Adventures in Perturbation Theory

Jacob Bourjaily
Penn State University &
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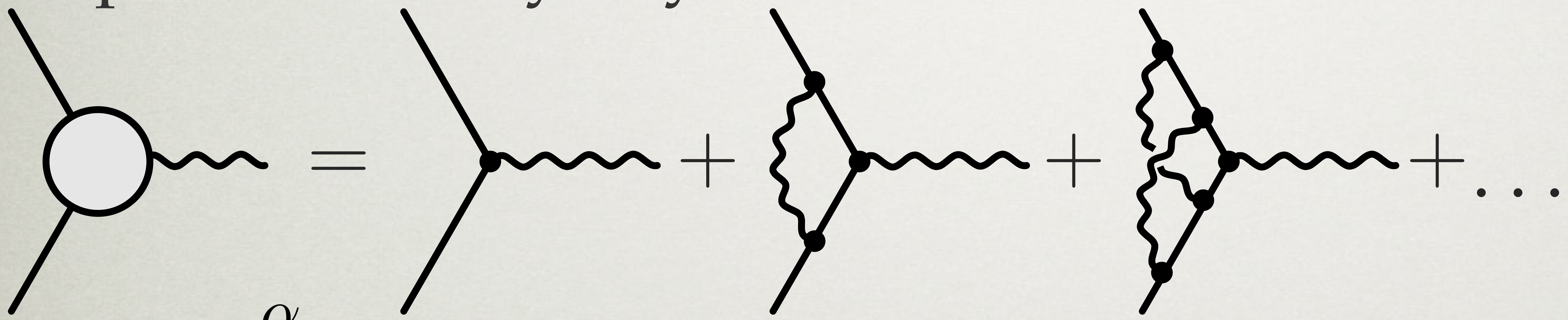
Current Themes in High Energy Physics and Cosmology

2 August, 2022

What Form do Observables Take?

- What is the *mathematical form* of the predictions made by QFT? (perturbatively, say?)



$$g_e = \;\; 2 + \frac{\alpha}{\pi}(1)$$

[Dirac (1933)]

[Feynman; Schwinger; Tomanaga (1947)]

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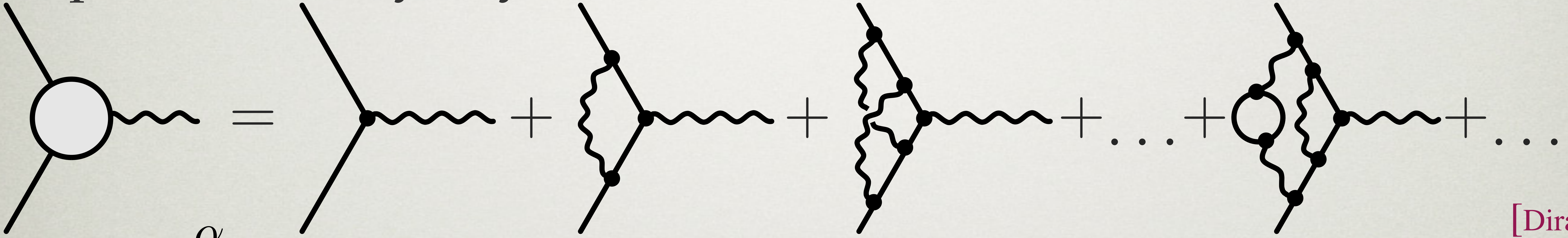


$$g_e = \; 2 + \frac{\alpha}{\pi}(1)$$
$$+\frac{\alpha^2}{\pi^2}\Big(\frac{3}{2}\zeta_3 - \pi^2\log(2) + \zeta_2 + \frac{197}{72}\Big)$$
$$-\frac{\alpha^3}{\pi^3}\Big(\frac{215}{12}\zeta_5 + \frac{100}{3}u_{3,1} + \frac{13}{4}\zeta_4 - \frac{139}{9}\zeta_3 + \frac{1192}{3}\zeta_2\log(2) - \frac{34202}{135}\zeta_2 - \frac{28259}{2592}\Big)$$

[Dirac (1933)]
[Feynman; Schwinger; Tomanaga (1947)]
[Laporta, Remiddi (1996)] [Petermann (1957)]
[Laporta (2017)] [Kinoshita (1990)]

* $(u_{3,1}$ is related to $\mathrm{Li}_4(1/2)$ and $\log(2))$

beyond three loops, no analytic expression is known

(CY periods appear)

What Form do Observables Take?

- What is the *mathematical form* of the predictions made by QFT? (perturbatively, say?)

e.g. maximally supersymmetric ($\mathcal{N}=4$) Yang-Mills theory (planar limit)

[Beisert, Eden, Staudacher (2007);…]

$$
\begin{aligned}
\gamma_{\text{cusp}} = \;& a\times 1\\
&+a^2\times 2\,\zeta_2\\
&+a^3\times 22\,\zeta_4\\
&+a^4\times 2\big(24\,\zeta_2^3+4\,\zeta_3^2+2\,\zeta_2\,\zeta_4+\zeta_6\big)\\
&+a^5\times 8\big(252\,\zeta_4^2+20\,\zeta_3\,\zeta_5+4\,\zeta_2\,\zeta_3^2+\zeta_2\,\zeta_6\big)\\
&+a^6\times 8\big(282\,\zeta_2^5+\zeta_2^3\zeta_4+4\,\zeta_2\,\zeta_4^2+80\,\zeta_2\,\zeta_3\,\zeta_5+5\,\zeta_2^2\,\zeta_6+48\,\zeta_3^2\,\zeta_4+102\,\zeta_5^2+210\,\zeta_3\,\zeta_7+3\,\zeta_{10}\big)\\
&+\ldots
\end{aligned}
$$

$\Rightarrow$ "maximal *transcendentality*" of planar $\mathcal{N}=4$ super Yang-Mills (?)

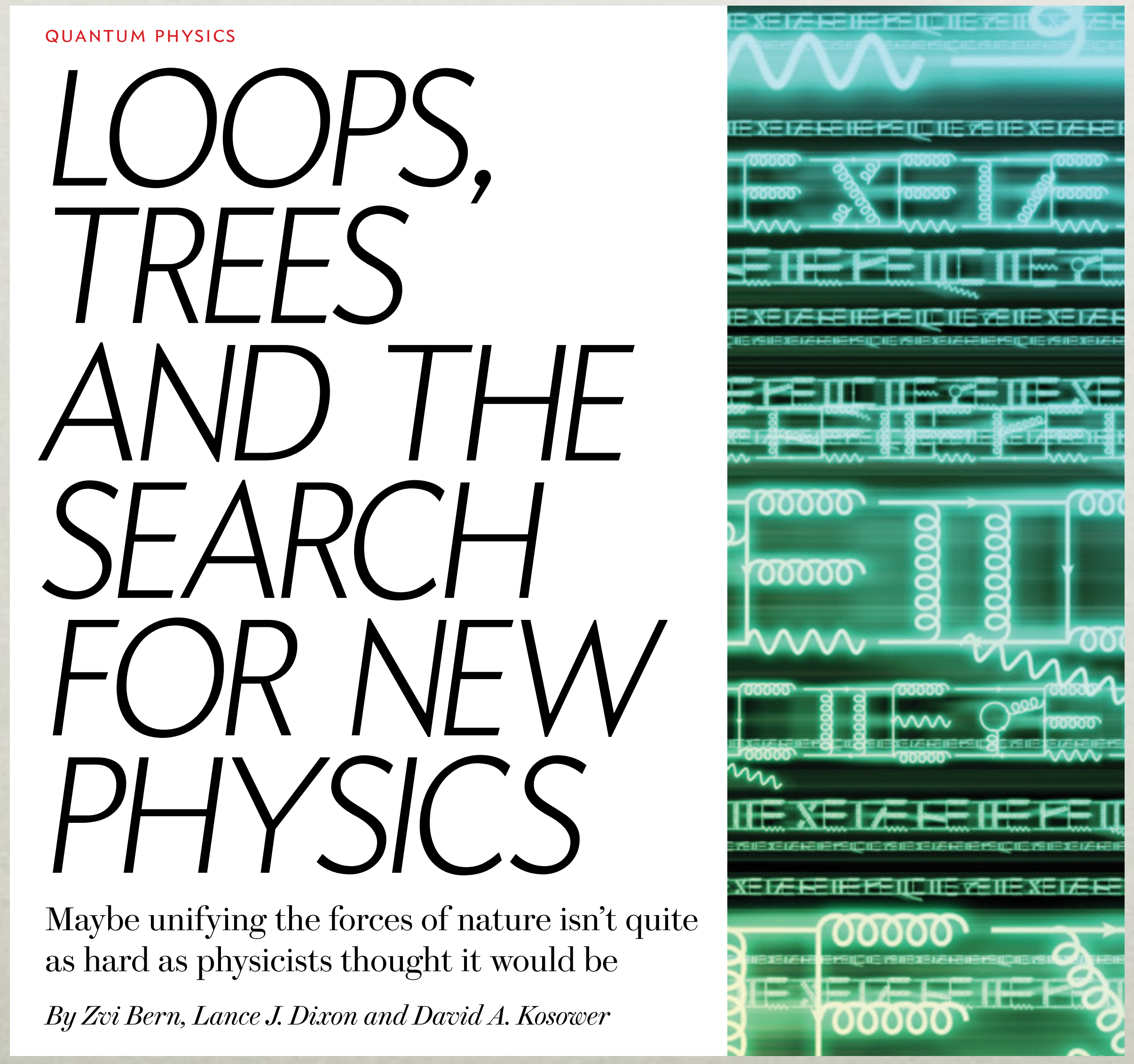
Explosions of Complexity

- While ultimately correct, the Feynman expansion renders *all but the **most trivial*** predictions—

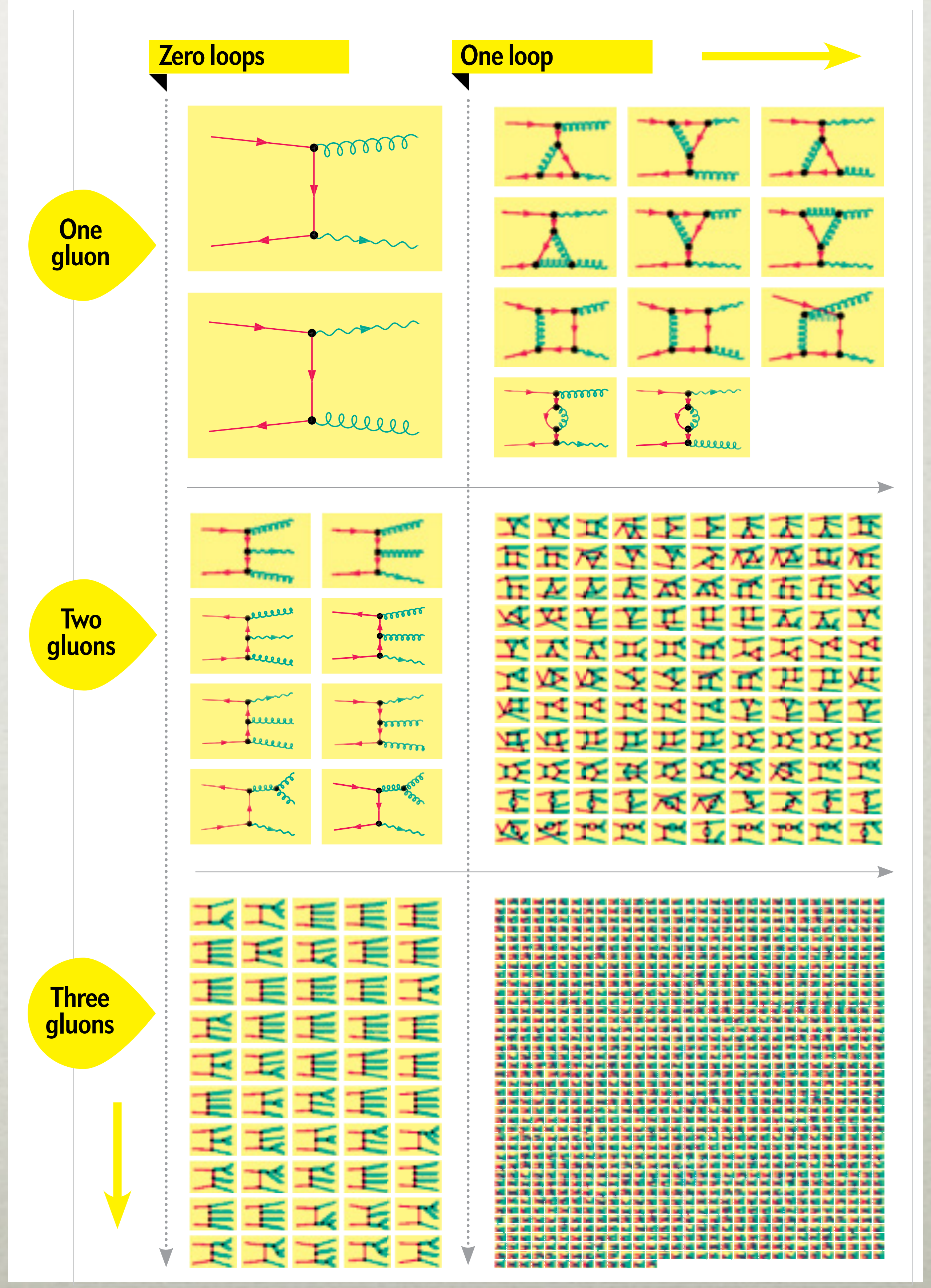
  involving the fewest particles, at the **lowest orders** of perturbation—

  computationally ***intractable***

  or theoretically ***inscrutable***


QUANTUM PHYSICS
LOOPS, TREES AND THE SEARCH FOR NEW PHYSICS
Maybe unifying the forces of nature isn't quite as hard as physicists thought it would be
By Zvi Bern, Lance J. Dixon and David A. Kosower

[Bern, Dixon, Kosower, *Scientific American* (2012)]



Needs (Once) Beyond Our Reach

- Background amplitudes **crucial** for *e.g.* colliders

Supercollider physics [*Rev.Mod.Phys*. **56** (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

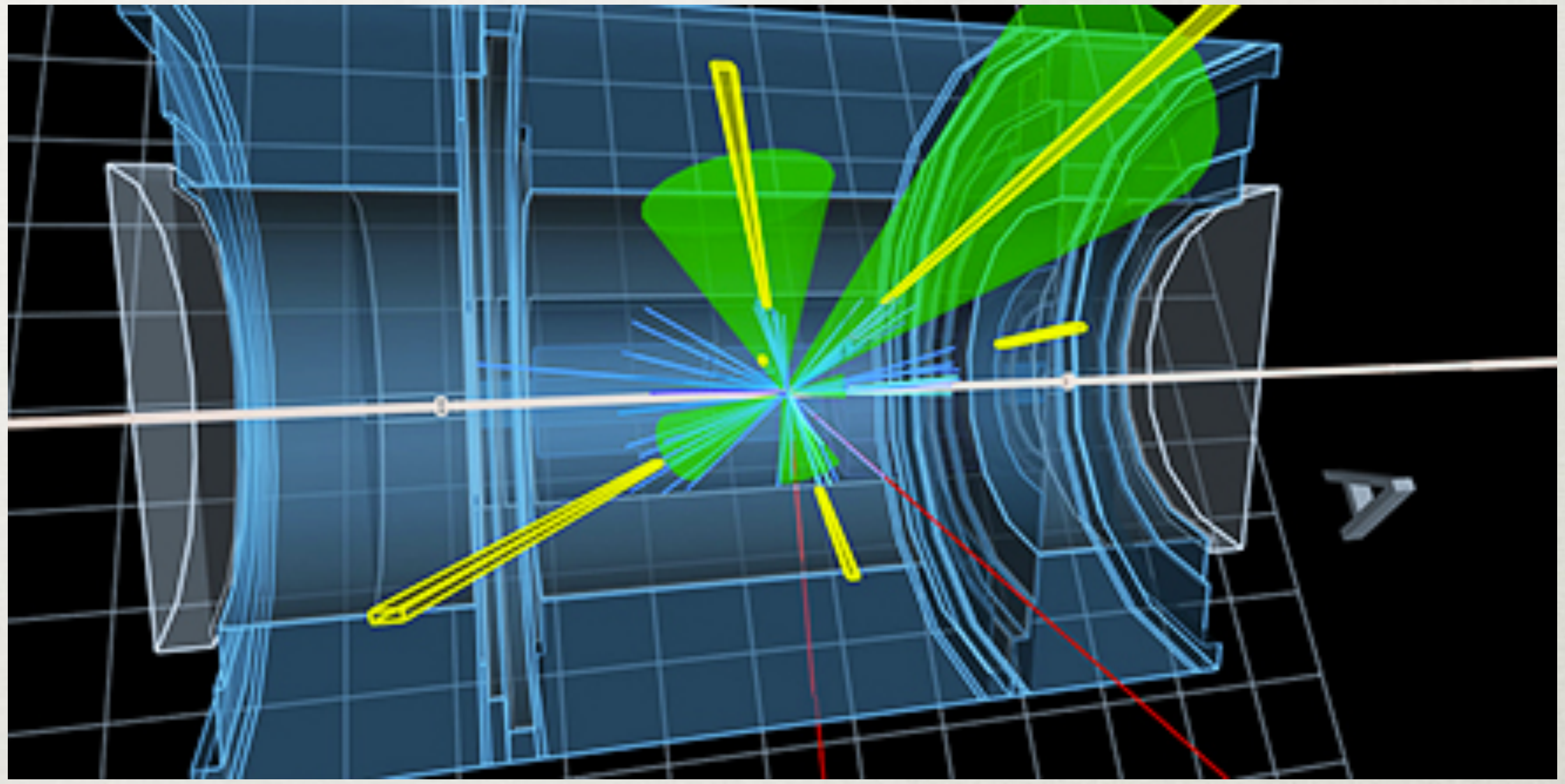
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



- Once considered ***computationally intractable***

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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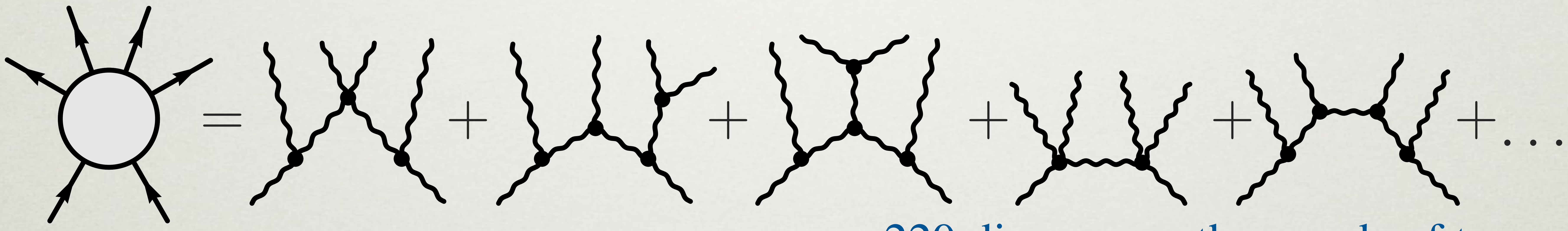

220 diagrams — thousands of terms

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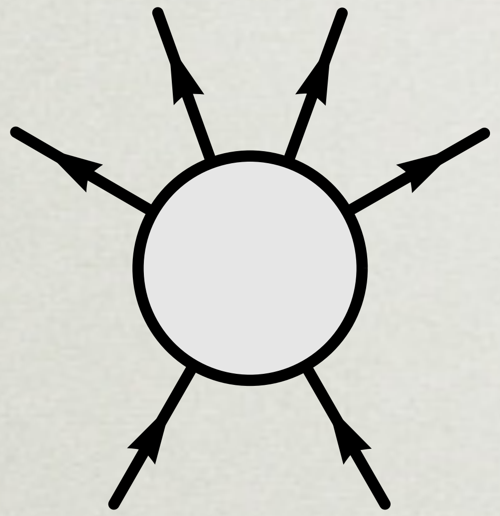

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON–GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys*. **B269** (1985)]

- Once considered ***computationally intractable***

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

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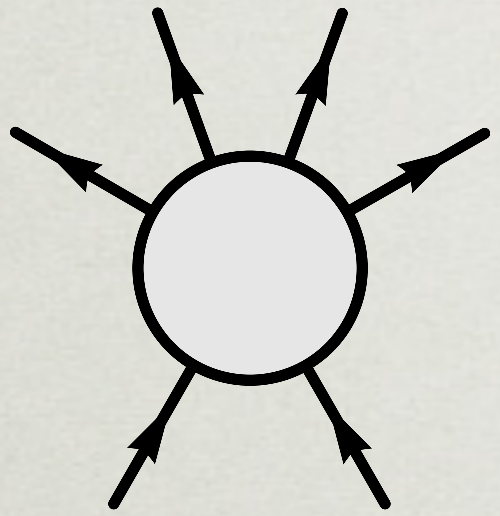

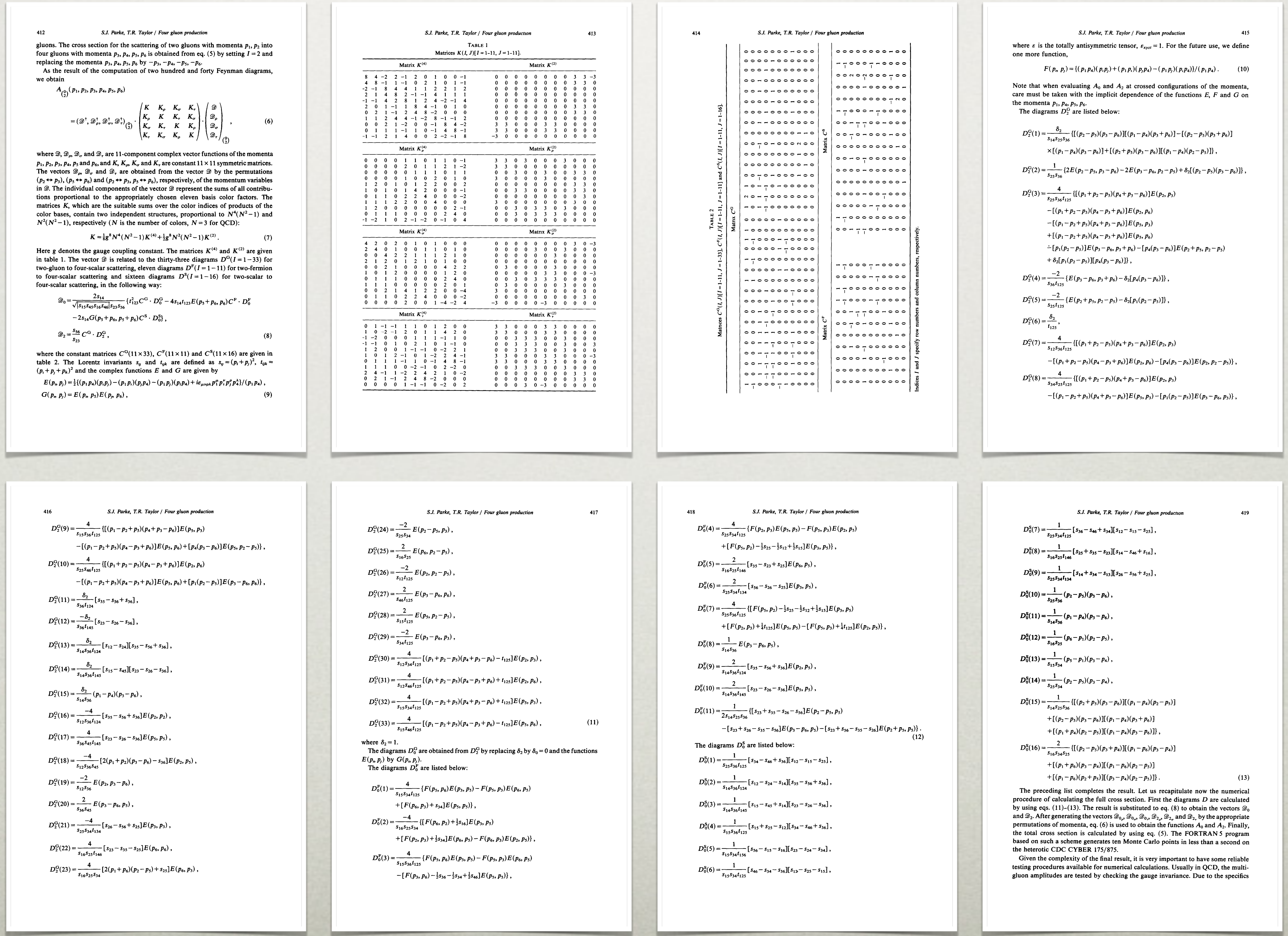

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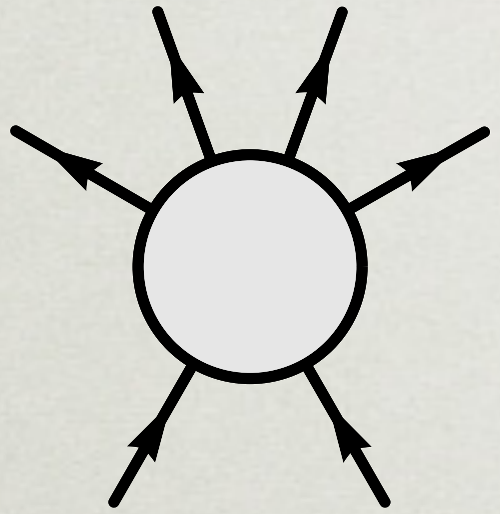
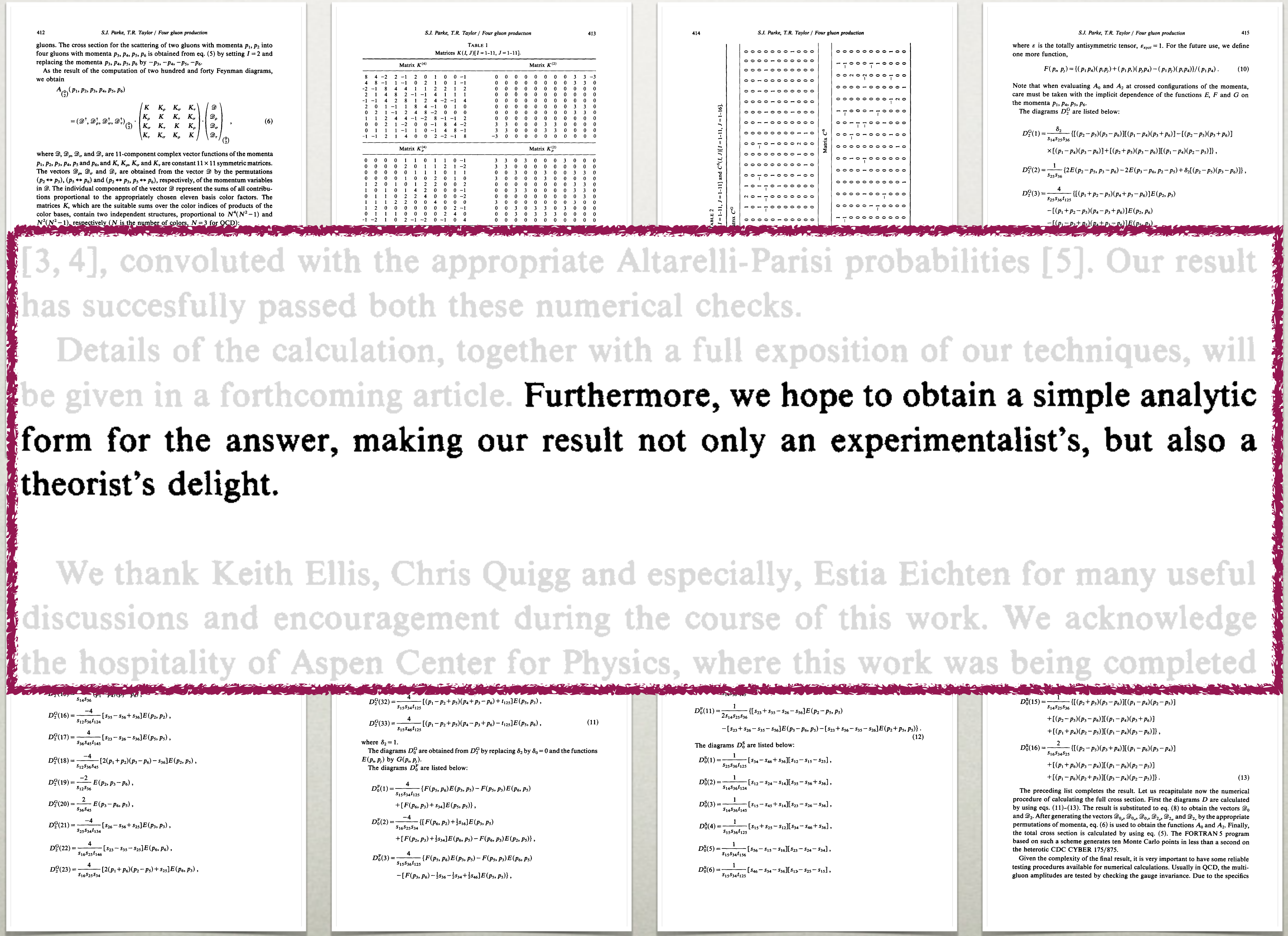




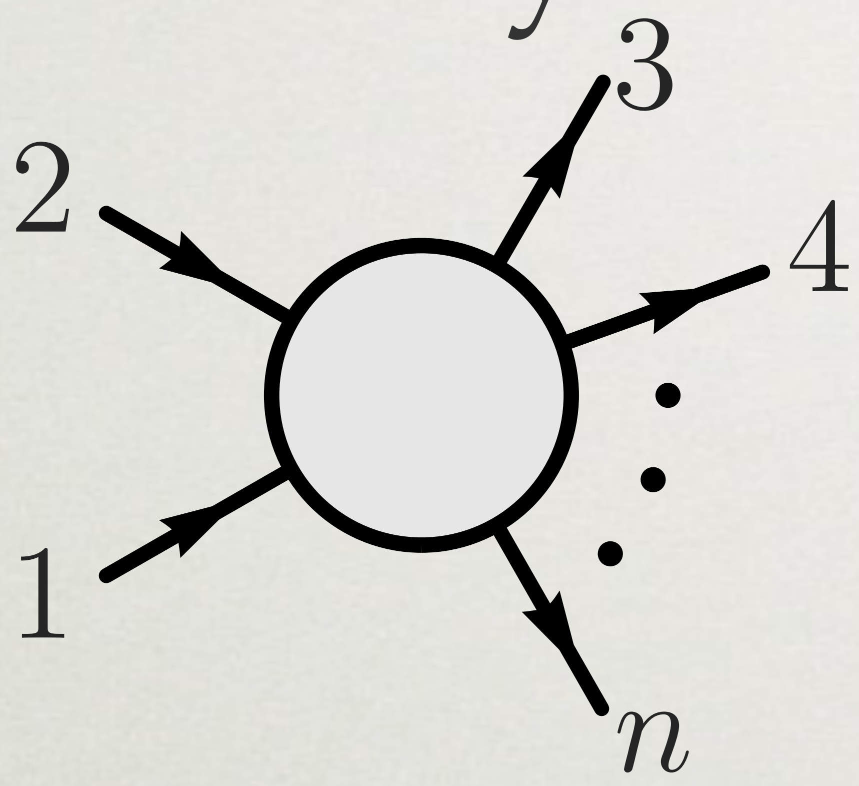
[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. **Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.**

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

Discovery of Shocking Simplicity

- Within six months, Parke-Taylor stumbled on a simple guess—unquestionably a *theorist's delight*:



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle}$$

Amplitude for *n*-Gluon Scattering [*PRL* **56** (1986)]

Stephen J. Parke and T. R. Taylor

Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^\mu \equiv \sigma^\mu_{\alpha\dot\alpha}\lambda_a^\alpha\widetilde{\lambda}_a^{\dot\alpha}$$

$$\langle a\,b\rangle \equiv \det(\lambda_a,\lambda_b)$$

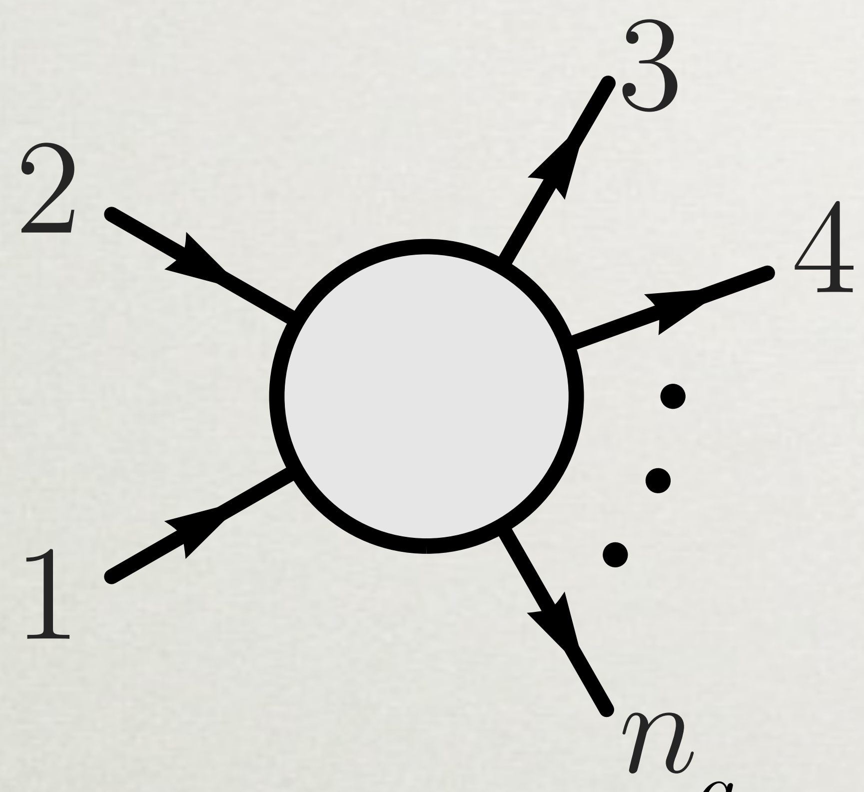
$$[a\,b] \equiv \det(\widetilde{\lambda}_a,\widetilde{\lambda}_b)$$

[van der Waerden (1929)]

Perturbations of Parke/Taylor's Guess

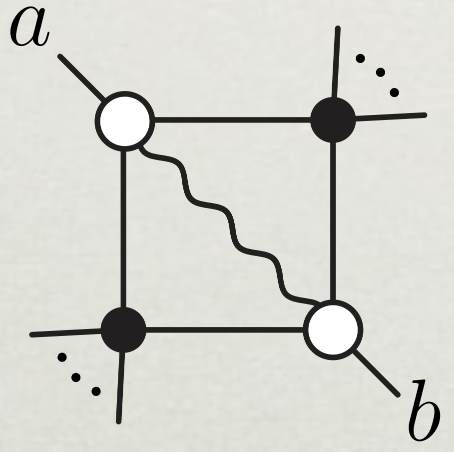
- What about beyond the leading order of approximation?

[Bern, Dixon, Dunbar, Kosower (1994)]



$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle} \times$$

$$\left\{1 + \sum_{a<b}\right.$$

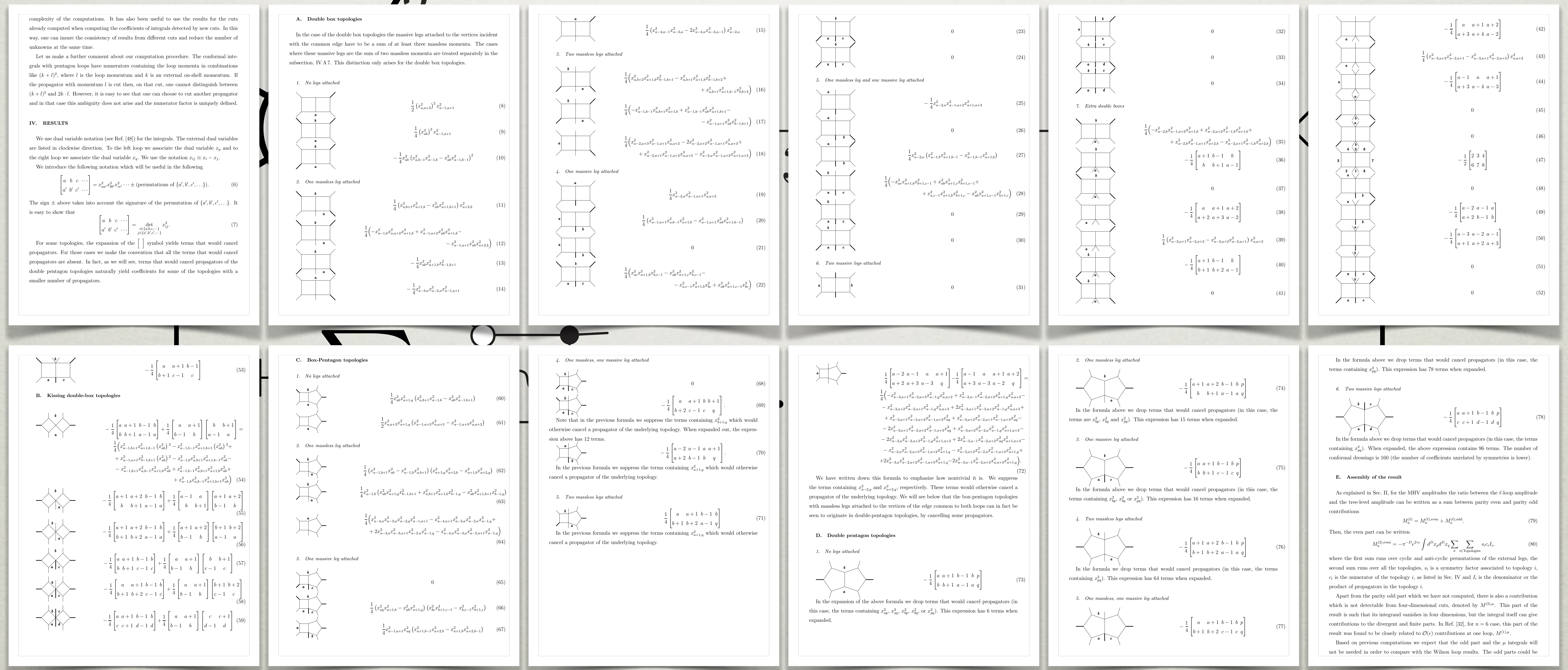


$$\left. + \ldots \right\}$$

Perturbations of Parke/Taylor's Guess

- What about beyond the leading order of approximation?

[Vergu (2009)]



Perturbations of Parke/Taylor's Guess

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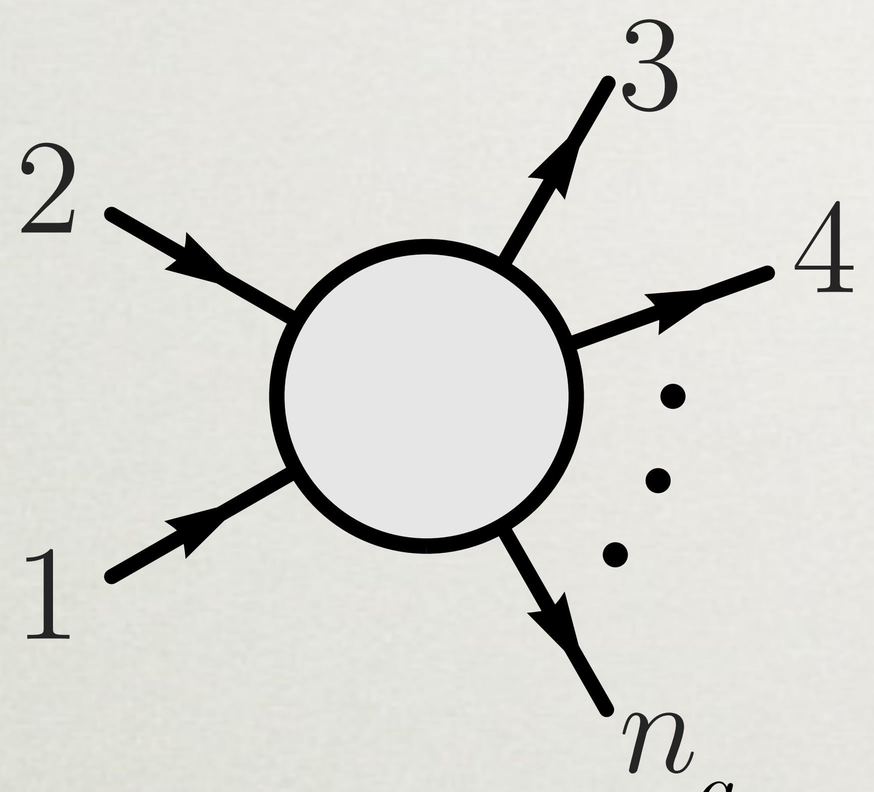
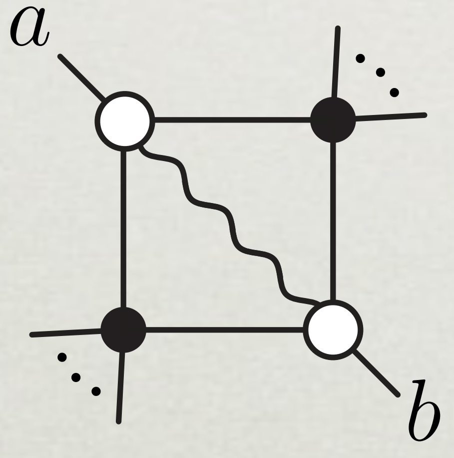
[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]

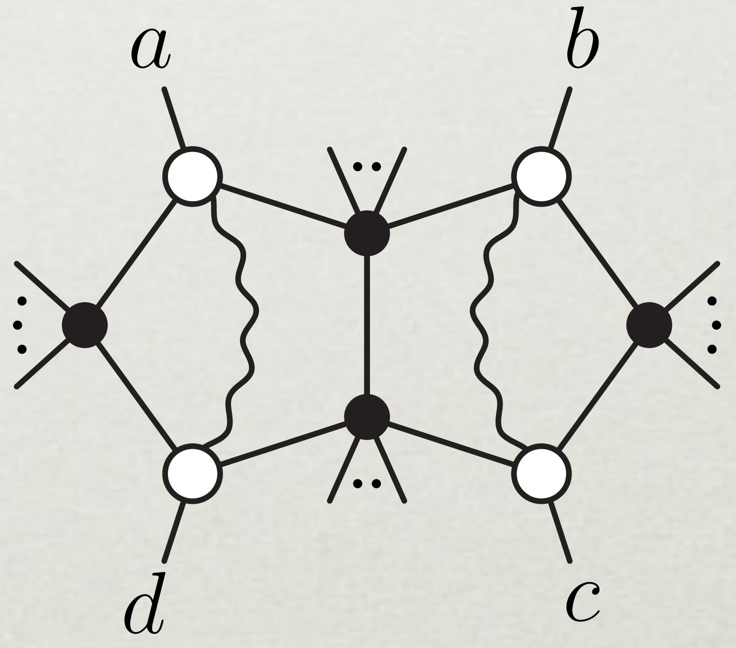


$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\,\cdots\,\langle n\,1\rangle} \times$$

$$\Bigg\{ 1 + \sum_{a<b}$$



$$+ \sum_{a<b<c<d}$$



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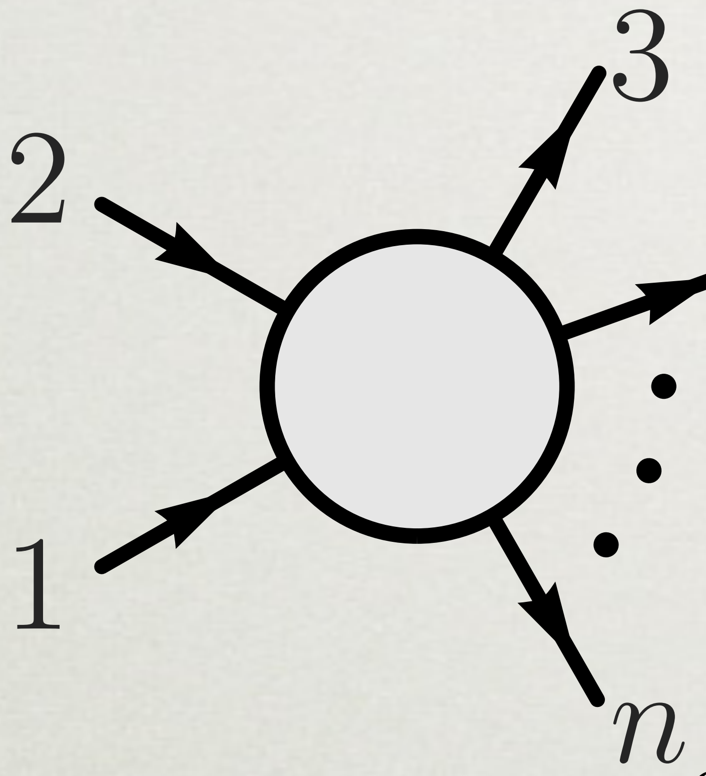
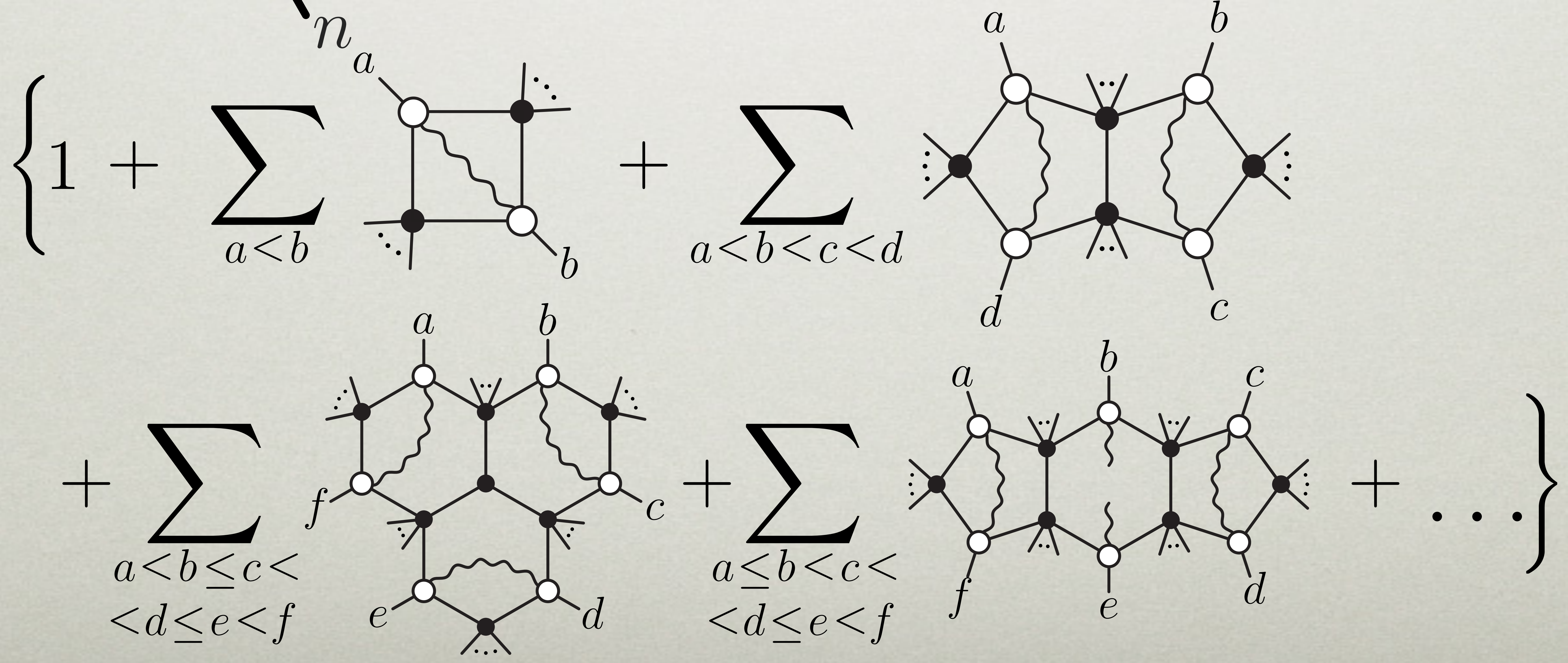


$$= \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\;\cdots\;\langle n\,1\rangle}\times$$



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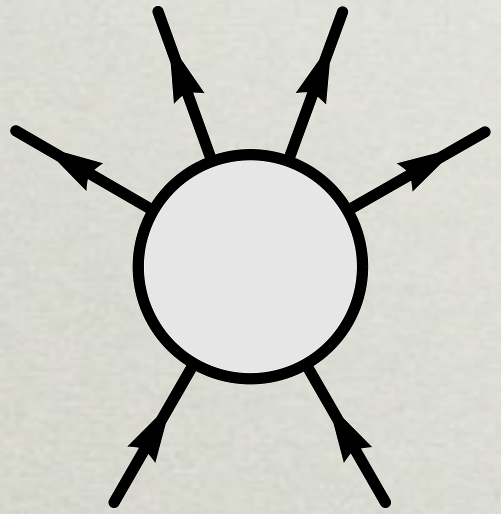
- But what about after **regularization** and ***loop integration***?
  What is the ***mathematical form*** of the predictions made by QFT?



The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

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Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia
E-mail: smirnov@theory.sinp.msu.ru

[Del Duca, Duhr, Smirnov (2010)]

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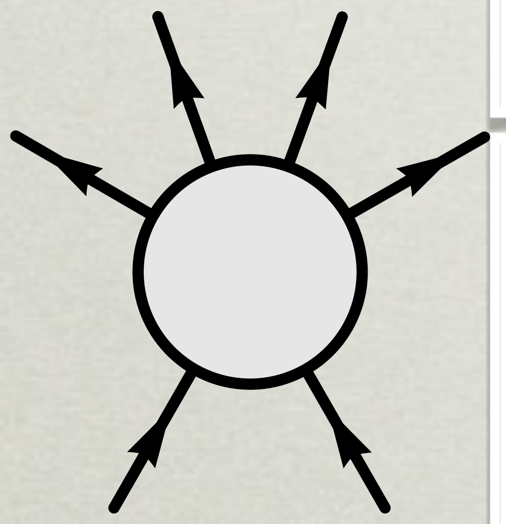

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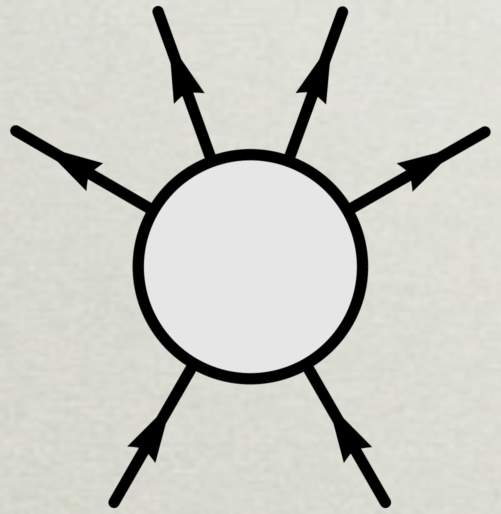

Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,[1] M. Spradlin,[2] C. Vergu,[2] and A. Volovich[2]

[1] *Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*
[2] *Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R(u_1,u_2,u_3)=\sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-)-\frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2+\frac{J^4}{24}+\frac{1}{2}\zeta_2(J^2+\zeta_2)$$

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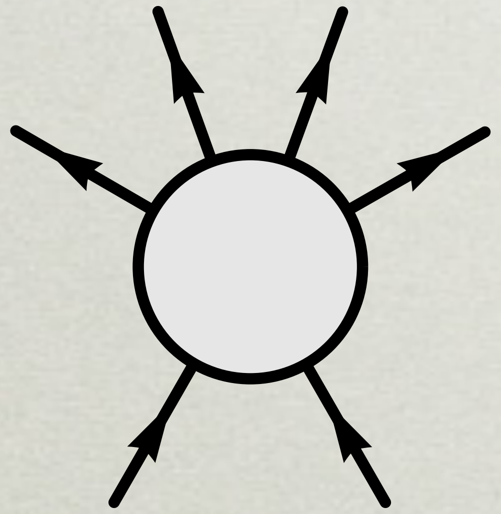

<u>State of the art:</u>

6-point (N)MHV @ (6) 7 loops(!!!)

7-point (N)MHV @ 4 loops (symbol-level) [Dixon, *et al* (2019);…]

Amplitudes: a Virtuous Cycle

Compute Something
beyond the reach of
recent imagination

Discover Simplicity
beyond expectations

Understand Why
study it, understand it,
& explore consequences

Exploit Simplicity
to build **more powerful**
computational technology

What Form do Observables Take?

- In a general (say, 4d) QFT, it would have ***recently been*** expected by "experts" that observables took the following general form:

(general dimension d: $2 \mapsto \lfloor d/2 \rfloor$)

$$\mathcal{A} = \mathcal{A}^{\text{tree}} + \hbar\,\mathcal{A}^{(L=1)} + \hbar^2\,\mathcal{A}^{(L=2)} + \ldots + \hbar^L\,\mathcal{A}^{(L)} + \ldots$$

$\mathcal{A}^{\text{tree}}$	$\hbar\,\mathcal{A}^{(L=1)}$	$\hbar^2\,\mathcal{A}^{(L=2)}$	…	$\hbar^L\,\mathcal{A}^{(L)}$	
rational	+ (weight-2 polylogs)	+ (weight-4 polylogs)	+ …	+ (weight-2L polylogs)	+ …
	+ (weight-1 polylogs)	+ ⋮	+ …	+ ⋮	
	+ rational	+ (weight-1 polylogs)	+ …	+ ⋮	
		+ rational			

coefficients:
leading singularities

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$$\boxed{\text{rational} + \begin{pmatrix}\text{weight-2}\\ \text{polylogs}\end{pmatrix} + \begin{pmatrix}\text{weight-4}\\ \text{polylogs}\end{pmatrix} + \cdots + \begin{pmatrix}\text{weight-2L}\\ \text{polylogs}\end{pmatrix} + \cdots}$$

planar $\mathcal{N}=4$?

$$+ \begin{pmatrix}\text{weight-1}\\ \text{polylogs}\end{pmatrix} + \vdots + \cdots + \vdots$$

$$+ \text{rational} + \begin{pmatrix}\text{weight-1}\\ \text{polylogs}\end{pmatrix} + \cdots + \vdots$$

$$+ \text{rational}$$

coefficients:
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	+	(weight-1 polylogs)	+	⋮	+ ⋯	+	⋮	
	+	rational	+	(weight-1 polylogs)	+ ⋯	+	⋮	
			+	rational				

planar $\mathcal{N}=4$?

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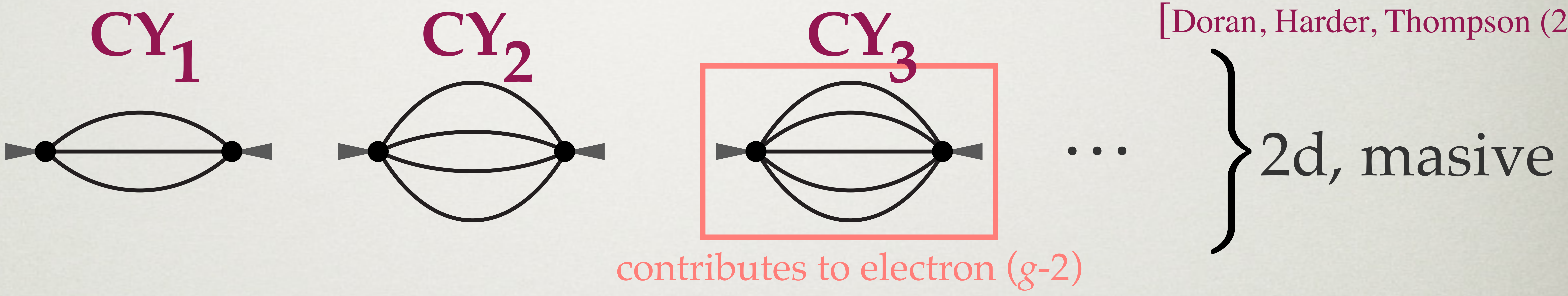
What Form do Observables Take?

- Unfortunately, many pesky counterexamples were to be found:

[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;…]
[Doran, Harder, Thompson (2019)]

sunrises:



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CY_1 CY_2 CY_3

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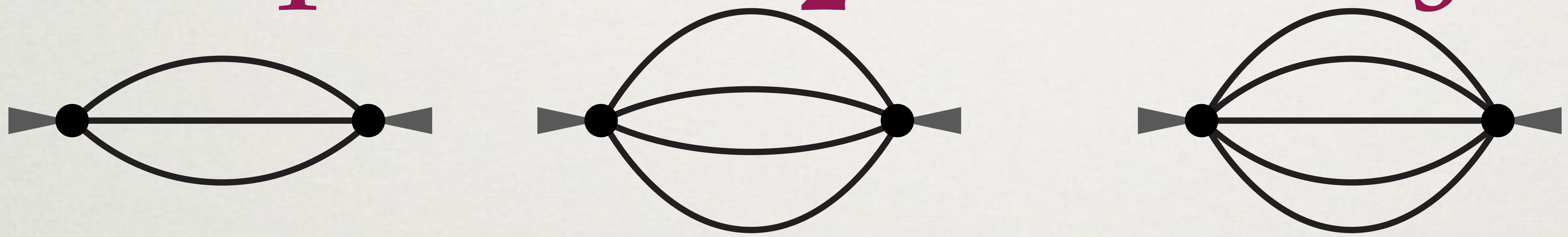
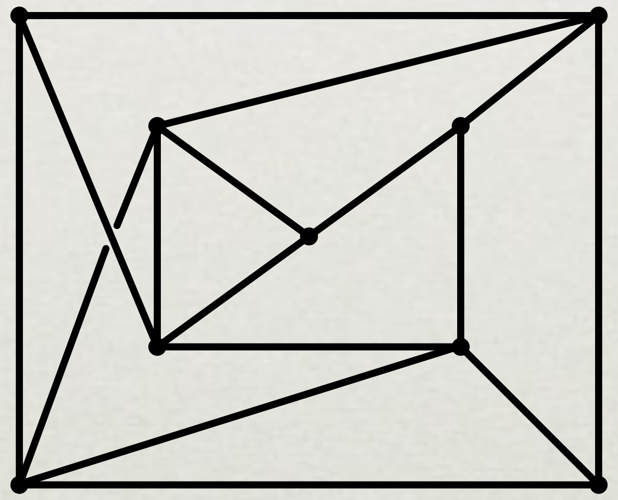


⋯ } 2d, masive



an 8-loop vacuum graph
evaluating to a **K3 period**

} 4d, massless

[Brown, Schnetz (2011)]

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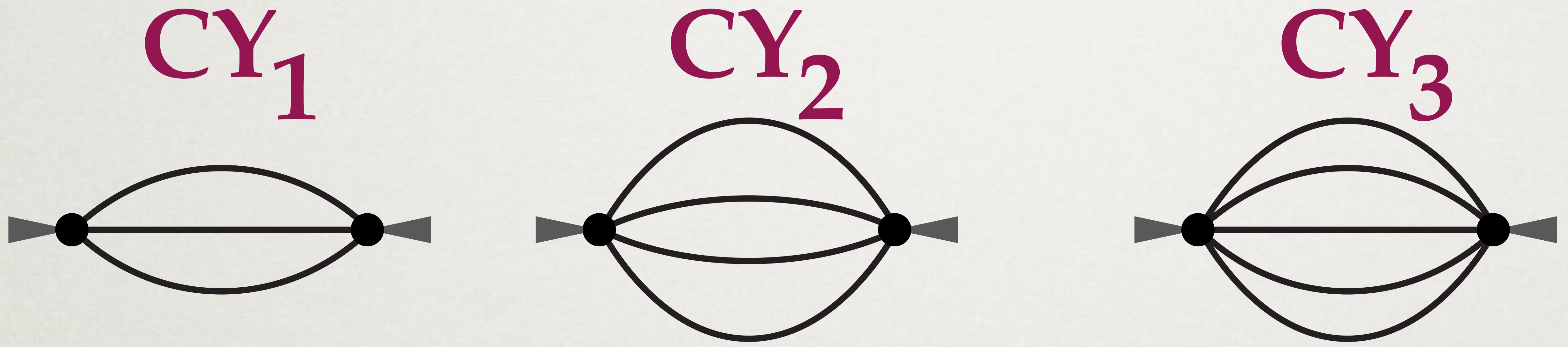
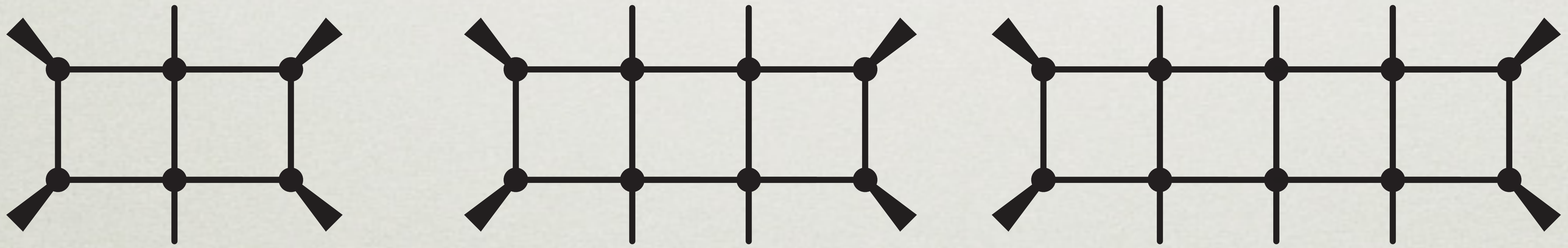


$\cdots$ } 2d, masive

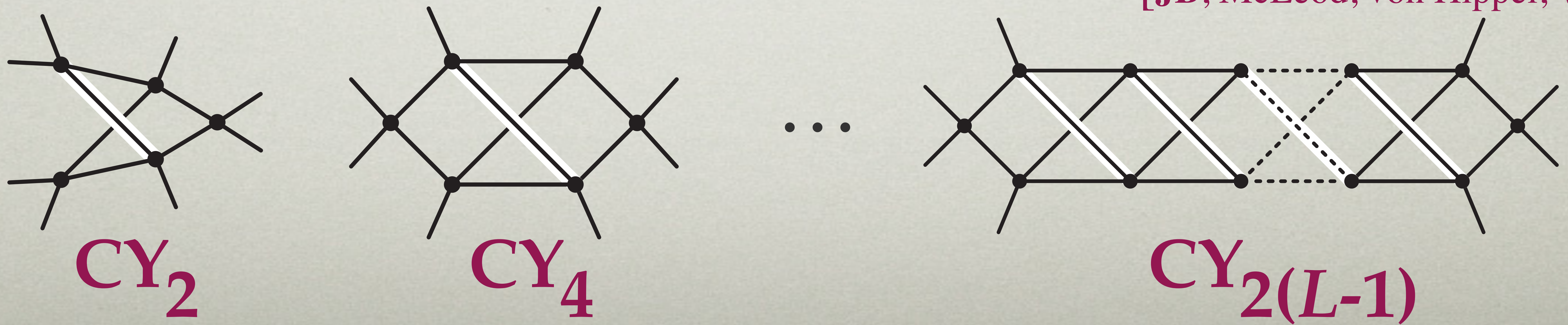
traintracks:



$\cdots$ } 4d, massless

[**JB**, McLeod, von Hippel, Wilhelm (2018)]

tardigrades:



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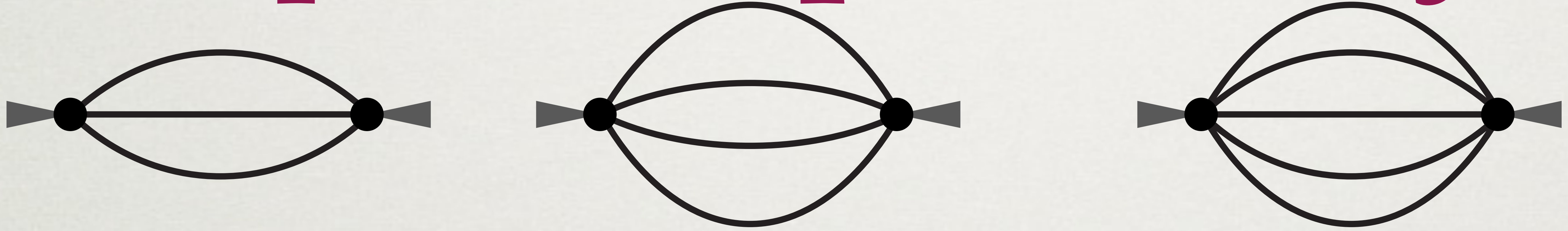


… } 2d, masive

traintra

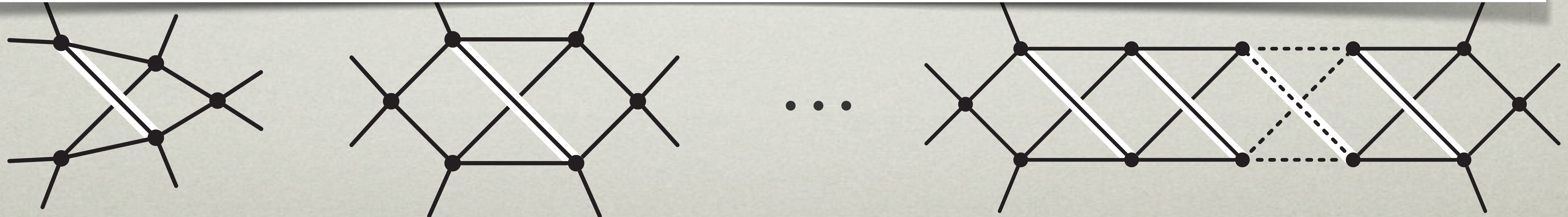
Outside a *small* list of **extremely** limited / simple cases (which are widely expected* to be exceptions):

almost all observables in ***almost all*** QFTs
are expected to be non-polylogarithmic

nassless

, Wilhelm (2018)]

tardigrades:



CY_2 CY_4 … $CY_{2(L\text{-}1)}$

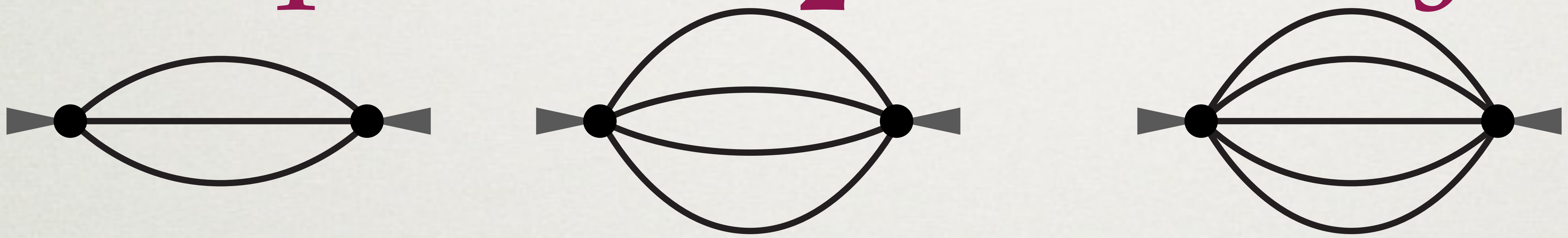
What Form do Observables Take?

✦ Unfortunately, many pesky counterexamples were to be found:

[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;…]
[Doran, Harder, Thompson (2019)]

CY_1 CY_2 CY_3

sunrises:



… } 2d, masive

tra

ss

(2018)]

✦ In every instance known, the Calabi-Yau itself is **very** simple:

- degree-$(d{+}1)$ hypersurface in $\mathbb{P}^d$ (or multiple-cover thereof)

✦ kinematic data (momenta/masses) control the moduli

- often **extremely** *singular*

tardigrades:

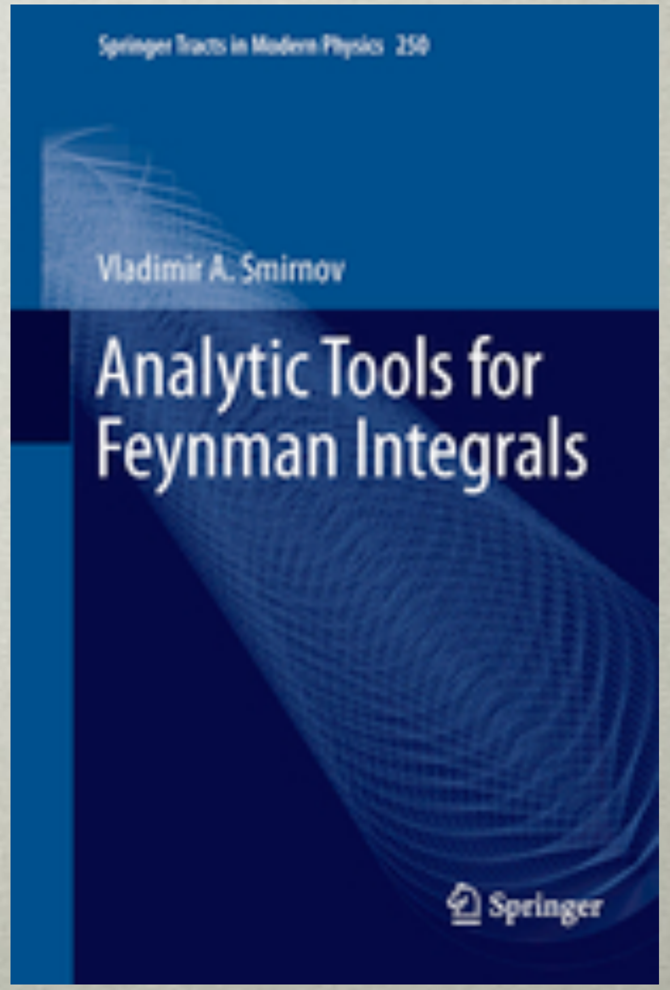
…

CY_2 CY_4 $\mathrm{CY}_{2(L\text{-}1)}$

Why is Perturbation Theory so Hard?

- **Feynman diagrams** (esp. with *scalar* numerators) are ***horrible***
  - difficult to integrate, explosive in number, non-physical,…
- **Regularization** obscures symmetries (+is technically difficult)
- Most familiar **mater integrand bases** are ***unnecessarily bad*:**
  - don't satisfy nice/canonical differential equations
  - contain ***multiple*** **elliptic**(+worse(!)) geometries,
  - …

$$\mathrm{I}_{a_1a_2a_3a_4a_5a_6a_7a_8a_9} = e^{2\gamma_E\epsilon}\int\!\!\int \frac{d^Dk_1 d^Dk_2}{(i\pi^{d/2})^2}\frac{P_8^{-a_8}\,P_9^{-a_9}}{P_1^{a_1}P_2^{a_2}P_3^{a_3}P_4^{a_4}P_5^{a_5}P_6^{a_6}P_7^{a_7}}$$



How can We Make it Easier?

- **Use unitarity** to choose the ***nicest / easiest*** integrals to integrate (***of course,*** integration "ease" changes with time and new methods)
  - search for as many ***pure*** integrals as you can —those which satisfy nice (canonical) differential equations

Definition: a function $f(s)$ is called ***pure*** if:

- there exists a grading of functions by "transcendental" ***weight***
- any derivative of $f(s)$ *is* ***strictly lower in weight***

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e.g. $g(s)\log\big(f(s)\big)$ would be ***impure***

$$\frac{\partial}{\partial s}\Big[g(s)\log\big(f(s)\big)\Big]=g'(s)\log\big(f(s)\big)+g(s)f'(s)/f(s)$$

How can We Make it Easier?

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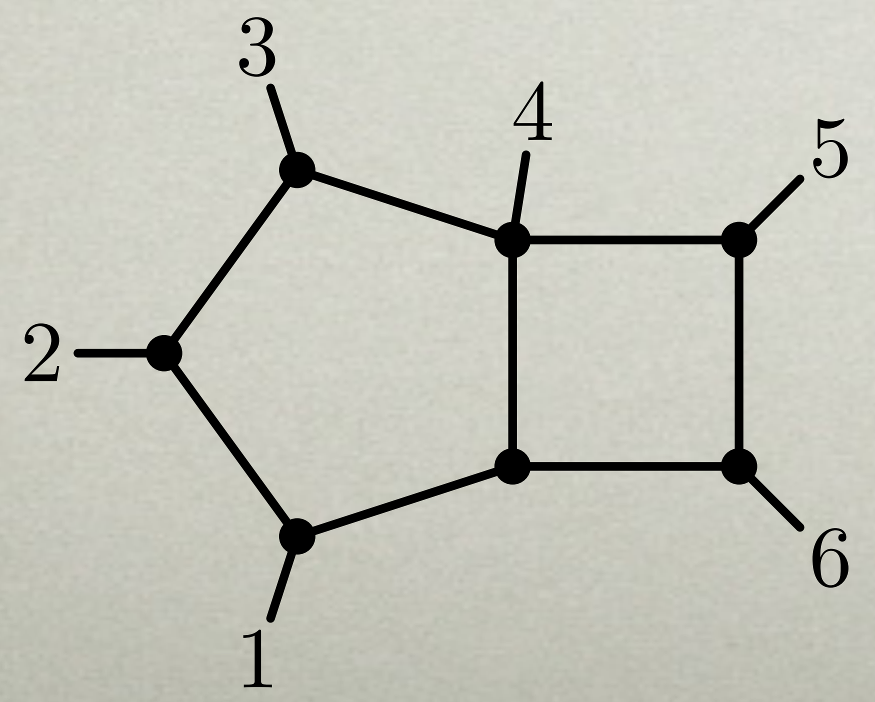

$$\Rightarrow \frac{s_{13}}{s_{12}s_{23}s_{56}}\left(\begin{array}{r}\mathrm{tr}_+[p_3,p_{12},p_6,p_1]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\\ +\mathrm{tr}_+[p_{12},p_6,p_1,p_3]\big(\mathrm{Li}_4(\cdots)\text{'s}+\ldots\big)\end{array}\right)$$

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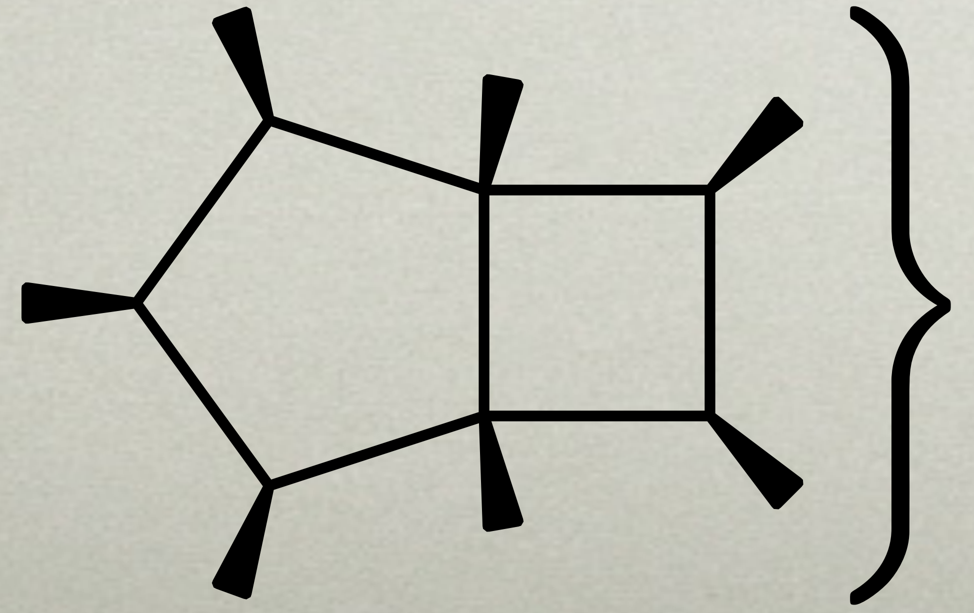
- **Avoid regularization** when**ever** possible:
  - can all(?) ***finite*** quantities be computed *without regularization*? —without expanding them in terms of divergent integrals?

(**Answer**: sometimes)

[**JB**, Langer, Patatoukos (2021); …]



with numerator `1′, this integrates to a sum of (an ***impure*** combination of) **polylogarithms** & **elliptic-polylogarithms** involving ***4 elliptic curves***

Unitarity-Based Strategies:

a modern perspective

Generalized Unitarity: a modern take

- The basic idea behind **unitarity-based methods** is that any ***Feynman integrand*** is a *rational differential form on loop momenta*
  - as such, it can be expanded into a **basis** $\mathfrak{B}$ of such forms:

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

- For any fixed QFT (spacetime dimension, particle content), the space of **all amplitude integrands** is **finite-dimensional**
  - ***all-multiplicity amplitudes*** can be expressed in a ***finite basis***!

- **Key observation:** viewed as a potential element of *some* basis, *every* **Feynman** integrand can be interesting!
  - Why not try to find the *best/easiest* integrands—and use these?

Stratifying Quantum Field Theories

- **QFTs** can be **partially** ***ordered*** by the scope of the basis required to represent their amplitudes

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

$$[\text{Standard Model}] \succ [(\text{Standard Model} \backslash \text{Higgs})] \succ [\text{QCD}] \succ [\text{Yang-Mills}]$$

$$[\text{Yang-Mills}] \succ [\mathcal{N}=2 \text{ super-Yang-Mills}] \succ [\mathcal{N}=4 \text{ super-Yang-Mills}]$$

$$[\mathcal{N}=4 \text{ Yang-Mills}] \succ [\text{planar } \mathcal{N}=4 \text{ super-Yang-Mills}] \succ \cdots \succ [\text{fishnet theory}]$$

This reflects UV behavior (“*power-counting*”) of theories; it suggests a possible ***stratification of integrand bases***

$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$

Stratifying *Integrand Bases*

- Suppose that a basis could be carved up into subspaces (by any arbitrary means):

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

$$\mathfrak{B} =: \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \qquad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p \coloneqq \mathcal{A} \bigcap \mathfrak{B}_p \coloneqq \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \, \mathfrak{b}_p^i$$

- ¿Is it possible to stratify integrand bases by ***physical structure***?

$$\big\{\text{finite}\big\} \oplus \big\{\text{divergent}\big\}$$

[**JB**, Langer, Zhang, (2021)]
[**JB**, Herrmann, Langer, Patatoukos, *et al* (2021)]

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- ¿Is it possible to stratify integrand bases by ***physical** structure*?

$$\Big\{\text{finite}\Big\} \oplus \Big\{\Big(\text{UV-divergent}\Big)\Big\} \oplus \Big\{\Big(\text{IR-divergent}\Big)\Big\}$$

[**JB**, Langer, Zhang, (2021)]
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Stratifying ***Integrand Bases***

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- ¿Is it possible to stratify integrand bases by ***physical*** *structure*?

$$\Big\{\text{finite}\Big\}\oplus\Big\{\Big(\mathcal{O}(1/\epsilon^{2L})\text{-divergent}\Big)\oplus\Big(\mathcal{O}(1/\epsilon^{2L-1})\text{-divergent}\Big)\oplus\cdots\oplus\Big(\mathcal{O}(1/\epsilon)\text{-divergent}\Big)\Big\}$$

$$\oplus\Big\{\Big(\log(m)^{2L}\text{-divergent}\Big)\oplus\Big(\log(m)^{2L-1}\text{-divergent}\Big)\oplus\cdots\oplus\Big(\log(m)\text{-divergent}\Big)\Big\}$$

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- ¿Can we further stratify each part by ***transcendental** structure*?

$$\overbrace{\left\{\text{max-weight}\right\} \oplus \left\{\text{next-to-max-weight}\right\}}^{\left\{\text{finite}\right\}} \oplus \cdots \oplus \left\{\text{rational}\right\}$$

$$\overbrace{\left\{\text{polylogs}\right\} \oplus \left\{\text{elliptic-polylogs}\right\}}^{\left\{\text{max-weight}\right\}} \oplus \left\{\text{K3-polylogs}\right\} \oplus \cdots$$

[**JB**, Langer, Zhang, (2021)]

[**JB**, Herrmann, Langer, Patatoukos, *et al* (2021)]

Prescriptive ***Integrand Bases***

- How ***generalized unitarity*** has been used to match amplitudes:

$$\mathcal{A} = \sum_i c_i \, \mathcal{I}_i^0$$

with coefficients c_i determined by **cuts**: a spanning set of cycles $\{\Omega_j\}$

$$\oint_{\Omega_j} \mathcal{I}_i^0 =: \mathbf{M}_{i,j}$$

$$a_j \coloneqq \oint_{\Omega_j} \mathcal{A} = \sum_i c_i \, \mathbf{M}_{i,j} \Rightarrow c_j = \sum_i a_i \, (\mathbf{M}^{-1})_{i,j}$$

Prescriptive *Integrand Bases*

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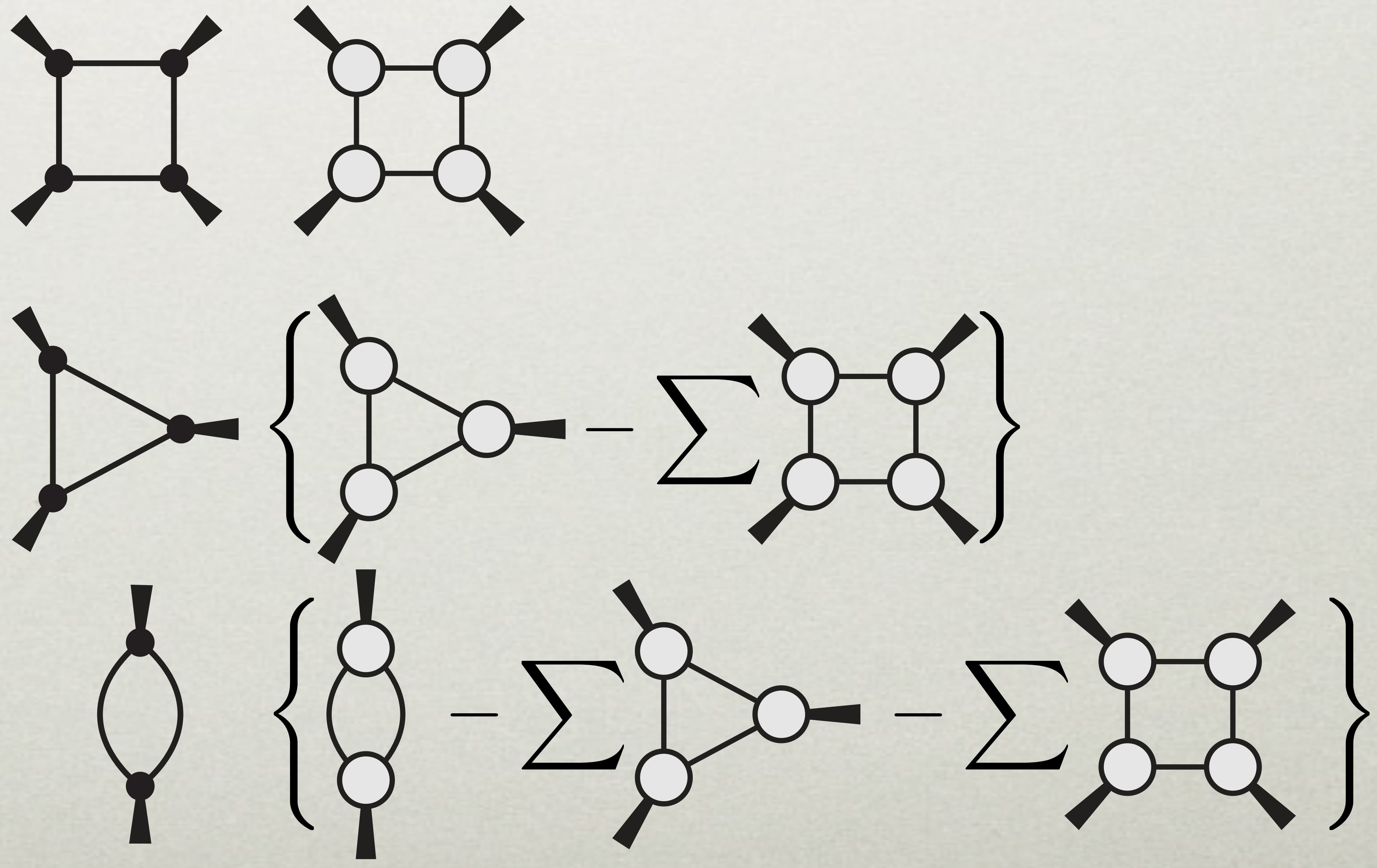

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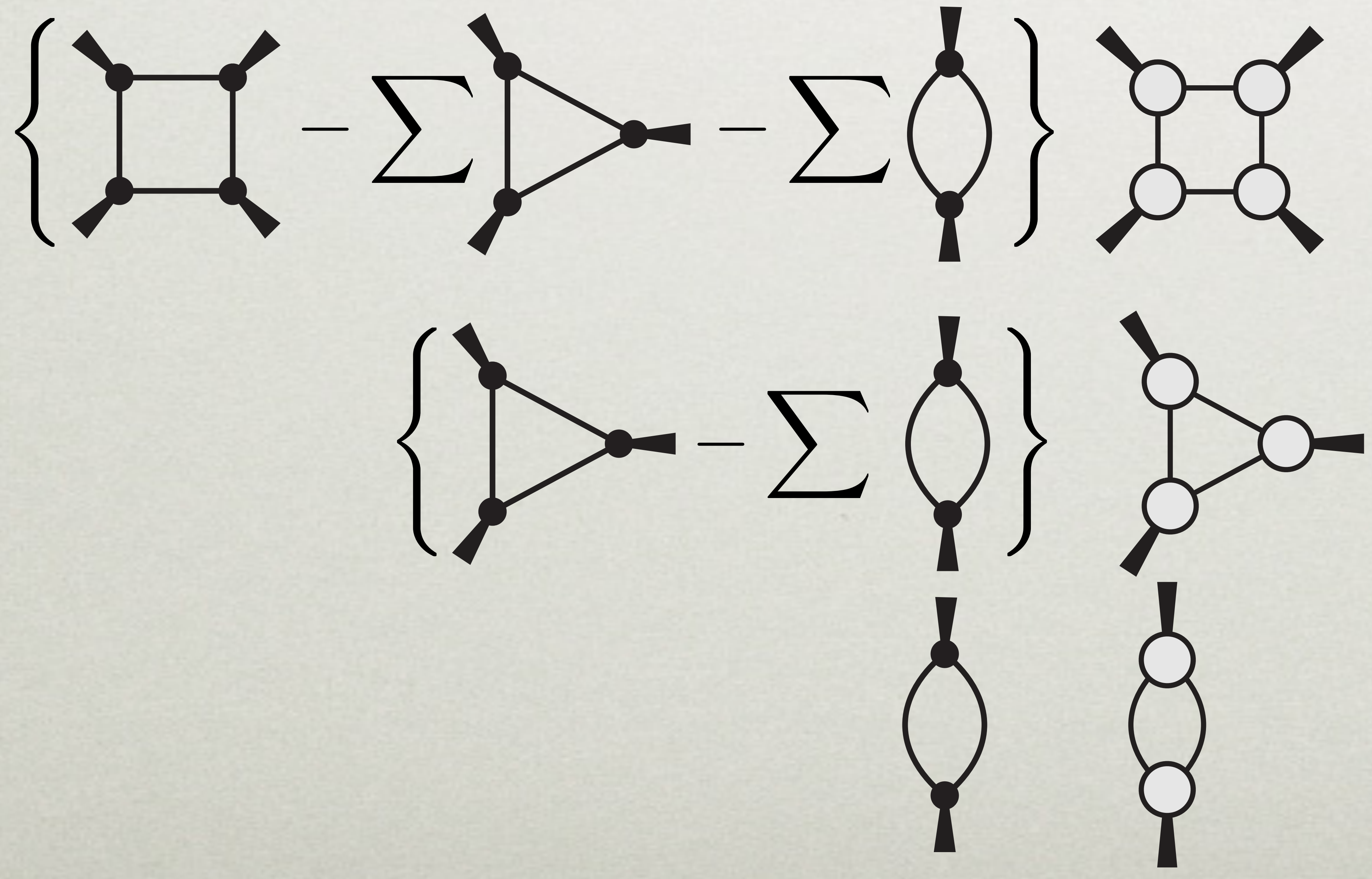

Prescriptive *Integrand Bases*

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with coefficients c_i determined by **cuts**: a spanning set of cycles $\{\Omega_j\}$

$$\mathcal{I}_j \coloneqq \sum_i \mathcal{I}_i^0 \left(\mathbf{M}^{-1}\right)_{i,j} \qquad \oint_{\Omega_j} \mathcal{I}_i = \delta_{i,j}$$

- A basis is called ***prescriptive*** if it is the **cohomological dual** of a spanning set of **cycles** $\{\Omega_j\}$

Strategies for ***Building*** ***Bases***

- Given ***some*** integrand basis (or strata thereof), one should ***diagonalize*** the space of integrands according to a

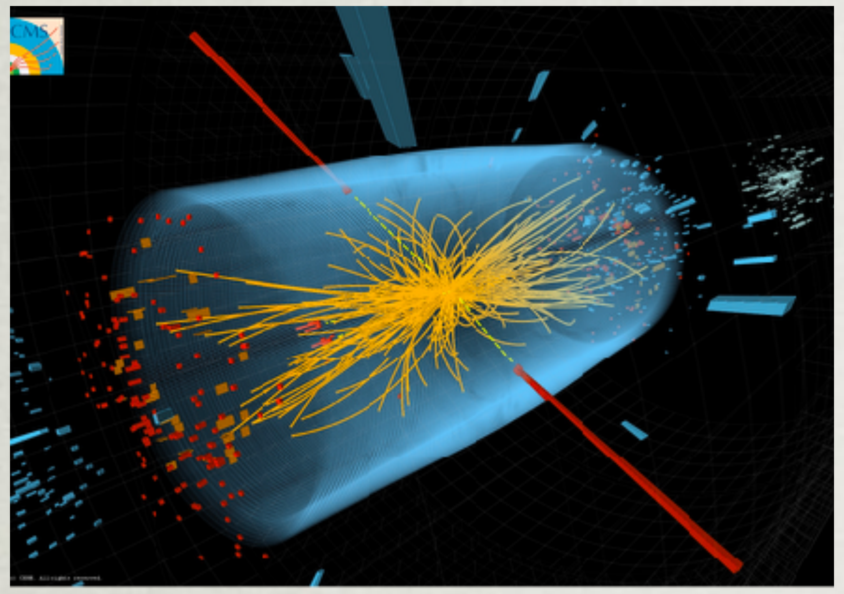
  homological/cohomological pairing:

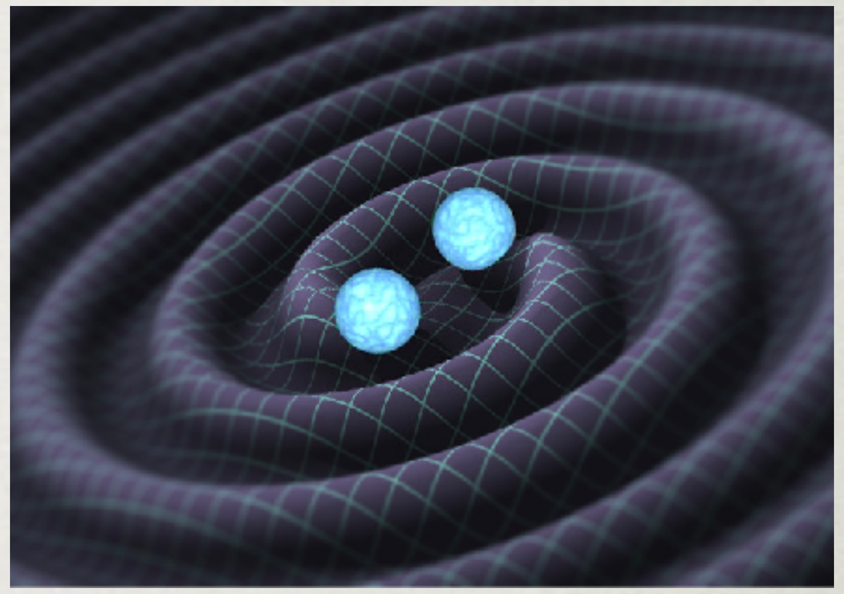
  - choose a *spanning-set* of **compact, max-dimensional** contours Ω_j
  - normalize and diagonalize the basis by the requirement

$$\int_{\Omega_j} \mathfrak{b}_i = \delta_{ij} \qquad \Omega_j : 4L\text{-dimensional compact contours} \begin{cases} \text{“residues”} \\ \text{elliptic periods} \\ \text{K3 periods, etc.} \end{cases}$$

- This ***trivializes*** the representation of amplitudes:
  - the **coefficient** of any amplitude in this basis will simply be the *on-shell function* evaluated on the contour (a **leading singularity**)
- Choosing a maximal set of IR/UV-divergence-probing contours ensures(?) that the basis is split into finite/divergent subspaces

Amplitudes: a Virtuous Cycle





Compute Something

beyond the reach of
recent imagination

Discover Simplicity

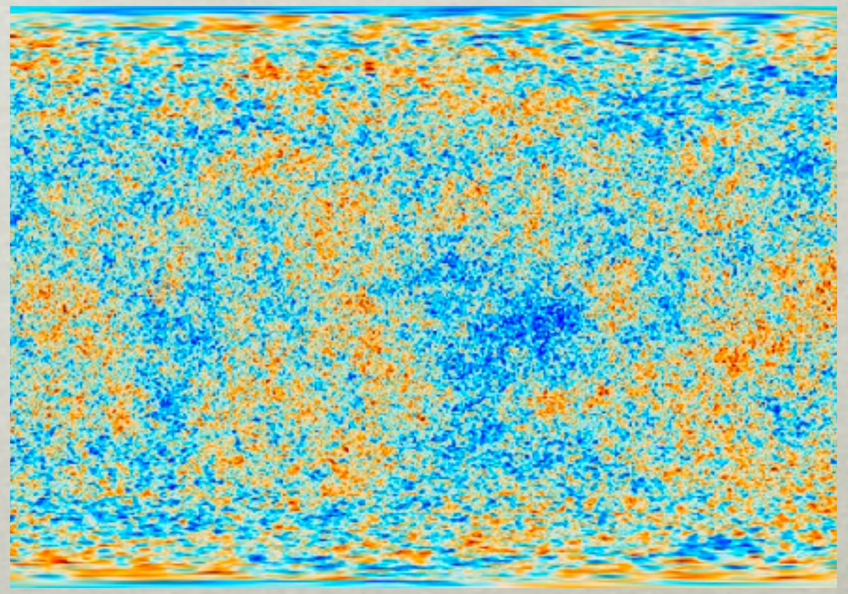
beyond expectations

Understand Why

study it, understand it,
& explore consequences

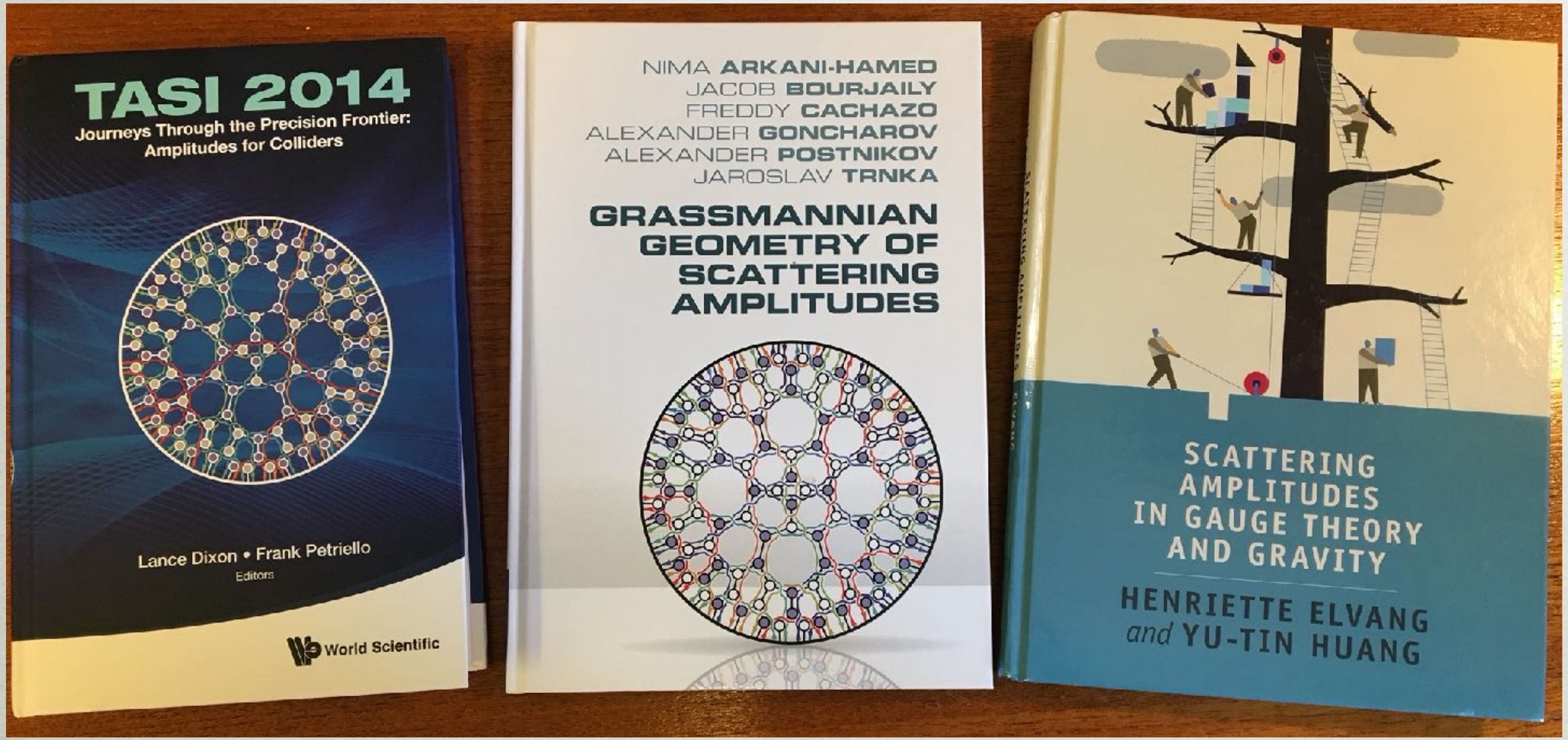
Exploit Simplicity

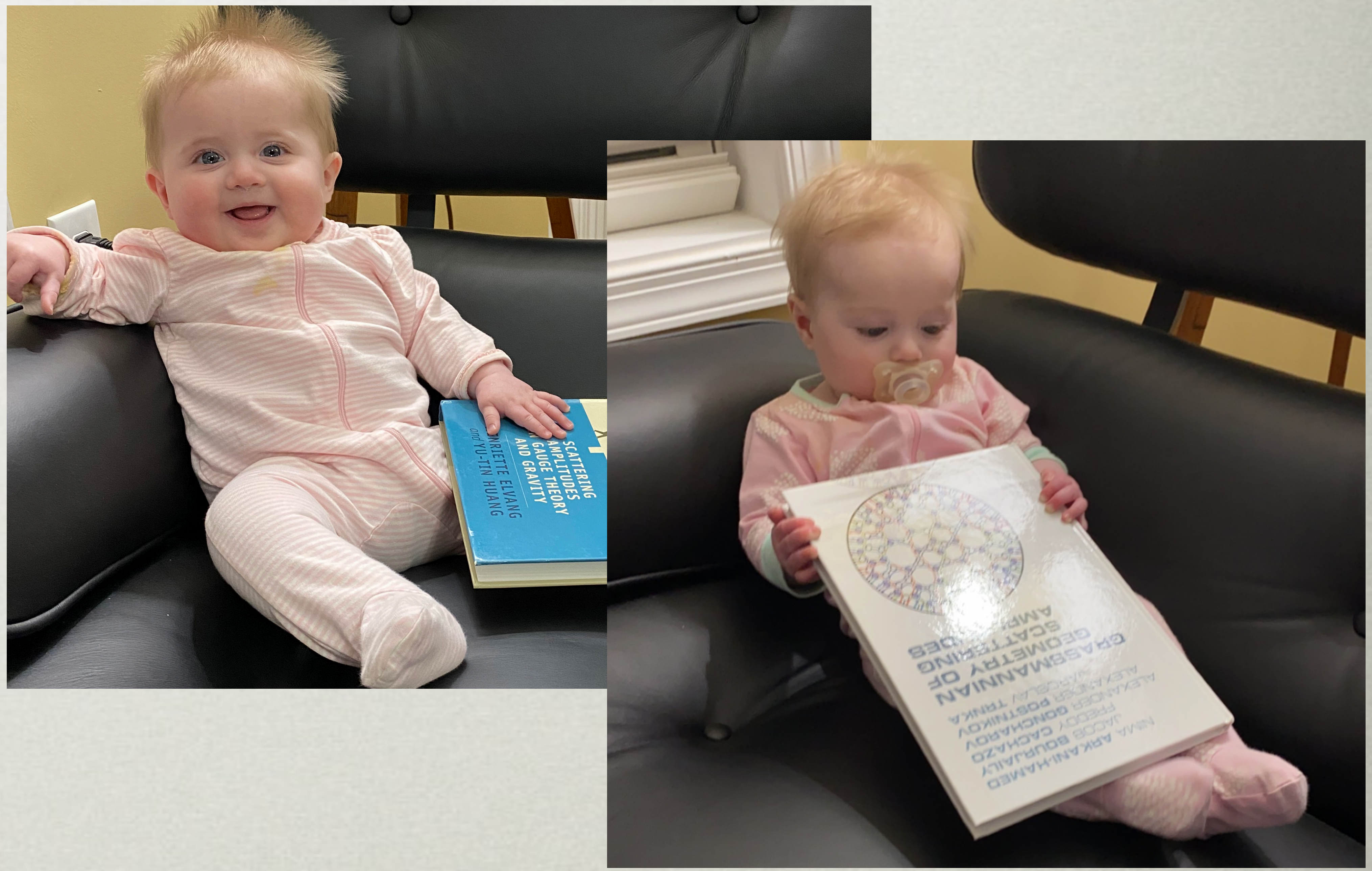
to build **more powerful**
computational technology



Today's Revolution in QFT

- Defining an ***ongoing revolution*** in science:
  - when ***all** the textbooks* of a field become ***obsolete***
- Experts who ***need*** QFT no longer use **textbook tools**





- The foundations for *the **future's textbooks***
are still being **discovered** every day (it's **exciting!**)

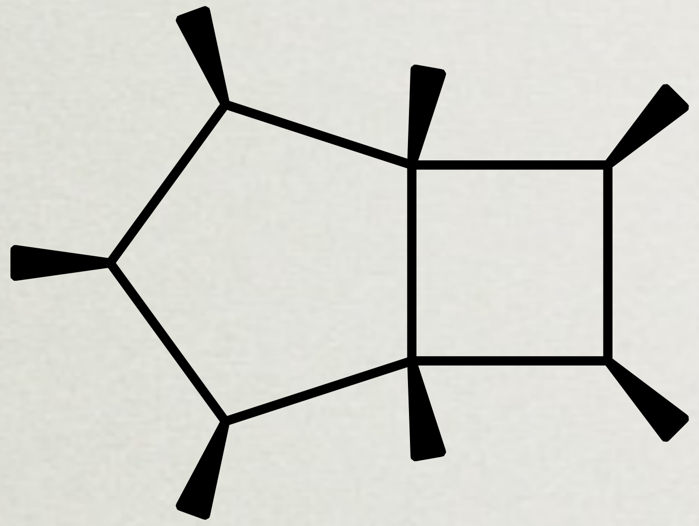
Thank you!

Stratifying Rigidity

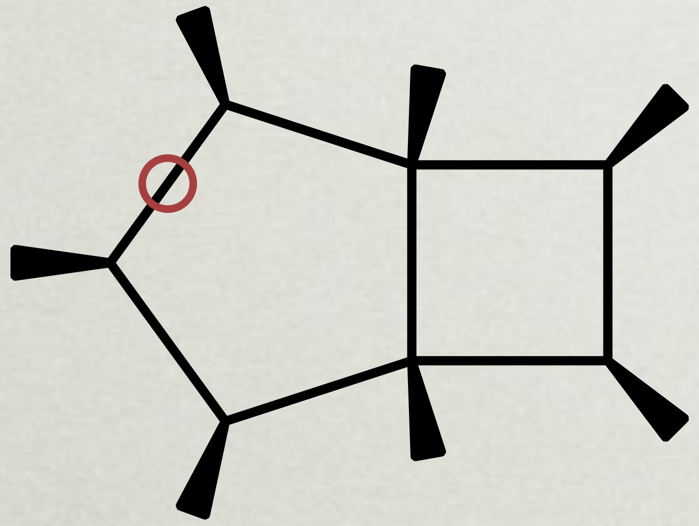
Diagonalization of Rigidity

- Consider the following sets of pentabox integrands

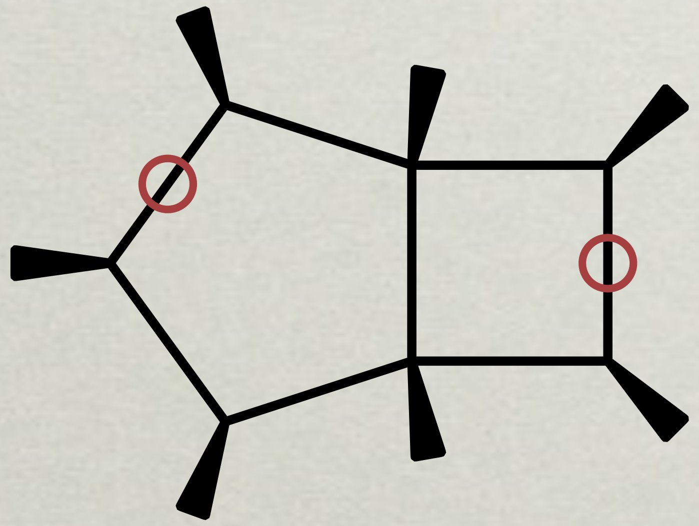
[**JB**, Kalyanapuram (2022)]



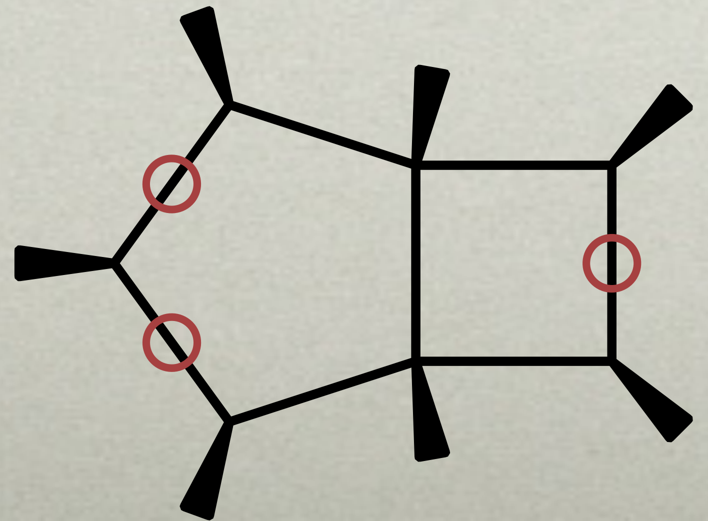
$$\Leftrightarrow \mathcal{I}_0 := \frac{(\ell_1|X)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_1(N) := \frac{(\ell_1|N)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_2(N,M) := \frac{(\ell_1|N)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



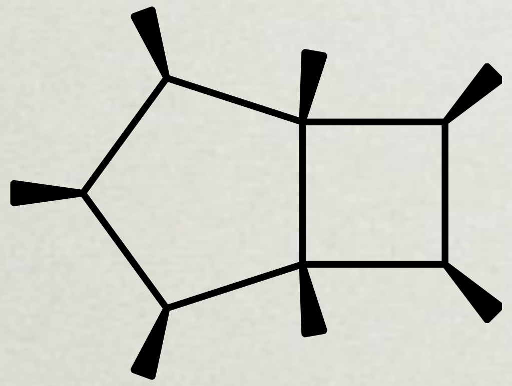
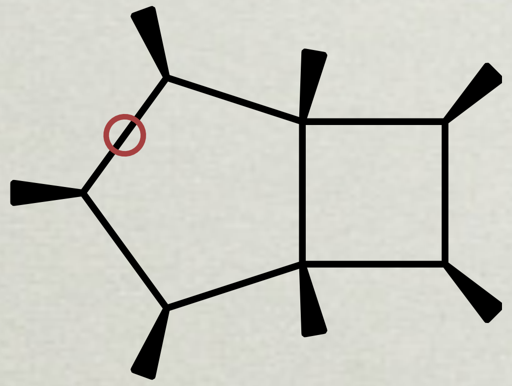
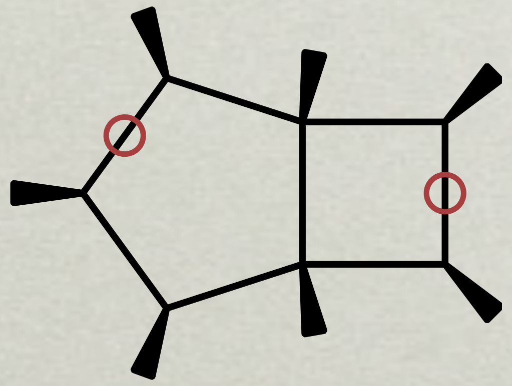
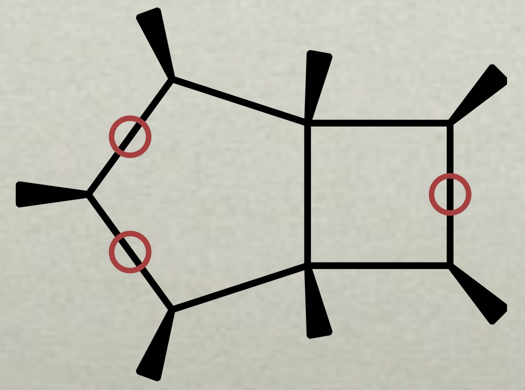
$$\Leftrightarrow \mathcal{I}_3(\vec{N},M) := \frac{(\ell_1|N_1)(\ell_1|N_2)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|X)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$

Diagonalization of Rigidity

- Consider the following sets of pentabox integrands

[**JB**, Kalyanapuram (2022)]

pure

	# ints	# polylogs	# elliptics	# impure/mixed
	1	0	0	1
	6	0	4 (4)	2
	36	10	12 (12)	14
	120	76	44 (22)	0