z \approx 1$

For example, below the first massive threshold

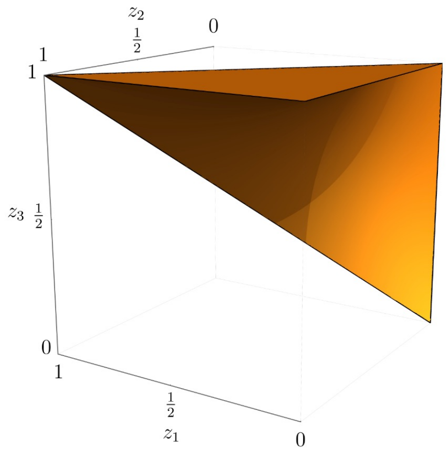
$$q^{s z_{41} z_{32} - t z_{21} z_{43}} \left(\frac{\vartheta_1(z_{21}\tau,\tau)\vartheta_1(z_{43}\tau,\tau)}{\vartheta_1(z_{31}\tau,\tau)\vartheta_1(z_{42}\tau,\tau)} \right)^{-s} \left(\frac{\vartheta_1\vartheta_1}{\vartheta_1\vartheta_1} \right)^{-t} \sim q^{-s(1-z_{41})z_{32} - t z_{21} z_{43}} \, (1-q^{z_{21}})^{-s}(1-q^{z_{43}})^{-s}$$

exact computation supported in



Identify the variables

$$\alpha_{\rm L} = z_{21} \,, \qquad \alpha_{\rm R} = z_{43} \,, \qquad t_{\rm L} = -s z_{32} + t z_{43}$$



and integrate in

$$1 = s\sqrt{\frac{-i\tau}{2stu}} \int_{-\infty}^{\infty} \mathrm{d}t_{\rm R} \, q^{-\frac{1}{4st(s+t)}(st_{\rm R} - (s+2t)t_{\rm L} + 2t(s+t)\alpha_{\rm R} - st)^2}$$

Gives exactly the same formula we've derived before from unitarity

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}\Big|_{s<1} = \frac{N}{64\sqrt{2stu}} \int_{\longrightarrow} \frac{\mathrm{d}\tau}{(-i\tau)^{\frac{3}{2}}} \int_{\mathcal{R}} \mathrm{d}\alpha_{\mathrm{L}}\, \mathrm{d}\alpha_{\mathrm{R}}\, \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\; q^{-t_{\mathrm{L}}\alpha_{\mathrm{L}}-t_{\mathrm{R}}\alpha_{\mathrm{R}}-P(t_{\mathrm{L}},t_{\mathrm{R}})}(1-q^{\alpha_{\mathrm{L}}})^{-s}(1-q^{\alpha_{\mathrm{R}}})^{-s}$$

$$= \frac{N\pi}{60\sqrt{stu}} \int_{P>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\; P(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{L}})}{\Gamma(1-s-t_{\mathrm{L}})} \frac{\Gamma(1-s)\Gamma(-t_{\mathrm{R}})}{\Gamma(1-s-t_{\mathrm{R}})}$$

Tropical analysis previously featured in

- $\alpha' \to 0$ limit of string amplitudes

  [Tourkine ‘13]

- $\alpha' \to 0$ limit of tree-level amplitudes and loop integrands

  [Arkani-Hamed, He, Lam, Frost, Salvatori, Plamondon, Thomas ‘19-22]
  [talk by Arkani-Hamed]

- UV/IR divergences of individual Feynman integrals

  [Panzer, Borinsky, Arkani-Hamed, Hillman, SM ‘19-22]

But here it has a different role: we’re doing an *exact* computation!

Stringy Landau analysis

When does a new contribution to Trop < 0 appear?

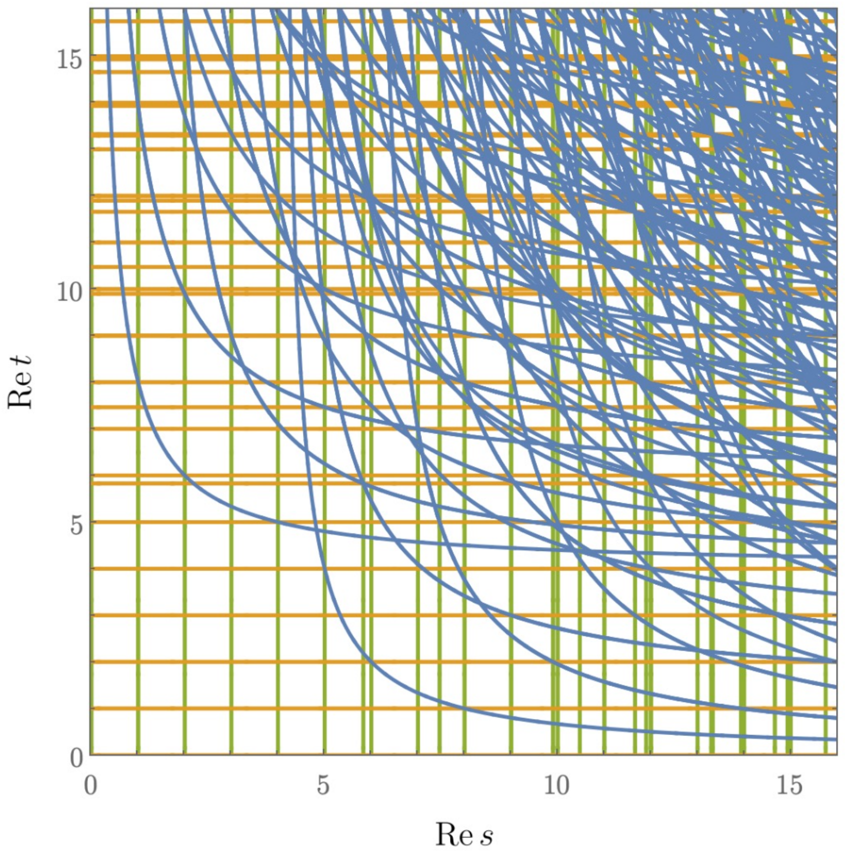
Normal thresholds at

$$s, t, u = (\sqrt{n_1} + \sqrt{n_2})^2$$

Anomalous thresholds at

$$\det \begin{bmatrix} 2n_1 & n_1{+}n_2 & n_1{+}n_3{-}s & n_1{+}n_4 \\ n_1{+}n_2 & 2n_2 & n_2{+}n_3 & n_2{+}n_4{-}t \\ n_1{+}n_3{-}s & n_2{+}n_3 & 2n_3 & n_3{+}n_4 \\ n_1{+}n_4 & n_2{+}n_4{-}t & n_3{+}n_4 & 2n_4 \end{bmatrix} = 0 \qquad n_i \in \mathbb{Z}_{\geqslant 0}$$

Analytic structure away from physical regions is complicated, but consistent with field theory expectations



Branch cuts

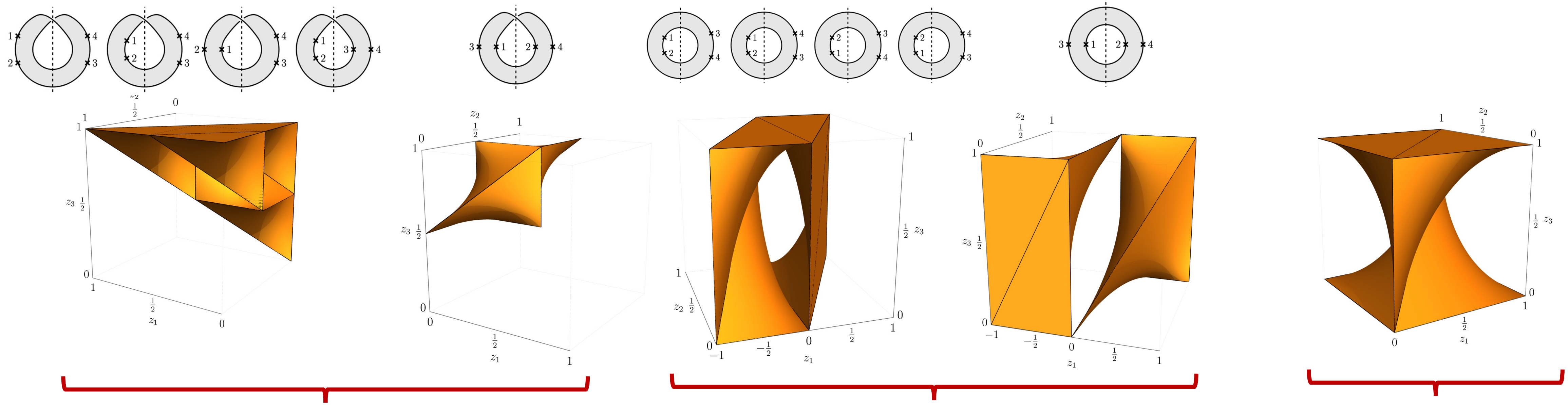
This strategy allows us to go to higher energies bypassing summing over states

$$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi N}{60} \frac{\Gamma(1-s)^2}{\sqrt{stu}} \sum_{n_1 \geqslant n_2 \geqslant 0} \theta\big[s - (\sqrt{n_1}+\sqrt{n_2})^2\big] \int_{P_{n_1,n_2}>0} \mathrm{d}t_{\mathrm{L}}\, \mathrm{d}t_{\mathrm{R}}\, P_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}})^{\frac{5}{2}}$$
$$\times Q_{n_1,n_2}(t_{\mathrm{L}}, t_{\mathrm{R}}) \frac{\Gamma(-t_{\mathrm{L}})\Gamma(-t_{\mathrm{R}})}{\Gamma(n_1+n_2+1-s-t_{\mathrm{L}})\Gamma(n_1+n_2+1-s-t_{\mathrm{R}})}$$

where the first few polynomials are

$$\begin{aligned}
Q_{0,0} &= 1\ , \\
Q_{1,0} &= 2\left(-2st_{\mathrm{L}}t_{\mathrm{R}} - s^2 t_{\mathrm{L}} + st_{\mathrm{L}} - s^2 t_{\mathrm{R}} + st_{\mathrm{R}} + s^2 t - 2st + t\right)\ , \\
Q_{2,0} &= 2s^4 t_{\mathrm{L}} t_{\mathrm{R}} + 4s^3 t_{\mathrm{L}} t_{\mathrm{R}}^2 + 4s^3 t_{\mathrm{L}}^2 t_{\mathrm{R}} - 4s^3 t t_{\mathrm{L}} t_{\mathrm{R}} - 12 s^3 t_{\mathrm{L}} t_{\mathrm{R}} + 4s^2 t_{\mathrm{L}}^2 t_{\mathrm{R}}^2 - 10 s^2 t_{\mathrm{L}} t_{\mathrm{R}}^2 \\
&\quad - 10 s^2 t_{\mathrm{L}}^2 t_{\mathrm{R}} + 12 s^2 t t_{\mathrm{L}} t_{\mathrm{R}} + 18 s^2 t_{\mathrm{L}} t_{\mathrm{R}} - 2st_{\mathrm{L}}^2 t_{\mathrm{R}}^2 + 4st_{\mathrm{L}} t_{\mathrm{R}}^2 + 4st_{\mathrm{L}}^2 t_{\mathrm{R}} - 12stt_{\mathrm{L}} t_{\mathrm{R}} \\
&\quad - 6st_{\mathrm{L}} t_{\mathrm{R}} + 4tt_{\mathrm{L}} t_{\mathrm{R}} + s^4 t_{\mathrm{L}}^2 - 2s^4 t t_{\mathrm{L}} - s^4 t_{\mathrm{L}} - 4s^3 t_{\mathrm{L}}^2 + 10 s^3 t t_{\mathrm{L}} + 4s^3 t_{\mathrm{L}} + 5s^2 t_{\mathrm{L}}^2 \\
&\quad - 18 s^2 t t_{\mathrm{L}} - 5s^2 t_{\mathrm{L}} - 2st_{\mathrm{L}}^2 + 14 stt_{\mathrm{L}} + 2st_{\mathrm{L}} - 4tt_{\mathrm{L}} + s^4 t_{\mathrm{R}}^2 - 2s^4 t t_{\mathrm{R}} - s^4 t_{\mathrm{R}} \\
&\quad - 4s^3 t_{\mathrm{R}}^2 + 10 s^3 t t_{\mathrm{R}} + 4s^3 t_{\mathrm{R}} + 5s^2 t_{\mathrm{R}}^2 - 18 s^2 t t_{\mathrm{R}} - 5s^2 t_{\mathrm{R}} - 2st_{\mathrm{R}}^2 + 14 stt_{\mathrm{R}} \\
&\quad + 2st_{\mathrm{R}} - 4tt_{\mathrm{R}} + s^4 t^2 + s^4 t - 6s^3 t^2 - 6s^3 t + 13 s^2 t^2 + 13 s^2 t - 12 st^2 - 12 st \\
&\quad + 4t^2 + 4t\ .
\end{aligned}$$

Similar analysis for other genus-one topologies in all kinematic channels



Möbius strip **Non-planar annulus** **Torus**

Outline of the talk

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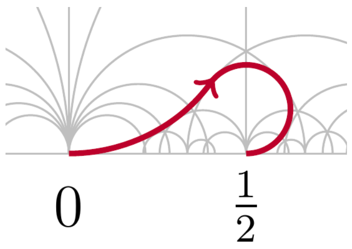

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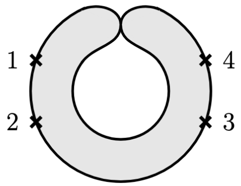

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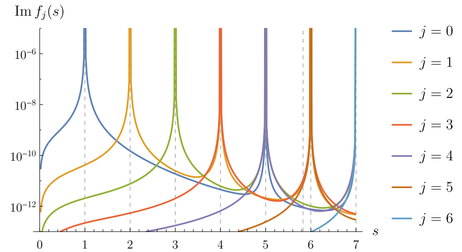

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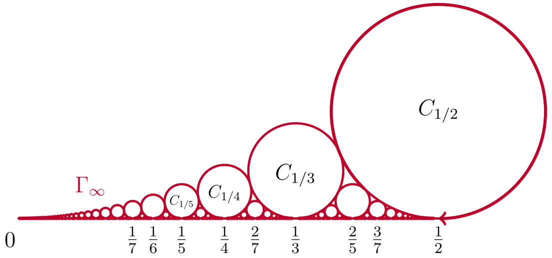
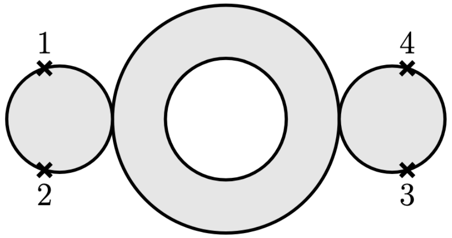


We can now analyze the results

(this talk: planar annulus in the s-channel only)

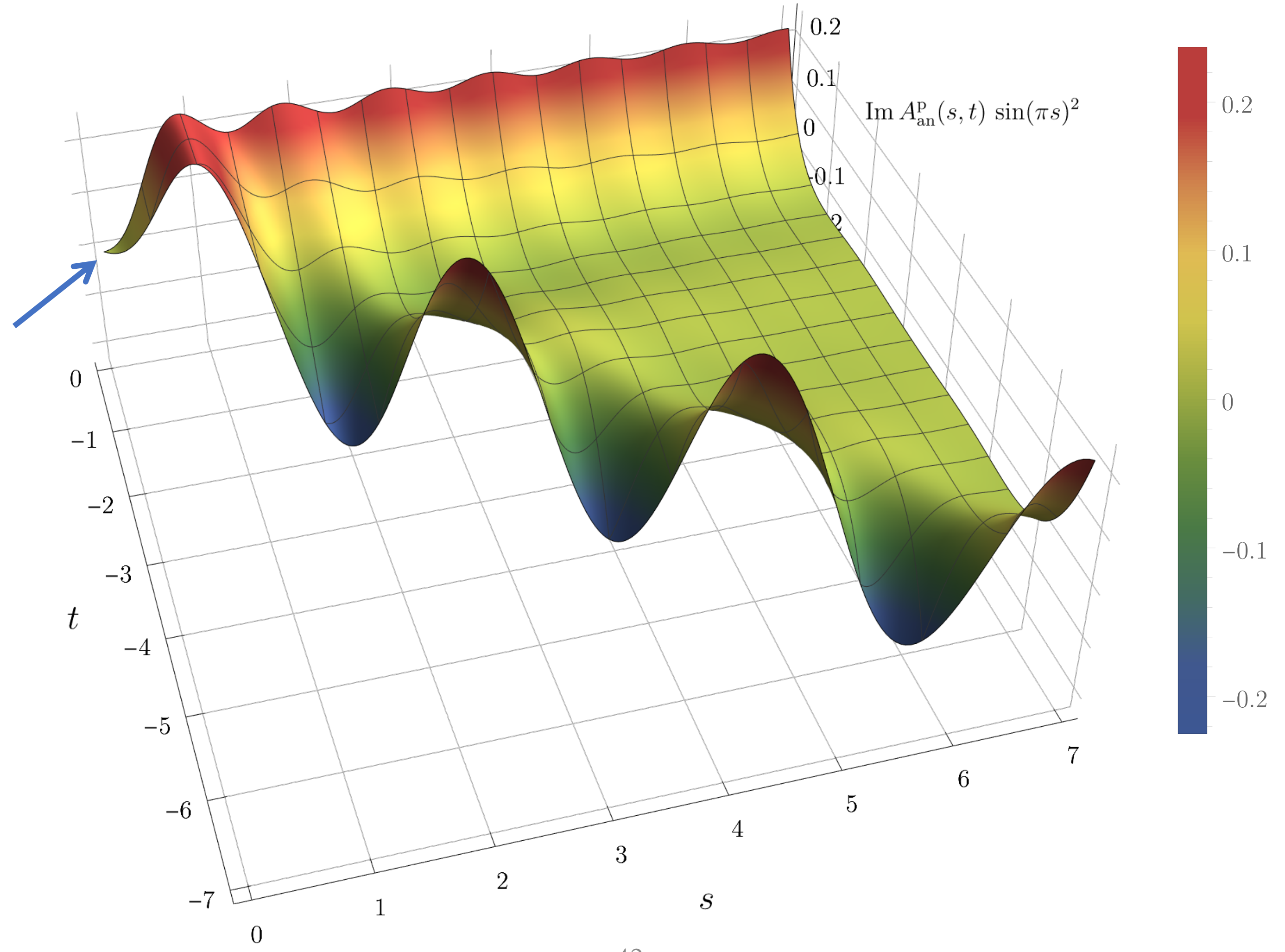
We often normalize by $\sin(\pi s)^2$ to remove the double poles



$\mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,t)$ does not include the t_8 tensor

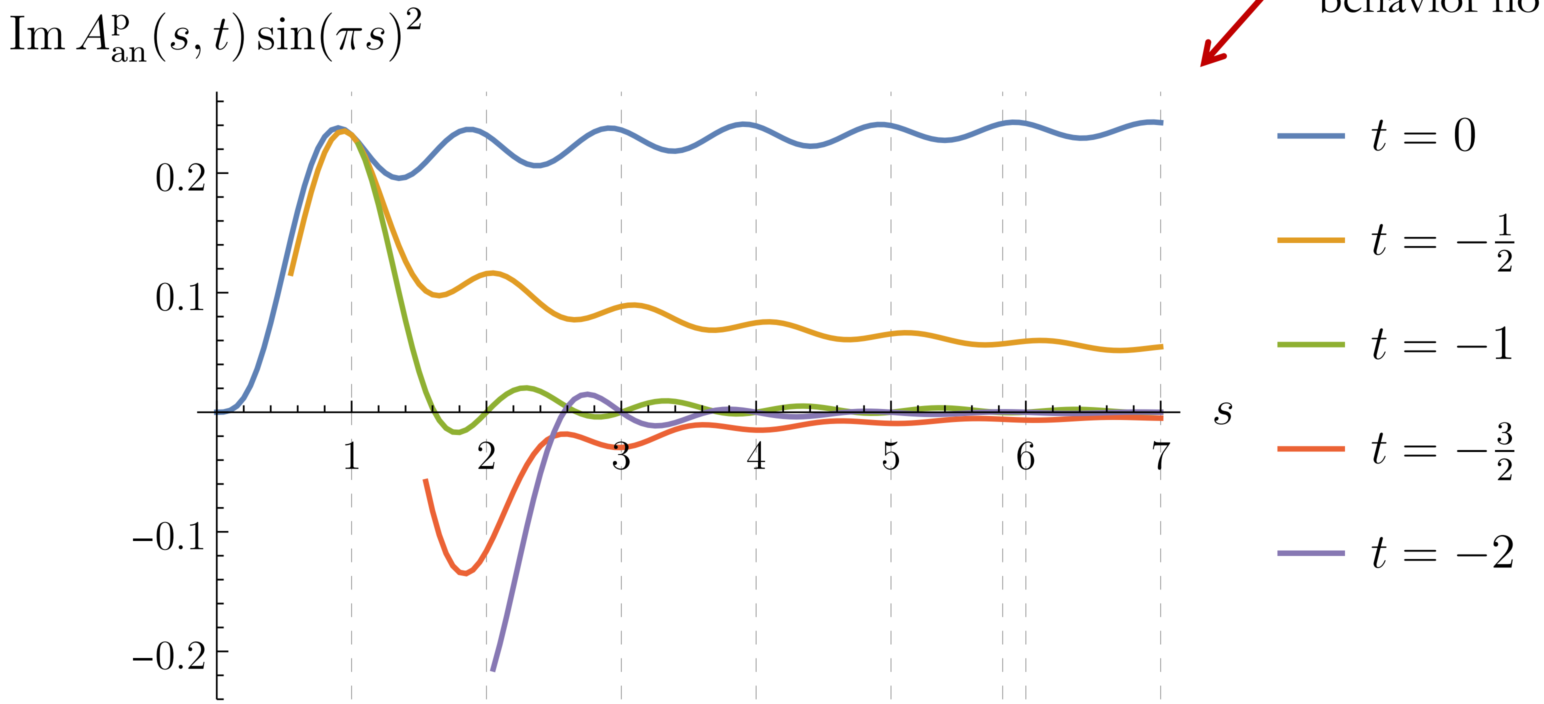
The imaginary part of the planar annulus

Most of the known results

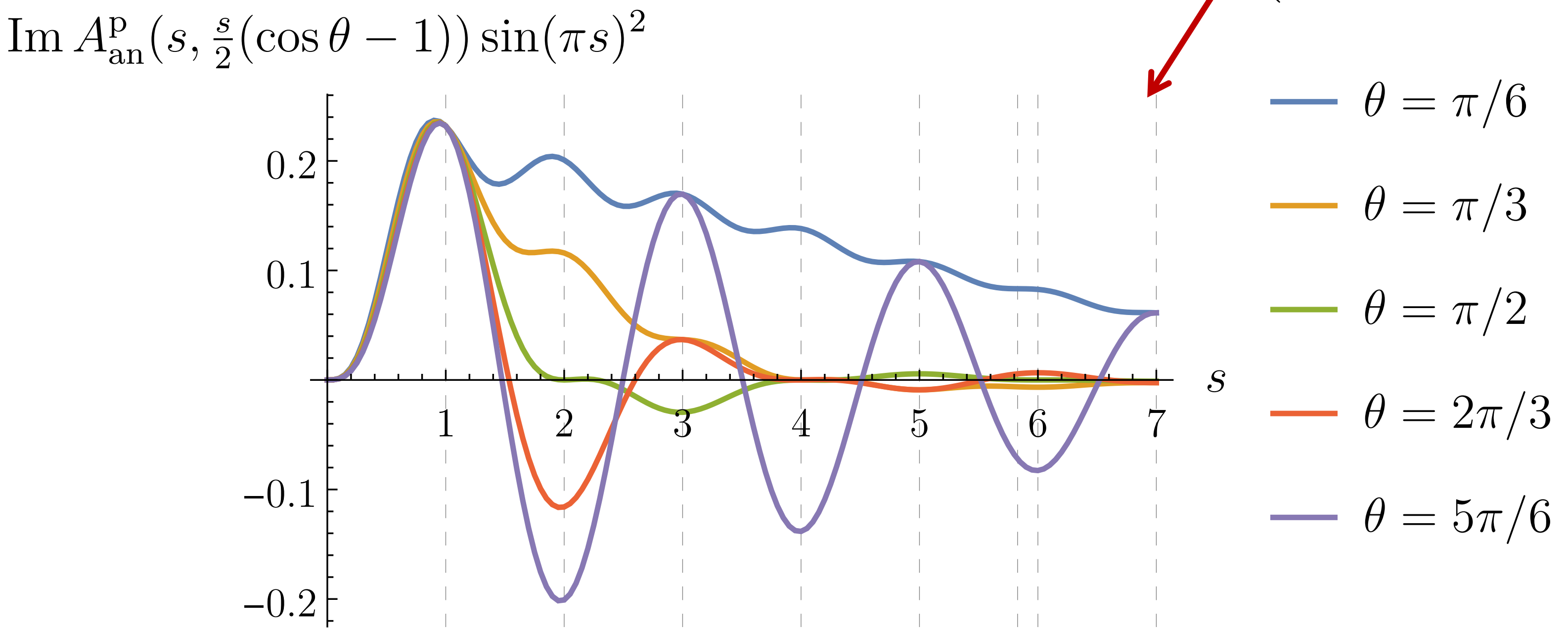


Fixed momentum transfer

Asymptotic behavior not reached



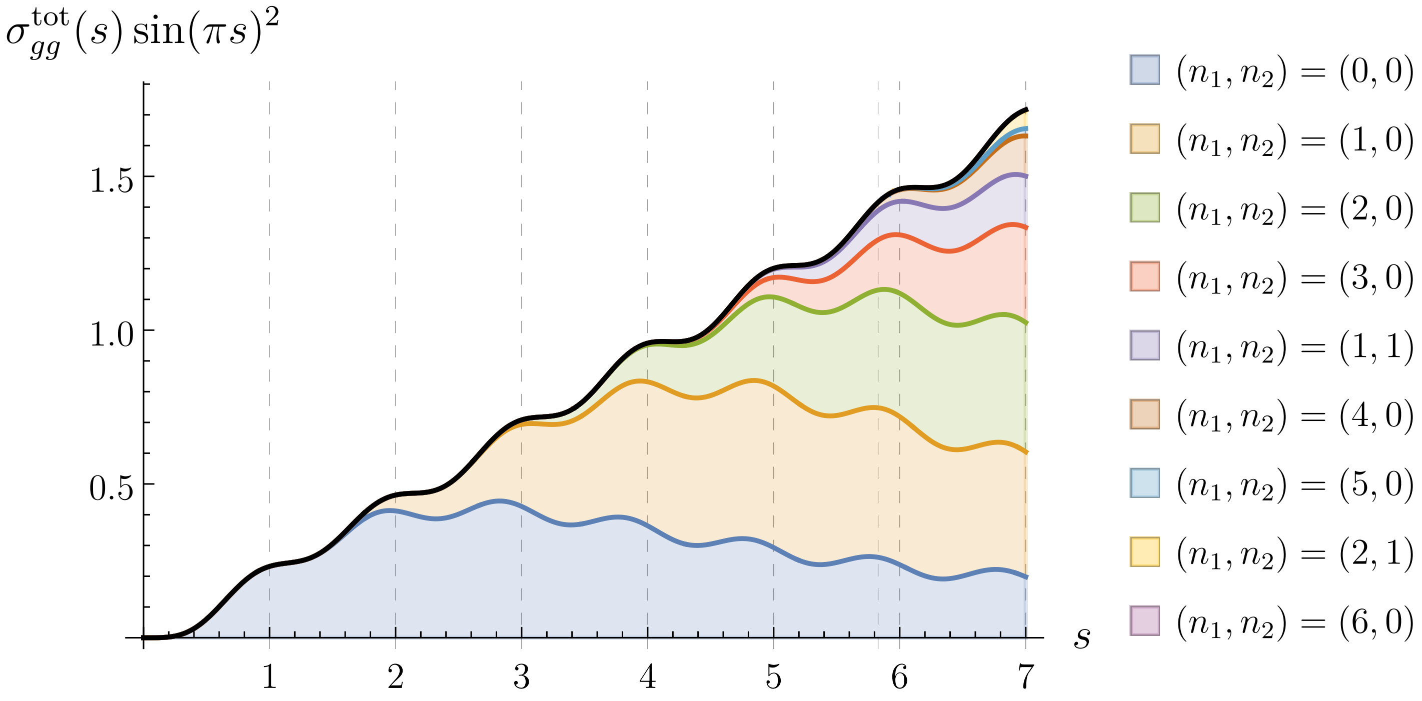
Fixed angle



Qualitatively exponential decay (cf. [Gross-Manes '89])

Total cross section

$\propto \frac{t_8}{s} \mathrm{Im}\, A^{\mathrm{p}}_{\mathrm{an}}(s,0)$

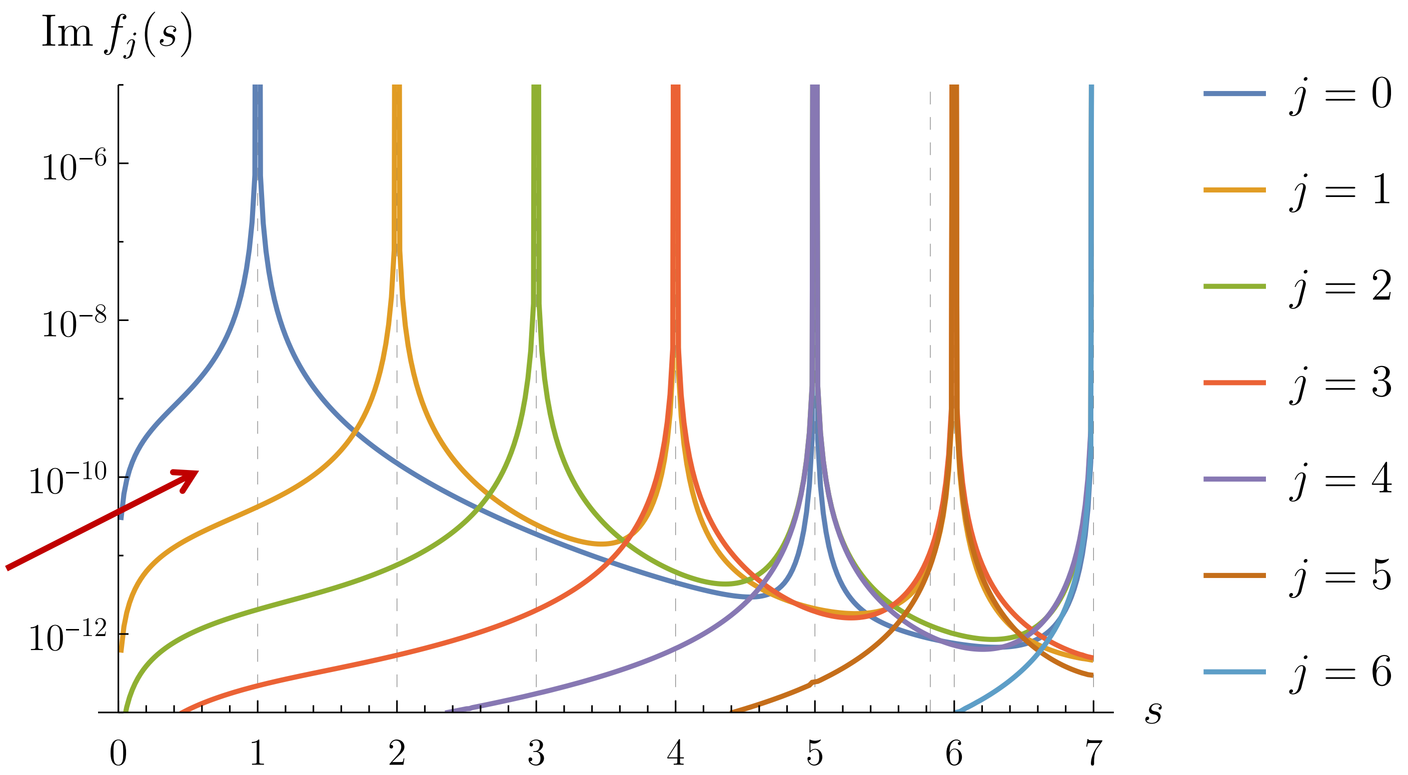


New thresholds opening up very slowly

Low-spin dominance

(cf. [Arkani-Hamed, Huang, Huan ’20], [Bern, Kosmopoulos, Zhiboedov ‘21] at tree level)

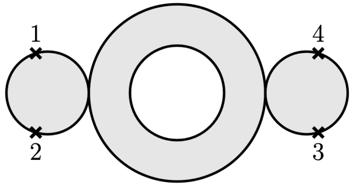
Partial wave coefficients $f_j(s) \propto \int_{-1}^{1} \mathrm{d}z\,(1-z^2)^3\,G_j^{(10)}(z) A_{\mathrm{an}}^{\mathrm{p}}(s,t(z))$



Almost all contributions from spins $j+1 \lesssim s$

Decay widths

Coefficient of the double residue computes decay widths



In agreement with
[Okada, Tsuchiya '89]

$$\underset{s=1}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi^2}{420}\ ,$$

$$\underset{s=2}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{\pi^2(t+1)}{420}\ ,$$

$$\underset{s=3}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = \frac{10883\pi^2(t+1)(t+2)}{8981280}\ ,$$

$$\vdots$$

$$\underset{s=10}{\mathrm{DRes}}\,\mathrm{Im}\,A^{\mathrm{p}}_{\mathrm{an}} = 6.8078\cdot 10^{-8}\times(t+1.00045)(t+2.00087)(t+3.0015)(t+4.0028)$$
$$\times(t+5)(t+5.9972)(t+6.9985)(t+7.99913)(t+8.99955)\ .$$

Finally, α' expansion is straightforward

$$
\begin{aligned}
\mathrm{Im}\, A_{\mathrm{I}} = \pi^2 g_s^4 t_8 \mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \Bigg[& \frac{\alpha' \mathrm{Im}\left[(N{-}4)\mathcal{I}_{\mathrm{box}}(s,t) - 2\mathcal{I}_{\mathrm{box}}(s,u)\right]}{120} \\
& + \frac{\zeta_2}{180}\alpha'^3 (N{-}3)s^3 + \frac{\zeta_3}{1260}\alpha'^4 s^3((4N{-}22)s + (N{-}2)t) \\
& + \frac{\zeta_2^2}{50400}\alpha'^5 s^3 \left(2(92N{-}219)s^2 + (15{-}8N)st + (4N{-}9)t^2\right) \\
& + \frac{\zeta_5}{15120}\alpha'^6 s^3 \left((38N{-}208)s^3 + 6(2N{-}5)s^2 t + 3(N{-}4)st^2 + (N{-}2)t^3\right) \\
& + \frac{\zeta_2\zeta_3}{5040}\alpha'^6 s^4 \left(12(N{-}3)s^2 + t((N{-}2)u + t) + st\right) \\
& + \frac{\zeta_3^2}{30240}\alpha'^7 s^4 \left(4(5N{-}28)s^3 + 2(N{+}1)s^2 t - 3(N{-}4)st^2 - (N{-}2)t^3\right) \\
& + \frac{\zeta_2^3}{5292000}\alpha'^7 s^3 \big(70(176N{-}383)s^4 + 25(9{-}11N)s^3 t + 3(119N{-}347)s^2t^2 \\
& \qquad + 4(17{-}9N)st^3 + 2(16N{-}33)t^4\big) \\
& + \frac{\zeta_7}{831600}\alpha'^8 s^3 \big(20(83N{-}452)s^5 + 5(129N{-}368)s^4 t + 2(148N{-}593)s^3t^2 \\
& \qquad + (137N{-}362)s^2t^3 + 4(9N{-}29)st^4 + 10(N{-}2)t^5\big) \\
& + \frac{\zeta_2^2\zeta_3}{756000}\alpha'^8 s^4 \big(60(20N{-}47)s^4 + 5(66{-}23N)s^3 t + 6(33{-}8N)s^2t^2 \\
& \qquad - (N{-}6)st^3 + 2(9{-}4N)t^4\big) \\
& + \frac{\zeta_2\zeta_5}{37800}(N{-}3)\alpha'^8 s^4 \big(70s^4 - 5s^3t - 6s^2t^2 - 2st^3 - t^4\big) + \mathcal{O}(\alpha'^9)\Bigg] + \ldots .
\end{aligned}
$$

Coefficient of N in agreement with [Edison, Guillen, Johansson, Schlotterer, Teng '21]

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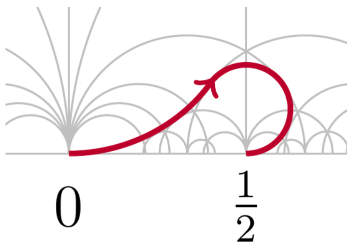

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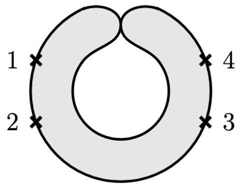

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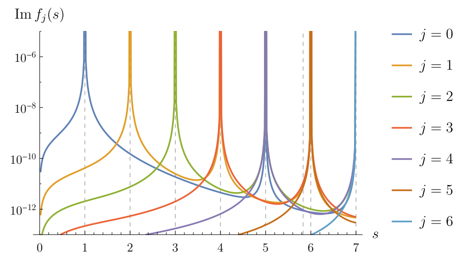

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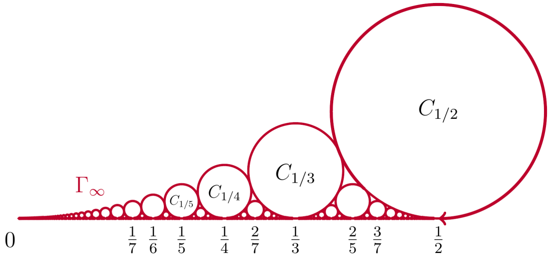


The idea is to recycle the computation of a single circle (infinitely) many times

0

Farey sequence

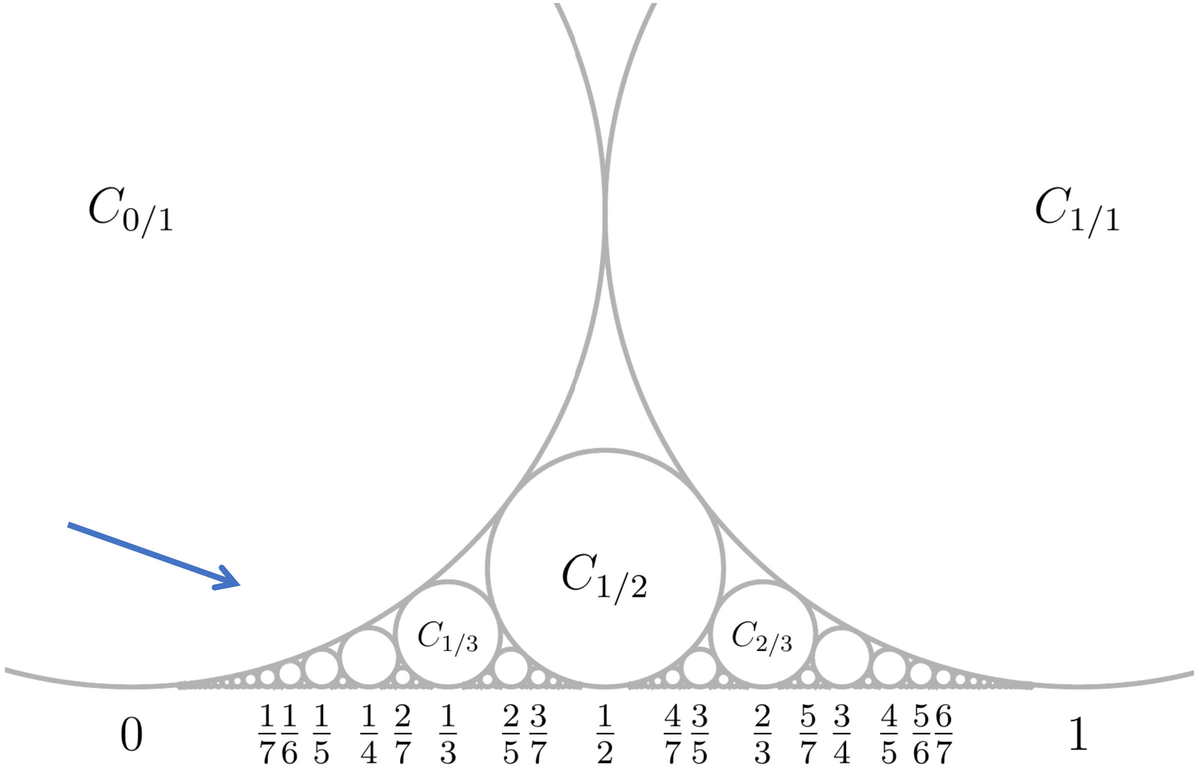
$F_q =$ all irreducible fractions between 0 and 1 with the denominator $\leqslant q$

$$
\begin{aligned}
F_1 &= \left(\tfrac{0}{1}, \tfrac{1}{1}\right) \\
F_2 &= \left(\tfrac{0}{1}, \tfrac{1}{2}, \tfrac{1}{1}\right) \\
F_3 &= \left(\tfrac{0}{1}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{1}{1}\right) \\
F_4 &= \left(\tfrac{0}{1}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{1}{1}\right) \\
F_5 &= \left(\tfrac{0}{1}, \tfrac{1}{5}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{5}, \tfrac{1}{2}, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{1}{1}\right) \\
&\vdots
\end{aligned}
$$

Ford circles

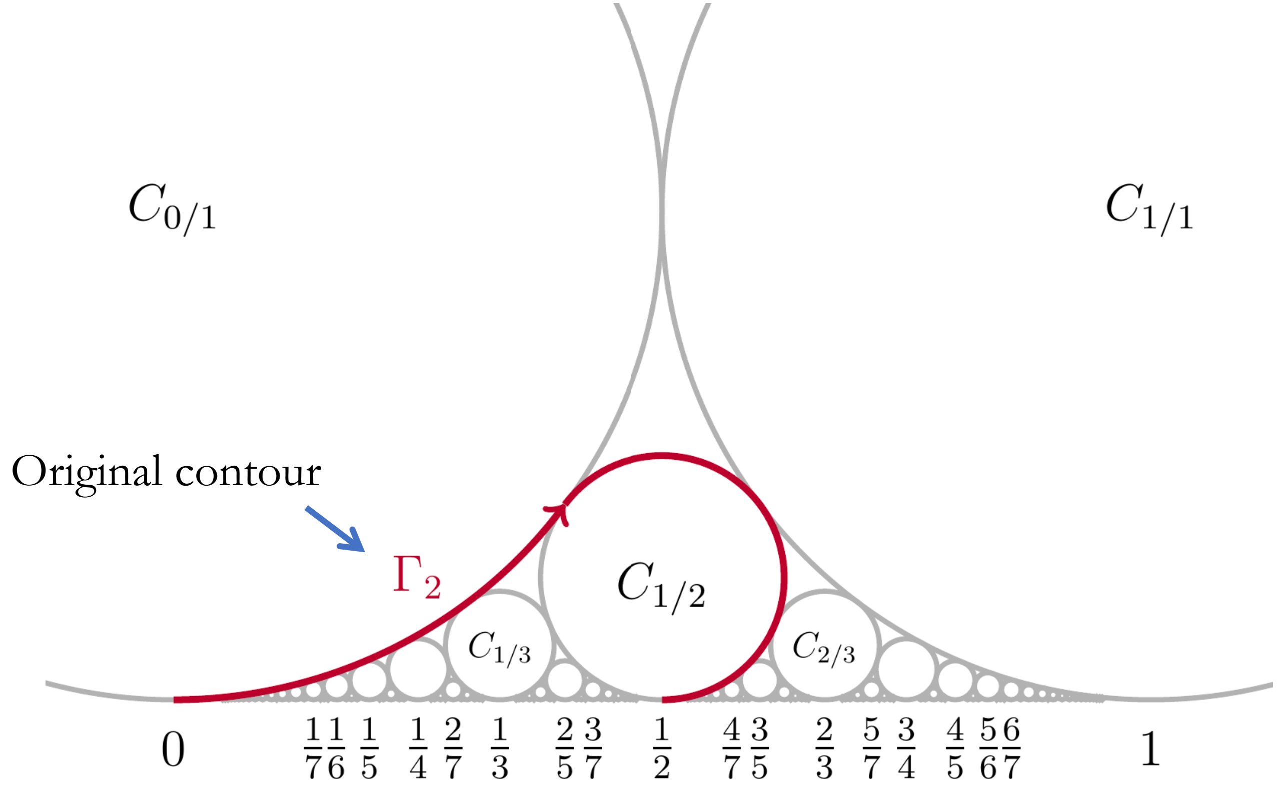
$$C_{p/q} = \text{ circle touching the real axis at } \frac{p}{q} \text{ with radius } \frac{1}{2q^2} \text{ in the } \tau \text{ plane}$$

Each one is a modular transform of $C_{0/1}$

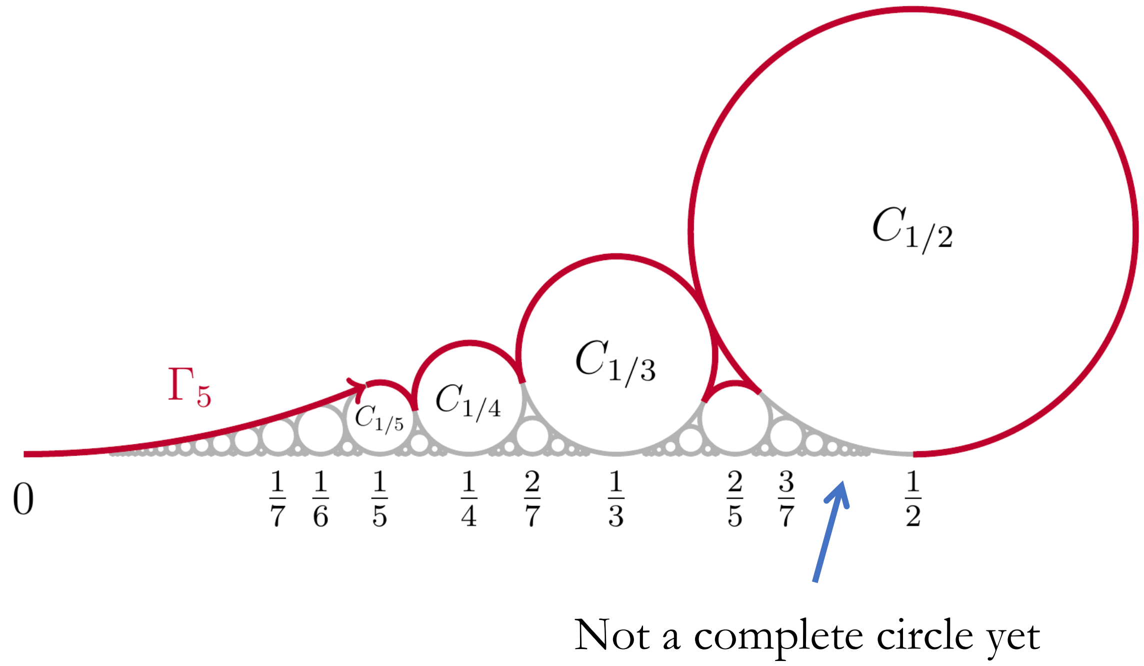


Rademacher contour

$\Gamma_q =$ follow all the Ford circles in the Farey sequence F_q from 0 to ½

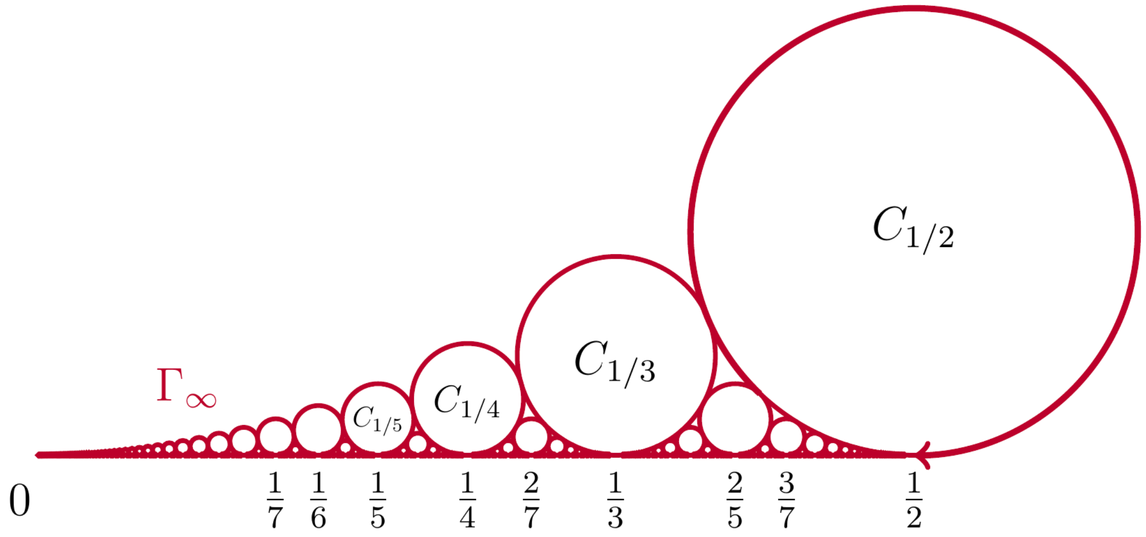


... and so on



Γ_5
$C_{1/5}$
$C_{1/4}$
$C_{1/3}$
$C_{1/2}$
0
$\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{2}{7}$ $\frac{1}{3}$ $\frac{2}{5}$ $\frac{3}{7}$ $\frac{1}{2}$
Not a complete circle yet

In the limit, we enclose all the circles



In all cases we observed that this series converges!
Stay tuned

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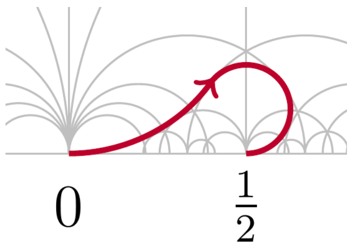

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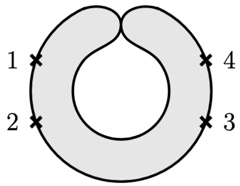

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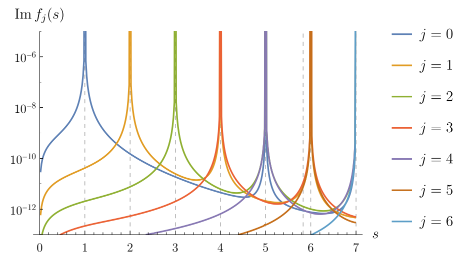

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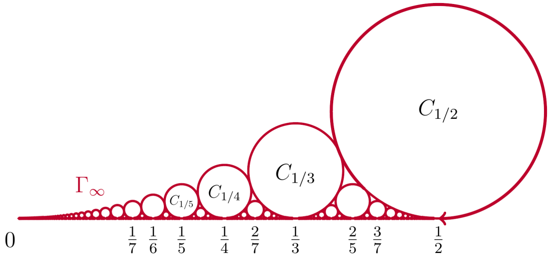

Thank you!