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The Niels Bohr International Academy

PennState

Adventures in erturbation Theory

Current Themes in High Energy Physics and Cosmology

2 August, 2022





Dirac (1933)

Feynman; Schwinger; Tomanaga (1947)





What is the mathematical form of the predictions made by QFT? (perturbatively, say?)

e.g. maximally supersymmetric $(\mathcal{N}=4)$ Yang-Mills theory (planar limit)

 $\gamma_{\rm cusp} = a \times 1$ $+a^2 \times 2\zeta_2$ $+a^3 \times 22 \zeta_4$ $+a^4 \times 2(24\zeta_2^3 + 4\zeta_3^2 + 2\zeta_2\zeta_4 + \zeta_6)$ $+a^5 \times 8(252\,\zeta_4^2 + 20\,\zeta_3\,\zeta_5 + 4\,\zeta_2\,\zeta_3^2 + \zeta_2$ $+a^{6} \times 8 (282 \zeta_{2}^{5} + \zeta_{2}^{3} \zeta_{4} + 4 \zeta_{2} \zeta_{4}^{2} + 80 \zeta_{2}$ +...

 \Rightarrow "maximal transcendentality" of planar $\mathcal{N} = 4$ super Yang-Mills (?)



Beisert, Eden, Staudacher (2007);...

$$\zeta_{6})$$

$$\zeta_{3}\zeta_{5} + 5\zeta_{2}^{2}\zeta_{6} + 48\zeta_{3}^{2}\zeta_{4} + 102\zeta_{5}^{2} + 210\zeta_{3}\zeta_{7} + 3$$

Explosions of Complexity

 While ultimately correct, the Feynman expansion renders all but the most trivial predictions involving the fewest particles, at the lowest orders of perturbation computationally intractable theoretically inscrutable or



Maybe unifying the forces of nature isn't quite as hard as physicists thought it would be By Zvi Bern, Lance J. Dixon and David A. Kosower



Bern, Dixon, Kosower, Scientific American (2012)





Background amplitudes crucial for e.g. colliders

Supercollider physics [*Rev.Mod.Phys.* 56 (1984)]

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Eichten *et al.* summarize the motivation for exploring the 1-TeV ($=10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two \rightarrow four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.





Background amplitudes crucial for e.g. colliders



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220 diagrams—thousands of terms



Background amplitudes crucial for e.g. colliders

THE CROSS SECTION FOR FOUR-GLUON PRODUCTION **BY GLUON-GLUON FUSION**

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

Nucl.Phys. **B269** (1985) Once considered computationally intractable

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two \rightarrow four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.





Background amplitudes crucial for e.g. colliders



S.J. Parke, T.R. Taylor / Four gluon production

gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3 , p_4 , p_5 , p_6 is obtained from eq. (5) by setting I = 2 and replacing the momenta p_3 , p_4 , p_5 , p_5 by $-p_3$, $-p_4$, $-p_5$, $-p_6$. As the result of the computation of two hundred and forty Feynman diagrams, we obtain

 $A_{(0)}(p_1, p_2, p_3, p_4, p_5, p_6)$

 $= (\mathcal{D}^{\dagger}, \mathcal{D}^{\dagger}_{\rho}, \mathcal{D}^{\dagger}_{\sigma}, \mathcal{D}^{\dagger}_{\tau})^{(0)}_{(2)} \cdot \begin{pmatrix} K_{\rho} & \Lambda & \Lambda_{\tau} & \Lambda_{\sigma} \\ K_{\sigma} & K_{\tau} & K & K_{\rho} \end{pmatrix} \cdot \begin{pmatrix} \Sigma_{\rho} \\ \Sigma_{\sigma} \end{pmatrix}$

where $\mathfrak{D}, \mathfrak{D}_{p}, \mathfrak{D}_{q}$ and \mathfrak{D}_{r} are 11-component complex vector functions of the moment p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_p, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors \mathcal{D}_{α} \mathcal{D}_{α} and \mathcal{D}_{γ} are obtained from the vector \mathcal{D} by the permutations The rection \mathcal{D}_{p} the \mathcal{D}_{p} and \mathcal{D}_{p} are obtained from the rector \mathcal{D} by the permutations $(p_{2} \leftrightarrow p_{2})$ and $(p_{2} \leftrightarrow p_{3})$, $p_{3} \leftrightarrow p_{4}$, respectively, of the momentum variables in \mathcal{D} . The individual components of the vector \mathcal{D} represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, N=3 for QCD):

 $K = \frac{1}{8}g^8 N^4 (N^2 - 1)K^{(4)} + \frac{1}{2}g^8 N^2 (N^2 - 1)K^{(2)}.$ Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector \mathcal{D} is related to the thirty-three diagrams $D^G(I=1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^{F}(I = 1 - 11)$ for two-fermior to four-scalar scattering and sixteen diagrams $D^{S}(I = 1 - 16)$ for two-scalar to four-scalar scattering, in the following way:

 $\mathcal{D}_{0} = \frac{2s_{14}}{\sqrt{|s_{15}s_{45}s_{16}s_{46}|s_{23}s_{56}}} \{t_{123}^{2}C^{G} \cdot D_{0}^{G} - 4s_{14}t_{123}E(p_{5} + p_{6}, p_{6})C^{F} \cdot D_{0}^{F}\}$ $-2s_{14}G(p_5+p_6,p_5+p_6)C^{\rm S}\cdot D_0^{\rm S}\},$

 $\mathscr{D}_2 = \frac{s_{56}}{s} C^{\mathrm{G}} \cdot D_2^{\mathrm{G}},$

where the constant matrices $C^{G}(11 \times 33)$, $C^{F}(11 \times 11)$ and $C^{S}(11 \times 16)$ are given in table 2. The Lorentz invariants s_v and t_{ok} are defined as $s_v = (p_t + p_t)^2$, $t_{ijk} = (p_t + p_t)^2$ and the complex functions E and G are given by $E(p_{i}, p_{j}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{i}, p_{j}) - (p_{1}, p_{i})(p_{j}, p_{4}) - (p_{1}, p_{j})(p_{i}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{j}) - (p_{1}, p_{j})(p_{j}, p_{4}) - (p_{1}, p_{j})(p_{j}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{j}) - (p_{1}, p_{j})(p_{j}, p_{4}) - (p_{1}, p_{j})(p_{j}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda} p_{4}^{\nu} p_{i}^{\nu} p_{j}^{\rho} p_{4}^{\lambda}\}/(p_{1}, p_{4}) = \frac{1}{2} \{(p_{1}, p_{4})(p_{1}, p_{4}) + i\varepsilon_{\mu\nu\rho\lambda} p_{1}^{\mu} p_{i}^{\nu} p_{j}^{\nu} p_{j}^{\lambda} p_{4}^{\lambda} p_{i}^{\nu} p_{j}^{\nu} p_{j}^{\lambda} p_{4}^{\lambda} p_{4}^{\nu} p_{j}^{\nu} p_{4}^{\nu} p_{j}^{\nu} p_{4}^{\nu} p_{4}^{\nu} p_{j}^{\nu} p_{4}^{\nu} p_$

 $G(p_{n}, p_{j}) = E(p_{n}, p_{5})E(p_{n}, p_{6}),$

S.J. Parke, T.R. Taylor / Four gluon production $D_2^{\rm G}(9) = \frac{4}{s_1 s_2 s_4 t_{125}} \{ [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6)] E(p_5, p_3) \}$ $-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_4(p_3-p_6)]E(p_5,p_2-p_5)\},$ $D_2^{\rm G}(10) = \frac{4}{s_{25}s_{46}t_{125}} \{ [(p_1 + p_2 - p_3)(p_4 - p_3 + p_6)] E(p_2, p_6) \}$ $-[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5,p_6)+[p_1(p_2-p_5)]E(p_3-p_6,p_6)\},$ $D_2^{\rm G}(11) = \frac{\delta_2}{s_{36}t_{124}} [s_{35} - s_{56} + s_{36}],$ $D_2^{\rm G}(12) = \frac{-\delta_2}{s_{36}t_{145}} [s_{23} - s_{26} - s_{36}],$ $D_2^{\rm G}(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}} [s_{12} - s_{24}] [s_{35} - s_{56} + s_{36}],$ $D_2^{\rm G}(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}} [s_{15} - s_{45}] [s_{23} - s_{26} - s_{36}],$ $D_2^{\rm G}(15) = \frac{\delta_2}{\delta_{14}\delta_{24}} (p_1 - p_4)(p_3 - p_6) ,$ $D_2^{\rm G}(16) = \frac{-4}{s_{12}s_{36}t_{124}} [s_{35} - s_{56} + s_{36}]E(p_2, p_2),$ $D_2^{\rm G}(17) = \frac{4}{s_{36}s_{45}t_{145}} [s_{23} - s_{26} - s_{36}] E(p_5, p_5) ,$ $D_2^{\mathbf{G}}(18) = \frac{-4}{s_{12}s_{34}s_{45}} [2(p_1 + p_2)(p_3 - p_6) - s_{36}] E(p_2, p_5),$ $D_2^{\rm G}(19) = \frac{-2}{s_{12}s_{24}} E(p_2, p_3 - p_6),$ $D_2^G(20) = \frac{2}{s_{34}s_{45}} E(p_3 - p_6, p_5),$ $D_2^{\rm G}(21) = \frac{-4}{s_{25}s_{14}t_{134}} [s_{26} - s_{56} + s_{25}] E(p_3, p_3),$

 $D_2^{\rm G}(22) = \frac{4}{s_{16}s_{25}t_{146}} [s_{23} - s_{35} - s_{25}]E(p_6, p_6),$

 $D_2^{\rm G}(23) = \frac{4}{s_{16}s_{25}s_{34}} [2(p_1 + p_6)(p_2 - p_5) + s_{25}]E(p_6, p_3),$

S.J. Parke, T.R. Taylor / Four gluo $D_2^{\rm G}(24) = \frac{-2}{s_2 + s_3} E(p_2 - p_5, p_3),$ $D_2^{\rm G}(25) = \frac{2}{S_{16}S_{25}} E(p_6, p_2 - p_5),$ $D_2^{\rm G}(26) = \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5),$ $D_2^{\rm G}(27) = \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6) ,$ $D_2^{\rm G}(28) = \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5),$ $D_2^{\rm G}(29) = \frac{-2}{s_{24}t_{125}} E(p_3 - p_6, p_3),$ $D_2^{\rm G}(30) = \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{12}]$ $D_2^{\rm G}(31) = \frac{4}{s_{12}s_{44}t_{12}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{12}]$ $D_2^{\rm G}(32) = \frac{4}{\sum_{s \in S_2, t_{12}}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{12}]$ $D_2^{\rm G}(33) = \frac{4}{s_{15}s_{46}t_{125}} \left[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{12} \right]$ where $\delta_2 = 1$. The diagrams D_0^G are obtained from D_2^G by repla $E(p_{i}, p_{j})$ by $G(p_{i}, p_{j})$. The diagrams D_{0}^{F} are listed below: $D_0^{\rm F}(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)\}$ + [$F(p_6, p_3) + s_{34}$] $E(p_5, p_5)$ }, $D_0^{\rm F}(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{ [F(p_6, p_2) + \frac{1}{2}s_{16}] E(p_3, p_5) \}$ $+[F(p_2, p_3)+\frac{1}{2}s_{34}]E(p_6, p_5)-F(p_6, p_3)E(p_6, p_3)$ $D_0^{\rm F}(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)\}$

S.J. Parke, T.R. Taylor / Four gluon production	413	414	S.J. Parke, T.R. Taylor / F	our gluon production
TABLE 1 Matrices $K(I, J)[I = 1-11, J = 1-11].$			000000-000	0000000 - 00
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S.J. Parke, T.R. Taylor / Four gluon production	417	418	S.J. Parke, T.R. Taylor / Fou	r gluon production
S.I. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$	417	418 $D_0^F(4) = -\frac{1}{s_2}$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{25^334^{1}25}$ { $F(p_2, p_3)E(p_5, p_5) - F(p_5, p_5)$	r gluon production $s, p_3)E(p_2, p_5)$
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$	417	418 $D_0^{\rm P}(4) = \frac{1}{s_2}$ +	S.J. Parke, T.R. Taylor / Fou $\frac{4}{25^{5}34^{1}25} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}\}E(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}) - \frac{1}{2}s_{15} + \frac{1}{2}s_{15} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}) - \frac{1}{2}s_{15} + $	r gluon production $(s, p_3)E(p_2, p_3)$ $(s, p_3)\},$
S.I. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{-2}{s_{16}s_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{E(p_5, p_7 - p_5)}.$	417	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ + $D_{0}^{F}(5) = \frac{1}{s_{3}}$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{25^{5}34^{1}125} \{F(p_2, p_3)E(p_5, p_5) - F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}\}E(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}\}E(p_5, p_5),$	r gluon production $(x, p_3)E(p_2, p_3)$ $(x, p_3)\}$,
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{s_{12}t_{25}} E(p_2, p_2 - p_5),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = -$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{2s^{5}4s^{1}125} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}\}E(p_{5}, p_{5}) + \frac{2}{1s^{5}2s^{1}46} [s_{35} - s_{23} + s_{23}]E(p_{6}, p_{5}) ,$ $\frac{2}{1s^{5}s_{5}s_{146}} [s_{36} - s_{26} - s_{23}]E(p_{3}, p_{5}) + \frac{2}{1s^{5}s_{16}} [s_{36} - s_{26} - s_{23}]E(p_{3}, p_{5}) + \frac{2}{1s^{5}s_{16}} [s_{36} - s_{36} - s_{35}]E(p_{3}, p_{5}) + \frac{2}{1s^{5}s_{16}} [s_{36} - s_{36} - s_{35}]E(p_{33}, p_{5}) + \frac{2}{1s^{5}s_{16}} [s_{16} - s_{16} - s$	r gluon production $(x, p_3) E(p_2, p_3)$ $(x, p_3) \},$
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{-2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5),$ $= \frac{-2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ $D_0^{\rm F}(2)$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{2s^{5}_{34}f_{125}} \{F(p_2, p_3)E(p_5, p_5) - F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_2, p_3) - \frac{1}{2}s_{25}s_{14} - \frac{1}{2}s_{15}s_{12} + \frac{1}{2}s_{15}]E(p_5, p_5) ,$ $\frac{2}{2s^{5}_{25}s_{14}}[s_{55} - s_{26} - s_{25}]E(p_3, p_5) ,$	r gluon production $(s, p_3)E(p_2, p_3)$ (s, p_3) }
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{s_{12}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_3),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ $D_0^{\rm F}(7) = \frac{1}{s_2}$	5.1. Parke, T.R. Taylor / Fou $\frac{4}{2s^{5}4t^{1}25} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}\}E(p_{5}, p_{5}),$ $\frac{2}{2s^{5}s_{4}t^{1}s_{4}} [s_{35} - s_{23} + s_{23}]E(p_{6}, p_{5}),$ $\frac{4}{2s^{5}s_{6}t^{1}s_{5}} \{[F(p_{5}, p_{2}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}) - \frac{1}{2}E(p_{5}, p_{5}) - \frac{1}$	$r gluon production s, p_3)E(p_2, p_3)s, p_3)\},(s_{13}]E(p_3, p_3)p_1 p_2 + (p_1, p_2)$
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5),$ $= \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{14}t_{125}} E(p_5, p_2 - p_3),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ $D_0^{\rm F}(7) = \frac{1}{s_2}$ + $D_0^{\rm F}(8) = -$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{2s5_3d^{1}125} \{F(p_2, p_3)E(p_5, p_5) - F(p_1) + F(p_2, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_2, p_3) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_3, p_3),$ $\frac{2}{2s5_3d^{1}134} [s_{36} - s_{26} - s_{25}]E(p_3, p_3),$ $\frac{4}{2s5_3d^{1}125} \{[F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_3, p_3) - F(p_3, p_3) - $	r gluon production $s_{1}, p_{3})E(p_{2}, p_{5})$ $s_{2}, p_{3})\},$ $s_{13}]E(p_{3}, p_{5})$ $p_{5}, p_{3}) + \frac{1}{4}t_{125}]E(p_{2}, p_{3})\},$
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{s_{12}t_{25}} E(p_2, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{s_{16}t_{125}} E(p_5, p_2 - p_5),$ $= \frac{-2}{s_{34}t_{125}} E(p_5 - p_6, p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ + $D_0^{\rm F}(0) = \frac{1}{s_1}$ + $D_0^{\rm F}(0) = \frac{1}{s_1}$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{25^{3}4^{1}125} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}5_{25} - \frac{1}{2}5_{12} + \frac{1}{2}s_{15}\}E(p_{5}, p_{5}),$ $\frac{2}{15^{5}25^{1}146} [s_{35} - s_{23} + s_{23}]E(p_{6}, p_{5}),$ $\frac{2}{25^{5}34^{1}134} [s_{56} - s_{26} - s_{25}]E(p_{3}, p_{5}),$ $\frac{4}{25^{5}36^{1}125} \{[F(p_{5}, p_{2}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}) - [F(p_{5}, p_{5}) + \frac{1}{4}t_{122}]E(p_{5}, p_{5}) + \frac{1}{4}t_{122}]E(p_{5}, p_{5}) + [F(p_{5}, p_{5}) + \frac{1}{4}t_{122}]E(p_{5}, p_{5}) + \frac{1}{4}t_{122}]E(p_{5}, p_{5}) + \frac{1}{4}t_{12}]E(p_{5}, p_{5}) + \frac{1}{4}t_{12}$	r gluon production $s, p_3 \in (p_2, p_3)$ $s, p_3 \in (p_2, p_3)$ $s, p_3 \in (p_3, p_3)$ $s_{13} \in (p_3, p_3)$ $p_5, p_3) + \frac{1}{4}t_{123} \in (p_2, p_3)$
S.I. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_5, p_3),$ $= \frac{-2}{5_{16}5_{125}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{-2}{s_{15}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{14}t_{125}} E(p_5 - p_6, p_3),$ $= \frac{-4}{s_{12}s_{14}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{-4}{s_{14}} [(p_1 + p_2 - p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_6),$	417	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ + $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ + $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(9) = \frac{1}{s_{1}}$	$S.J. Parke, T.R. Taylor / Fou \frac{4}{25^{5}34^{1}125} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_{5}, p_{5}), \frac{2}{25^{5}34^{1}134} [s_{56} - s_{26} - s_{25}]E(p_{5}, p_{5}), \frac{4}{25^{5}36^{1}(125)} \{[F(p_{5}, p_{2}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{12}]E(p_{5}, p_{5}) - [F(1)] \frac{1}{14^{5}36} E(p_{3} - p_{6}, p_{5}), \frac{2}{14^{5}36^{1}t_{12}} [s_{35} - s_{56} + s_{36}]E(p_{2}, p_{5}),$	r gluon production $(x_1, p_2) E(p_2, p_2)$ $(x_2, p_3) E(p_2, p_3)$ $(x_3, p_3) E(p_3, p_3)$ $(x_3, p_3) E(p_3, p_3) E(p_2, p_3) E(p_3, p_3)$
S.J. Parke, T.R. Taylor / Four gluon production $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{5_{16}5_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_2, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{s_{34}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_5 - p_6, p_3),$ $= \frac{-2}{s_{12}5_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{-4}{s_{12}5_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_6),$	417	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(9) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$	$\begin{split} & \text{S.J. Parke, T.R. Taylor / Fou} \\ & \frac{4}{25^{5}34^{1}125} \left\{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F($	r gluon production $s, p_3 \in (p_2, p_3)$ $b, p_3 \in (p_3, p_3)$ $s_{13} \in (p_3, p_3)$ $p_{5}, p_3) + \frac{1}{4}t_{125} \in (p_2, p_3)$
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{5_{16}5_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_2, p_2 - p_3),$ $= \frac{2}{5_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{4}{5_{12}5_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{4}{5_{12}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 + p_3 - p_6) + t_{123}]E(p_2, p_6),$ $= \frac{4}{5_{15}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 + p_3 - p_6) + t_{123}]E(p_5, p_3),$	417	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = -\frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(9) = \frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$	$\begin{split} & S.J. Parke, T.R. Taylor / Fou \\ & \frac{4}{25^{5}4^{1}125} \left\{ F(p_{2}, p_{3}) E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_$	r gluon production $s, p_3) E(p_2, p_5)$ $s, p_3) \},$ $s_{13} [E(p_3, p_3)$ $p_5, p_3) + \frac{1}{4} t_{123} [E(p_2, p_3)],$
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{5_{16}5_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{5_{12}t_{125}} E(p_2, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-4}{5_{12}5_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{125}]E(p_2, p_3),$ $= \frac{-4}{5_{12}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 + p_3 - p_6) + t_{123}]E(p_2, p_3),$ $= \frac{-4}{5_{15}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_3),$	417	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ $D_0^{\rm F}(0) = \frac{1}{s_2}$ $D_0^{\rm F}(9) = \frac{1}{s_1}$ $D_0^{\rm F}(10) = \frac{1}{s_2}$ $D_0^{\rm F}(11) = \frac{1}{2}$	$\begin{split} & \text{S.J. Parke, T.R. Taylor / Fou} \\ & \frac{4}{25^{3}4^{1}125} \left\{F(p_2, p_3)E(p_5, p_5) - F(p_5, p_5) - F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{15}\right]E(p_5, p_5) \\ & \cdot \left[F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{15}\right]E(p_6, p_5) \\ & \frac{2}{16^{3}25^{4}146} \left[s_{56} - s_{26} - s_{23}\right]E(p_6, p_5) \\ & \frac{4}{25^{3}36^{4}124} \left[s_{56} - s_{26} - s_{23}\right]E(p_5, p_5) - \left[F(1) + \frac{1}{14^{3}56} \left[E(p_5 - p_5) + \frac{1}{4}t_{125}\right]E(p_5, p_5) - \left[F(1) + \frac{2}{14^{3}56^{4}124} \left[s_{35} - s_{56} + s_{36}\right]E(p_2, p_5) \right] \\ & \frac{2}{14^{3}56^{4}144} \left[s_{33} - s_{26} - s_{36}\right]E(p_5, p_5) \\ & \frac{1}{14^{3}56^{5}1445} \left[s_{23} + s_{25} - s_{26} - s_{56}\right]E(p_5, p_5) \\ & - \left[s_{23} + s_{26} - s_{35} - s_{56}\right]E(p_5 - p_6, p_5) \end{split}$	$r gluon production$ $s, p_3) E(p_2, p_5)$ $s, p_3) \},$ $s_{13}[E(p_3, p_5)$ $p_5, p_3) + \frac{1}{4}t_{123}]E(p_2, p_3)\},$ $s_{2} = p_5, p_5)$ $s_{1} = [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)].$ (12)
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5),$ $= \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5),$ $= \frac{-2}{s_{46}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-4}{s_{12}s_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{125}]E(p_2, p_3),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_3),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{125}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$ $= \frac{-4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_3, p_6),$	417 (11)	418 $D_0^{\rm F}(4) = \frac{1}{s_2}$ + $D_0^{\rm F}(5) = \frac{1}{s_1}$ $D_0^{\rm F}(6) = \frac{1}{s_2}$ $D_0^{\rm F}(0) = \frac{1}{s_2}$ $D_0^{\rm F}(9) = \frac{1}{s_1}$ $D_0^{\rm F}(10) = \frac{1}{s_2}$ - The diagram	S.J. Parke, T.R. Taylor / Fou $\frac{4}{2s5_34t_{125}} \{F(p_2, p_3)E(p_5, p_5) - F(p_1)e_1 - F(p_2, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_1, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_2, p_3),$ $\frac{2}{2s5_34t_{134}} [s_{36} - s_{26} - s_{25}]E(p_3, p_3),$ $\frac{4}{2s5_36t_{125}} \{F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_5, p_5) - [F(1, \frac{1}{14s5_6}E(p_3 - p_6, p_5)),$ $\frac{2}{14s5_6t_{124}} [s_{35} - s_{56} + s_{36}]E(p_2, p_5),$ $\frac{2}{14s5_6t_{125}} [s_{23} - s_{26} - s_{36}]E(p_5, p_5),$ $\frac{1}{s_{14s5_2}s_{35}} \{s_{12} + s_{35} - s_{26} - s_{36}]E(p_5, p_6),$ $\frac{1}{s_{14s5_2}s_{35}} \{s_{12} + s_{35} - s_{26} - s_{36}]E(p_5, p_6, p_5),$ $\frac{1}{s_{14}s_{25}s_{35}} \{s_{16} - s_{35} - s_{36}]E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{35}} \{s_{16} - s_{35} - s_{36}]E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{15} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{25}s_{36}} \{s_{16} - s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{16}} \{s_{16} - s_{16} - s_{16} - s_{16}\}E(p_5 - p_6, p_5),$ $\frac{1}{s_{16}s_{16}} \{s_{16} - s_{16} - s_{16$	$r gluon production$ $s_{1}p_{3})E(p_{2}, p_{5})$ $s_{1}p_{3}]E(p_{3}, p_{5})$ $s_{13}]E(p_{3}, p_{5})$ $p_{5}, p_{3}) + \frac{1}{4}t_{125}]E(p_{2}, p_{3})\},$ $r_{2}-p_{5}, p_{3})$ $r_{-}[s_{23}+s_{56}-s_{35}-s_{26}]E(p_{2}+p_{5}, p_{3})\}.$ (12)
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{s_{12}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{s_{15}t_{125}} E(p_1 - p_2 - p_3)(p_4 + p_3 - p_6) - t_{125}]E(p_2, p_3),$ $= \frac{-4}{s_{12}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_6),$ $= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{125}]E(p_5, p_6),$ 1. Trams D_6^0 are obtained from D_2^0 by replacing δ_2 by $\delta_0 = 0$ and the function of the second sec	417 (11) ions	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(9) = \frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $-$ The diagrat $D_{0}^{S}(1) = \frac{1}{s_{2}}$	$\begin{split} & \text{S.J. Parke, T.R. Taylor / Fou} \\ & \frac{4}{2s^{5}4t^{1}25} \left\{F(p_{2},p_{3})E(p_{5},p_{5})-F$	r gluon production $s_{1}, p_{3}) E(p_{2}, p_{5})$ $s_{2}, p_{3}) E(p_{2}, p_{5})$ $s_{3}, p_{3}) \},$ $s_{13}[E(p_{3}, p_{5})$ $p_{5}, p_{5}) + \frac{1}{4}t_{125}]E(p_{2}, p_{5}) \},$ $t_{2} = p_{5}, p_{5})$ $t_{3} = [s_{23} + s_{56} - s_{35} - s_{26}]E(p_{2} + p_{5}, p_{5})].$ (12) $s_{23}],$
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_3, p_3),$ $= \frac{2}{5_{15}5_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_2, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-4}{5_{12}5_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{-4}{5_{15}5_{14}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_3, p_3),$ $= \frac{-4}{5_{15}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_5, p_3),$ $= \frac{-4}{5_{15}5_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_6),$ 1. The sums D_6^0 are obtained from D_2^0 by replacing δ_2 by $\delta_0 = 0$ and the funct of $O(p_0, p_1)$. The sums D_6^0 are listed below: $= -\frac{-4}{-5} \{F(p_5, p_6)E(p_1, p_2) - F(p_5, p_3)E(p_1, p_1)\}$	417 (11) ions	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(6) = \frac{1}{s_{1}}$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(9) = \frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ The diagram $D_{0}^{F}(1) = \frac{1}{s_{2}}$ $D_{0}^{F}(1) = \frac{1}{s_{2}}$ $D_{0}^{F}(1) = \frac{1}{s_{2}}$ $D_{0}^{F}(1) = \frac{1}{s_{2}}$ $D_{0}^{F}(1) = \frac{1}{s_{2}}$	$\begin{aligned} & \text{S.J. Parke, T.R. Taylor / Fou} \\ & \frac{4}{25^{5}34^{1}25} \left\{F(p_2, p_3)E(p_5, p_5) - F(p_1, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{15}\right]E(p_1, p_2) \\ & \cdot \left[F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{15}\right]E(p_1, p_3) , \\ & \frac{2}{25^{5}34^{1}134} \left[s_{56} - s_{26} - s_{23}\right]E(p_6, p_3) , \\ & \frac{4}{25^{5}36^{1}124} \left[s_{56} - s_{26} - s_{23}\right]E(p_5, p_5) - \left[F(1, \frac{1}{14556} E(p_3 - p_6, p_3) , \frac{2}{14556^{1}134} \left[s_{23} - s_{26} - s_{36}\right]E(p_2, p_3) , \\ & \frac{2}{14556^{1}134} \left[s_{23} - s_{26} - s_{36}\right]E(p_2, p_3) , \\ & \frac{2}{14556^{1}134} \left[s_{23} - s_{26} - s_{36}\right]E(p_2, p_5) , \\ & \frac{1}{1_{4555}}E(p_3 - p_6, p_3) , \\ & \frac{2}{1_{4556}}(s_{12} + s_{35} - s_{26} - s_{36})E(p_2, p_5) , \\ & \frac{1}{1_{225}36^{1}135} \left\{[s_{23} + s_{35} - s_{26} - s_{36}]E(p_2, p_6, p_3) , \\ & \frac{1}{1_{225}36^{1}125} \left[s_{34} - s_{46} + s_{36}\right][s_{12} - s_{15} - s_{16} - s_{16}]E(p_2, p_3 - s_{16}]E(p_3, p_3) \right] \\ & \frac{1}{1_{4556}}E(p_3 - p_3 - s_{16})E(p_3 - p_3) \\ & \frac{1}{1_{4556}}E(p_3 - p_3 - s_{16})E(p_3 - p_3) \\ & \frac{1}{1_{4556}}E(p_3 - p_3 - s_{16})E(p_3 - p_3) \\ & \frac{1}{1_{4556}}E(p_3 - p_3) \\ &$	r gluon production s, p_3) $E(p_2, p_3)$ s, p_3) $E(p_2, p_3)$ s, p_3)}, s_{13}] $E(p_3, p_3)$, s_{13}] $E(p_3, p_3)$, p_5, p_3) + $\frac{1}{4}t_{123}$] $E(p_2, p_3)$ }, p_5, p_3) + $\frac{1}{4}t_{123}$] $E(p_2, p_3)$ }, (12) s_{23}], s_{24}],
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3),$ $= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{s_{12}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{2}{s_{46}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_5, p_2 - p_3),$ $= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{4}{s_{12}s_{44}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{4}{s_{12}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_5, p_3),$ $= \frac{4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_5, p_3),$ $= \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_6),$ 1. Trans D_6^0 are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the function $G(p_n, p_1)$, rams D_6^0 are listed below: $= \frac{4}{s_{15}s_{44}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) + F(F(p_6, p_5) + s_3)E(p_5, p_5)\},$	417 (11) ions	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = -\frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(8) = -\frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{S}(11) = \frac{1}{s_{2}}$ $D_{0}^{S}(12) = -\frac{1}{s_{2}}$ $D_{0}^{S}(2) = -\frac{1}{s_{2}}$ $D_{0}^{S}(3) = -\frac{1}{s_{2}}$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{25^{3}4^{1}125} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5}) - F(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_{5}, p_{5}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_{6}, p_{5}),$ $\frac{2}{25^{5}4^{4}134} \{I_{55} - S_{26} - S_{25}\}E(p_{3}, p_{5}),$ $\frac{4}{25^{3}5^{4}125} \{[F(p_{5}, p_{2}) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{12}]E(p_{5}, p_{5}) - F(p_{12}, p_{13}) + \frac{1}{4}t_{123}]E(p_{5}, p_{5}) + \frac{2}{14s^{3}56}t_{125}} \{I_{33} - I_{356} + S_{36}]E(p_{2}, p_{5}),$ $\frac{2}{14s^{5}36t_{145}} \{I_{523} + S_{35} - S_{56} + S_{56}]E(p_{5}, p_{5}),$ $\frac{1}{125s^{5}6t_{125}}} \{I_{54} - I_{54} + S_{56}]E(p_{2} - p_{6}, p_{5})\}$ mms D_{0}^{5} are listed below: $\frac{1}{12s^{5}36t_{125}} \{I_{54} - s_{46} + s_{36}][s_{12} - s_{15} - \frac{1}{14s^{5}6t_{124}}} \{I_{512} - S_{24} - s_{14}][s_{35} - s_{56} + \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{45} + s_{14}][s_{23} - s_{26} - \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{45} + s_{14}][s_{23} - s_{26} - \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{45} + s_{14}][s_{23} - s_{26} - \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{45} + s_{14}][s_{23} - s_{26} - \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{54} + S_{54}][s_{51} - S_{56} + \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{54} + S_{54}][s_{51} - S_{56} - \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{54} + S_{54}][s_{51} - S_{56} + \frac{1}{16s^{5}6t_{124}}} \{I_{51} - S_{54} + S_{54}\} \}$	r gluon production s, P_3) $E(p_2, p_5)$ s, P_3) $E(p_2, p_5)$ s, p_3) , $s_{13}]E(p_3, p_3)$ p_5, p_3) $+\frac{1}{4}(1_{25}]E(p_2, p_3)$, $s_{2}=p_5, p_3$) $1-[s_{23}+s_{56}-s_{35}-s_{26}]E(p_2+p_3, p_3)$. (12) $s_{23}]$, $s_{36}]$, $s_{36}]$,
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_3, p_3),$ $= \frac{2}{5_{16}5_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}2t_{125}} E(p_2, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_6),$ $= \frac{2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-4}{5_{12}5_{34}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{4}{5_{15}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_6),$ $= \frac{4}{5_{15}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_5, p_3),$ $= \frac{4}{5_{15}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_6),$ 1. The sums D_6^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the function of $C(p_6, p_1)$. The sums D_6^G are obtained from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sums D_6^G are obtained from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sums D_6^G are balanced from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sums D_6^G are balanced from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sums D_6^G are balanced from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sums D_6^G are balanced from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$. The sum D_6^G are balanced from D_2^G by $replacing \delta_2$ by $\delta_0 = 0$ and the function of $C(p_6, p_2)$.	417 (11) ions	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(6) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{1}}$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{F}(12) = \frac{1}{s_{2}}$	S.J. Parke, T.R. Taylor / Fou $\frac{4}{2s^{5}s_{4}t_{125}} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p$	r gluon production s, p_3) $E(p_2, p_2)$ s, p_3) $E(p_2, p_3)$ s, p_3) $E(p_3, p_3)$ s ₁₃] $E(p_3, p_3)$ p ₅ , p_3) + $\frac{1}{4}t_{123}$] $E(p_2, p_3)$ }, p ₅ , p_3) + $\frac{1}{4}t_{123}$] $E(p_2, p_3)$ }, (12) s ₂₃], s ₃₆], s ₃₆], s ₃₆],
$\begin{split} S.J. Parke, T.R. Taylor / Four gluon production \\ &= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3), \\ &= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_3), \\ &= \frac{-2}{s_{12}t_{125}} E(p_5, p_2 - p_3), \\ &= \frac{-2}{s_{14}t_{125}} E(p_5, p_2 - p_3), \\ &= \frac{-2}{s_{14}t_{125}} E(p_5, p_2 - p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_5, p_2 - p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_5, p_2 - p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3), \\ &= \frac{-2}{s_{14}t_{125}} E(p_1 - p_2 + p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3), \\ &= \frac{-4}{s_{12}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_3), \\ &= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_3), \\ &= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_6), \\ 1. \\ rams D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by replacing \delta_2 by \delta_0 = 0 and the function of the trans D_6^0 are obtained from D_2^G by $	417 (11)	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ $D_{0}^{S}(11) = \frac{1}{s_{2}}$ $D_{0}^{S}(10) = \frac{1}{s_{2}}$	$S.J. Parke, T.R. Taylor / Fou \frac{4}{2s^{5}4t^{1}25} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, p_{5$	r gluon production $s_{1}, p_{2} \in E(p_{2}, p_{3})$ $s_{2}, p_{3} \in E(p_{2}, p_{3})$ $s_{3}, p_{3} \in E(p_{3}, p_{3})$ $p_{3}, p_{3} + \frac{1}{4}t_{123} \in E(p_{2}, p_{3})$ $p_{3}, p_{3} + \frac{1}{4}t_{123} \in E(p_{2}, p_{3})$ (12) $s_{23} = 1, \dots, (12)$ $s_{23} = 1, \dots, (12)$ $s_{23} = 1, \dots, (12)$ $s_{24} = 1, \dots, (12)$ $s_{25} = 1, \dots, (12)$ $s_{26} = 1, \dots, (12$
$S.J. Parke, T.R. Taylor / Four gluon production$ $= \frac{-2}{5_{25}5_{34}} E(p_2 - p_3, p_3),$ $= \frac{2}{5_{16}5_{25}} E(p_6, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_2, p_2 - p_3),$ $= \frac{-2}{5_{12}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3, p_2 - p_3),$ $= \frac{-2}{5_{34}t_{125}} E(p_3 - p_6, p_3),$ $= \frac{-2}{5_{34}t_{125}} [(p_1 + p_2 - p_3)(p_4 + p_3 - p_6) - t_{123}]E(p_2, p_3),$ $= \frac{4}{5_{12}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_2, p_6),$ $= \frac{4}{5_{15}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) + t_{123}]E(p_5, p_3),$ $= \frac{4}{5_{15}5_{34}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{123}]E(p_5, p_6),$ 1. Tams D_6^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the function of $G(p_0, p_1)$. Tams D_6^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the function of $G(p_0, p_1)$. Tams D_6^G are obtained from D_2^G by $P_5(p_5, p_3)E(p_6, p_5)$ $+ [F(p_6, p_3) + s_{34}]E(p_5, p_5)] + [F(p_5, p_3) + s_{34}]E(p_5, p_5)],$ $= \frac{4}{5_{15}5_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_3) - F(p_5, p_3)E(p_6, p_5) + F(F(p_6, p_3) + s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_3, p_5)],$ $= \frac{4}{5_{15}5_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_3) - F(p_5, p_3)E(p_6, p_5) + F(F(p_6, p_3) + s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_3, p_5)],$ $= \frac{4}{5_{15}5_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_6, p_3)E(p_6, p_5) + F(F(p_6, p_3) + s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_6, p_5)],$	417 (11) ions	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(6) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{1}}$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{s_{2}}$ D	$S.J. Parke, T.R. Taylor / Fou \frac{4}{2s^{5}s^{4}t_{125}} \{F(p_{2}, p_{3})E(p_{5}, p_{5}) - F(p_{5}, $	r gluon production s, p_3) $E(p_2, p_2)$ s, p_3) $E(p_2, p_3)$ s, p_3) $E(p_2, p_3)$ s, p_3) $E(p_3, p_3)$ s, p_3 , p_3) $F_4(p_2, p_3)$, p_5, p_3) $F_4(p_2, p_3)$, p_5, p_3) $F_4(p_2, p_3)$, p_5, p_3) $P_5(p_3, p_3)$ $P_5(p_3, p_3)$ $P_5($
$\begin{split} & S.J. \ Parke, T.R. \ Taylor / \ Four \ gluon \ production \\ &= \frac{-2}{s_{25}s_{34}} E(p_2-p_5,p_3), \\ &= \frac{2}{s_{16}s_{25}} E(p_6,p_2-p_3), \\ &= \frac{-2}{s_{12}t_{125}} E(p_5,p_2-p_3), \\ &= \frac{-2}{s_{14}t_{125}} E(p_5,p_2-p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_5,p_2-p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_5,p_2-p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_5,p_2-p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_3-p_6,p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_3-p_6,p_3), \\ &= \frac{-2}{s_{34}t_{125}} E(p_1+p_2-p_3)(p_4+p_3-p_6)-t_{123}]E(p_2,p_3), \\ &= \frac{-4}{s_{12}s_{44}t_{125}} [(p_1-p_2+p_3)(p_4-p_3+p_6)+t_{123}]E(p_5,p_3), \\ &= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1-p_2+p_3)(p_4-p_3+p_6)-t_{123}]E(p_5,p_3), \\ &= \frac{-4}{s_{15}s_{44}t_{125}} [(p_1-p_2+p_3)(p_4-p_3+p_6)-t_{123}]E(p_5,p_6), \\ &= 1. \\ rams \ D_6^0 \ are \ obtained \ from \ D_2^G \ by \ replacing \ \delta_2 \ by \ \delta_0 = 0 \ and \ the \ function \ C(p_6,p_7), \\ rams \ D_6^0 \ are \ obtained \ from \ D_2^G \ by \ replacing \ \delta_2 \ by \ \delta_0 = 0 \ and \ the \ function \ C(p_6,p_7), \\ &= \frac{-4}{s_{15}s_{25}s_{41}t_{125}} [F(p_5,p_6)E(p_3,p_5)-F(p_5,p_3)E(p_6,p_5)], \\ &= \frac{-4}{s_{16}s_{25}s_{26}t_{125}} [F(p_5,p_6)E(p_3,p_5)-F(p_5,p_3)E(p_6,p_5)], \\ &= \frac{-4}{s_{16}s_{25}s_{46}t_{125}} F(p_5,p_6)E(p_3,p_5)-F(p_5,p_3)E(p_6,p_5), \\ &= \frac{-4}{s_{15}s_{26}t_{125}} F(p_5,p_6)E(p_3,p_5)-F(p_5,p_3)E(p_6,p_5), \\ &= \frac{-4}{s_{16}s_{25}s_{46}t_{125}} F(p_5,p_6)E(p_3,p_5)-F(p_5,p_5)E(p_6,p_5), \\ &= \frac{-4}{s_{16}s_{26}s_{125}} F(p_5,p_6)E(p_5,p_5)-F(p_5,p_5)E(p_6,p_5), \\ &= (-F(p_5,p_6)-\frac{1}{2}s_{56}-\frac{1}{2}s_{56}+\frac{1}{2}s_{56}+\frac{1}{2}s_{56}}E(p_5,p_5)), \\ &= (-F(p_5,p_6)-\frac{1}{2}s_{56}-\frac{1}{2}s_{56}+\frac{1}{2}s_{56}+\frac{1}{2}s_{56}}E(p_5,p_5)), \\ \end{array}$	417 (11)	418 $D_{0}^{F}(4) = \frac{1}{s_{2}}$ $+$ $D_{0}^{F}(5) = \frac{1}{s_{1}}$ $D_{0}^{F}(6) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(7) = \frac{1}{s_{2}}$ $D_{0}^{F}(8) = \frac{1}{s_{1}}$ $D_{0}^{F}(8) = \frac{1}{s_{2}}$ $D_{0}^{F}(10) = \frac{1}{s_{2}}$ $D_{0}^{F}(11) = \frac{1}{2}$ $-$ $The diagrad D_{0}^{S}(11) = \frac{1}{s_{2}} D_{0}^{S}(10) = \frac{1}{s_{2}}$	$\begin{split} & S.J. Parke, T.R. Taylor / Fou \\ & \frac{4}{2s^5y_4t_{125}} \left\{F(p_2, p_3)E(p_5, p_5) - F(p_1, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{13}\right]E(p_2, p_3) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{13}]E(p_3, p_3) \\ & -\frac{2}{2s^5y_4t_{124}}\left[s_{55} - s_{23} + s_{23}\right]E(p_3, p_3) \\ & \frac{4}{2s^5y_6t_{125}}\left[F(p_{51}, p_2) - \frac{1}{2}s_{25} - \frac{1}{3}s_{12} + \frac{1}{2}s_{13}\right]E(p_5, p_5) - F(p_1, p_3) + \frac{1}{4}t_{123}\right]E(p_5, p_5) \\ & -\frac{2}{1a^5y_6t_{124}}\left[s_{23} - s_{26} - s_{36}\right]E(p_2, p_5) \\ & -\frac{1}{1a^5y_5t_{125}}\left[s_{34} - s_{46} + s_{36}\right]E(p_2 - p_6, p_3) \\ & -\frac{1}{1a^5y_5t_{125}}\left[s_{15} - s_{45} + s_{14}\right]\left[s_{12} - s_{15} - \frac{1}{1a^5y_5t_{145}}\left[s_{15} - s_{45} + s_{14}\right]\left[s_{23} - s_{26} - \frac{1}{13s^5y_6t_{125}}\left[s_{15} - s_{15} - s_{16}\right]\left[s_{23} - s_{24} - \frac{1}{13s^5y_6t_{125}}\left[s_{46} - s_{34} - s_{36}\right]\left[s_{12} - s_{12} - \frac{1}{13s^5y_4t_{125}}\left[s_{46} - s_{34} - s_{36}\right]\left[s_{12} - s_{25} - \frac{1}{13s^5y_6t_{125}}\left[s_{46} - s_{44} - s_{36}\right]\left[s_{12} - s_{25} - \frac{1}{3s^5y_6t_{125}}\left[s_{46} - s_{46} - s_{36}\right]\left[s_{12} - s_{25} - \frac{1}{3s^5y_6t_{125}}\left[s_{45} - s_{45} - s_{45}\right]\left[s_{45} - s_{45} - s_{45}\right]\left$	r gluon production $s_{1}, p_{2}) E(p_{2}, p_{3})$ $s_{2}, p_{3}) E(p_{2}, p_{3})$ $s_{3}, p_{3}) \},$ $s_{3}]E(p_{3}, p_{3})$ $p_{5}, p_{3}) + \frac{1}{4}t_{123}]E(p_{2}, p_{3})\},$ $r_{2} = p_{5}, p_{3})$ $r_{1} = [s_{23} + s_{56} - s_{35} - s_{26}]E(p_{2} + p_{3}, p_{3})\}.$ (12) $s_{23}],$ $s_{36}],$ s_{3

S.J. Parke, T.R. Taylor / Four gluon production where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt} = 1$. For the future use, we define one more function $F(p_n, p_j) = \{(p_1 p_4)(p_i p_j) + (p_1 p_i)(p_j p_4) - (p_1 p_j)(p_i p_4)\}/(p_1 p_4).$ (10)

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6 . The diagrams D_2^G are listed below:

 $D_2^G(1) = \frac{\delta_2}{s_{14}s_{25}s_{36}} \{ [(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] - [(p_2 - p_5)(p_3 + p_6)] \}$ ×[$(p_1-p_4)(p_3-p_6)$]+[$(p_2+p_5)(p_3-p_6)$][$(p_1-p_4)(p_2-p_5)$]},

 $D_2^{\rm G}(2) = \frac{1}{\sum_{s_2, s_{22}}} \left\{ 2E(p_2 - p_5, p_3 - p_6) - 2E(p_3 - p_6, p_2 - p_5) + \delta_2[(p_2 - p_5)(p_3 - p_6)] \right\},\$

 $D_2^{\rm G}(3) = \frac{4}{s_{25}s_{36}t_{125}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$ $-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6)$ $-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3)$ $+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5, p_6)$ $-[p_1(p_2-p_5)]E(p_3-p_6, p_3+p_6)-[p_4(p_3-p_6)]E(p_2+p_5, p_2-p_5)$ $+ \delta_2[p_1(p_2 - p_5)][p_4(p_3 - p_6)]\},$

 $D_2^G(4) = \frac{-2}{\sum_{i=1}^{n}} \{ E(p_3 - p_6, p_3 + p_6) - \delta_2[p_4(p_3 - p_6)] \},\$

 $D_2^G(5) = \frac{-2}{s_{a_1} f_{a_2}} \{ E(p_2 + p_5, p_2 - p_5) - \delta_2[p_1(p_2 - p_5)] \},\$

 $D_2^{\rm G}(6) = \frac{\delta_2}{t},$

 $D_2^G(7) = \frac{4}{s_{12}s_{36}t_{125}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$ $-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)-[p_4(p_3-p_6)]E(p_2,p_2-p_5)\},$

 $D_2^{\rm G}(8) = \frac{4}{s_{14}s_{25}t_{125}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$ $-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5,p_3)-[p_1(p_2-p_5)]E(p_3-p_6,p_3)\},$

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 $D_0^{\rm S}(7) = \frac{1}{s_{25}s_{34}t_{125}} [s_{36} - s_{46} + s_{34}] [s_{12} - s_{15} - s_{25}],$ $D_0^{\rm S}(8) = \frac{1}{s_{16}s_{25}t_{146}} [s_{25} + s_{35} - s_{23}] [s_{14} - s_{46} + s_{16}],$

 $D_0^{\rm S}(9) = \frac{1}{s_{25}s_{34}t_{134}} [s_{14} + s_{34} - s_{13}][s_{26} - s_{56} + s_{25}],$

 $D_0^{\rm S}(10) = \frac{1}{S_{25}S_{36}} (p_2 - p_5)(p_3 - p_6) ,$

 $D_0^{\rm S}(11) = \frac{1}{s_{14}s_{36}} (p_1 - p_4) (p_3 - p_6) ,$

 $D_0^{\rm S}(12) = \frac{1}{s_{12}s_{22}} \left(p_6 - p_1\right) \left(p_2 - p_5\right),$

 $D_0^{\rm S}(13) = \frac{1}{s_{15}s_{24}} (p_5 - p_1)(p_3 - p_4) ,$

 $D_0^{\rm S}(14) = \frac{1}{s_{\rm SS}} (p_2 - p_5)(p_3 - p_4) ,$

 $D_0^{\rm S}(15) = \frac{1}{s_{14}s_{25}s_{36}} \left\{ \left[(p_2 + p_5)(p_3 - p_6) \right] \left[(p_1 - p_4)(p_2 - p_5) \right] \right\}$ +[$(p_2-p_5)(p_3-p_6)$][$(p_1-p_4)(p_3+p_6)$] + [$(p_1+p_4)(p_2-p_5)$][$(p_1-p_4)(p_3-p_6)$]},

 $D_0^{\rm S}(16) = \frac{2}{s_{16}s_{34}s_{25}} \left\{ \left[(p_2 - p_5)(p_3 + p_4) \right] \left[(p_1 - p_6)(p_3 + p_4) \right] \right] \right\}$ +[$(p_1+p_6)(p_3-p_4)$][$(p_1-p_6)(p_2-p_5)$]

+[$(p_1-p_6)(p_2+p_5)$][$(p_3-p_4)(p_2-p_5)$]}

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors \mathcal{D}_{0} and \mathcal{D}_2 . After generating the vectors $\mathcal{D}_{0,r}$, $\mathcal{D}_{0,r}$, $\mathcal{D}_{2,r}$, $\mathcal{D}_{2,r}$, and $\mathcal{D}_{2,r}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2 . Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multigluon amplitudes are tested by checking the gauge invariance. Due to the specifics



Background amplitudes crucial for e.g. colliders



production	S.J. Parke, T.R. Taylor / Fo
gluons with momenta p_1, p_2 into from eq. (5) by setting $I = 2$ and	TABLE I Matrices K(I, J)[I = 1
$p_5, -p_6.$	Matrix K ⁽⁴⁾
d and forty Feynman diagrams, $ \begin{array}{c} K_{r} \\ K_{\sigma} \\ K_{\rho} \\ K \\ K \end{array}, (6) $	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$
vector functions of the momenta	Matrix K _p ⁽⁴⁾
stant 11 × 11 symmetric matrices. e vector \mathscr{D} by the permutations vely, of the momentum variables present the sums of all contribu- eleven basis color factors. The color indices of products of the proportional to $N^4(N^2-1)$ and	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$
rs, $N = 3$ for QCD):	

[3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.
Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

$$\begin{split} D_2^G(16) &= \frac{-4}{s_{12}s_{36}t_{124}} \left[s_{35} - s_{56} + s_{36} \right] E(p_2, p_2) , \\ D_2^G(17) &= \frac{4}{s_{36}s_{45}t_{145}} \left[s_{23} - s_{26} - s_{36} \right] E(p_5, p_5) , \\ D_2^G(18) &= \frac{-4}{s_{12}s_{56}s_{45}} \left[2(p_1 + p_2)(p_3 - p_6) - s_{36} \right] E(p_2, p_3) , \\ D_2^G(19) &= \frac{-2}{s_{12}s_{56}} E(p_2, p_3 - p_6) , \\ D_2^G(20) &= \frac{2}{s_{26}s_{45}} E(p_3 - p_6, p_5) , \\ D_2^G(21) &= \frac{-4}{s_{25}s_{54}t_{134}} \left[s_{23} - s_{56} + s_{25} \right] E(p_3, p_3) , \\ D_2^G(22) &= \frac{4}{s_{16}s_{25}s_{146}} \left[s_{23} - s_{35} - s_{25} \right] E(p_6, p_6) , \\ D_2^G(23) &= \frac{4}{s_{16}s_{25}s_{54}} \left[2(p_1 + p_6)(p_2 - p_5) + s_{25} \right] E(p_6, p_3) , \end{split}$$

S.J. Parke, T.R. Taylor / Four glue

gluons. The cross section for the scattering of two

four gluons with momenta p_3 , p_4 , p_5 , p_6 is obtained

replacing the momenta p_3 , p_4 , p_5 , p_6 by $-p_3$, $-p_4$,

As the result of the computation of two hur

where $\mathcal{D}, \mathcal{D}_{\rho}, \mathcal{D}_{\sigma}$ and \mathcal{D}_{τ} are 11-component com-

 p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_p, K_σ and K_τ are contained by the second second

The vectors \mathcal{D}_{α} , \mathcal{D}_{α} and \mathcal{D}_{γ} are obtained from t

in D. The individual components of the vector D i

is proportional to the appropriately cho

color bases, contain two independent structure

 (N^2-1) , respectively (N is the number of c

rices K, which are the suitable sums over the

 $(p_2 \leftrightarrow p_3), (p_5 \leftrightarrow p_6) \text{ and } (p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6), \text{ resp}$

 $= (\mathcal{D}^{\dagger}, \mathcal{D}^{\dagger}_{\rho}, \mathcal{D}^{\dagger}_{\sigma}, \mathcal{D}^{\dagger}_{\tau})_{\binom{0}{2}} \cdot \begin{vmatrix} \mathbf{r}_{\rho} \\ K_{\sigma} & K_{\tau} \end{vmatrix}$

 $A_{(0)}(p_1, p_2, p_3, p_4, p_5, p_6)$

we obtain

$$\begin{split} D_2^G(32) &= \frac{4}{s_{15}s_{34}t_{125}} \left[(p_1 - p_2 + p_3)(p_4 + p_3 - p_6) + t_{125} \right] E(p_5, p_3) , \\ D_2^G(33) &= \frac{4}{s_{15}s_{46}t_{125}} \left[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125} \right] E(p_5, p_6) , \end{split}$$

where $\delta_2 = 1$. The diagrams D_0^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the function $E(p_*, p_i)$ by $G(p_*, p_i)$. The diagrams D_0^F are listed below:

$$\begin{split} D_0^F(1) &= \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) \\ &+ [F(p_6, p_3) + s_{34}]E(p_5, p_5)\}, \\ D_0^F(2) &= \frac{-4}{s_{15}s_{15}s_{15}} \{[F(p_6, p_2) + \frac{1}{2}s_{16}]E(p_3, p_5)\} \end{split}$$

 $+[F(p_{2}, p_{3}) + \frac{1}{2}s_{34}]E(p_{6}, p_{5}) - F(p_{6}, p_{3})E(p_{2}, p_{5})\},$ $D_{0}^{F}(3) = \frac{4}{s_{15}s_{34}t_{155}}\{F(p_{5}, p_{6})E(p_{3}, p_{5}) - F(p_{5}, p_{3})E(p_{6}, p_{3})$

 $- [F(p_3, p_6) - \frac{1}{2}s_{36} - \frac{1}{2}s_{34} + \frac{1}{2}s_{46}]E(p_5, p_5)\},$



 $D_{0}^{5}(11) = \frac{1}{2s_{14}s_{25}s_{56}} \{[s_{23}+s_{35}-s_{26}-s_{56}]E(p_{2}-p_{5}, p_{5}) - [s_{23}+s_{56}-s_{55}-s_{26}]E(p_{2}+p_{5}, p_{5})] + [s_{23}+s_{56}-s_{55}-s_{56}]E(p_{2}+p_{5}, p_{5})] + [s_{23}+s_{56}-s_{56}-s_{56}-s_{56}] + [s_{23}+s_{56}-s_{56}-s_{56}-s_{56}]E(p_{2}+p_{5}-p_{56}-s_{56}-s_{56}-s_{56}] + [s_{23}+s_{56}-s_{56}-s_{56}-s_{56}]E(p_{2}+p_{5}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}] + [s_{23}+s_{56}-s_{56}-s_{56}-s_{56}-s_{56}]E(p_{2}+p_{5}-s_{56}-s_{56}-s_{56}-s_{56}]E(p_{2}+p_{5}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}] + [s_{23}+s_{56}-s_{56}-s_{56}-s_{56}-s_{56}-s_{56}]E(p_{2}+p_{56}-s_{56}-$

 $D_{0}^{S}(15) = \frac{1}{s_{14}s_{25}s_{36}} \{ [(p_{2}+p_{5})(p_{3}-p_{6})][(p_{1}-p_{4})(p_{2}-p_{5})] + [(p_{2}-p_{5})(p_{3}-p_{6})][(p_{1}-p_{4})(p_{3}+p_{6})] + [(p_{1}+p_{4})(p_{2}-p_{5})][(p_{1}-p_{4})(p_{3}-p_{6})] \},$

$$\begin{split} 6) &= \frac{1}{s_{16}s_{34}s_{25}} \{ [(p_2 - p_5)(p_3 + p_4)] [(p_1 - p_6)(p_3 - p_4)] \\ &+ [(p_1 + p_6)(p_3 - p_4)] [(p_1 - p_6)(p_2 - p_5)] \\ &+ [(p_1 - p_6)(p_2 + p_3)] [(p_3 - p_4)(p_2 - p_5)] \} \,. \end{split}$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors \mathcal{B}_0 and \mathcal{B}_2 . After generating the vectors $\mathcal{B}_{0,x} \mathcal{B}_{0,x} \mathcal{B}_{x,y} \mathcal{B}_{x,z}$ and \mathcal{B}_2 . by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2 . Finally, the total cross section is calculated by using eq. (5). The FORTRAN S program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

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where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt} = 1$. For the future use, we define

 $F(p_i, p_j) = \{(p_1, p_4)(p_i, p_j) + (p_1, p_i)(p_j, p_4) - (p_1, p_j)(p_i, p_4)\} / (p_1, p_4).$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta

 $D_2^{\rm G}(1) = \frac{\delta_2}{s_{14}s_{25}s_{36}} \{ [(p_2 - p_5)(p_3 - p_6)] [(p_1 - p_4)(p_3 + p_6)] - [(p_2 - p_5)(p_3 + p_6)] \}$

 $\times [(p_1 - p_4)(p_3 - p_6)] + [(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)]],$

 $D_2^{\rm G}(2) = \frac{1}{\sum_{s_1,s_2}} \left\{ 2E(p_2 - p_5, p_3 - p_6) - 2E(p_3 - p_6, p_2 - p_5) + \delta_2[(p_2 - p_5)(p_3 - p_6)] \right\},$

 $D_2^{\rm G}(3) = \frac{4}{s_{25}s_{36}t_{125}} \{ [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)] E(p_2, p_3) \}$

 $-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2,p_6)$

the momenta p_1, p_4, p_5, p_6 .

The diagrams D_2^G are listed below

care must be taken with the implicit dependence of the functions E, F and G on

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multigluon amplitudes are tested by checking the gauge invariance. Due to the specifics



Discovery of Shocking Simplicity

 Within six months, Parke-Taylor stumbled on a simple guess —unquestionably a theorist's delight:



Amplitude for *n*-Gluon Scattering*PRL* 56 (1986)

Stephen J. Parke and T. R. Taylor Fermi National Accelerator Laboratory, Batavia, Illinois 60510 (Received 17 March 1986)

2

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$\frac{\langle 1\,2\rangle^4}{\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,5\rangle\cdots\langle n\,1\rangle}$$

 $p_a^{\mu} \equiv \sigma^{\mu}_{\alpha\dot{\alpha}}\lambda^{\alpha}_a\lambda^{\dot{\alpha}}_a$ $\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$ $[a b] \equiv \det(\lambda_a, \lambda_b)$ van der Waerden (1929)



What about beyond the leading order of approximation?



Perturbations of Parke/Taylor's Guess

Bern, Dixon, Dunbar, Kosower (1994)

$$\frac{\langle 12\rangle^4}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle}\times$$



What about beyond the leading order of approximation?



Perturbations of Parke/Taylor's Guess



Perturbations of Parke/Taylor's Guess

What about beyond the leading order of approximation?



Arkani-Hamed, **JB**, Cachazo, Trnka (2010)

 $\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle$



Perturbations of Parke/Taylor's Guess

What about beyond the leading order of approximation?



Arkani-Hamed, **JB**, Cachazo, Trnka (2011)

 $\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle$ $a \leq b < c < < d \leq e < f$



But what about after regularization and loop integration? What is the *mathematical form* of the predictions made by QFT?



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Del Duca, Duhr, Smirnov (2010)



The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

But what about after regularization and loop integration? What is the *mathematical form* of the predictions made by QFT?

 $\begin{array}{c} u_{6_{0}}^{*}u_{7}(u_{1},u_{2},u_{3})=\\ \frac{1}{24}^{2}G\left(\frac{1}{(1-u_{1}},\frac{u_{2}-1}{u_{1}})+\frac{1}{24}^{2}G\left(\frac{1}{(u_{1}},\frac{1}{u_{1}+u_{2}})+\frac{1}{24}^{2}G\left(\frac{1}{(u_{1}},\frac{1}{u_{1}+u_{3}})\right)+\\ \frac{1}{24}^{2}G\left(\frac{1}{(1-u_{2}},\frac{u_{2}-1}{u_{2}+u_{2}})+\frac{1}{24}^{2}G\left(\frac{1}{(u_{2}},\frac{1}{u_{1}+u_{3}})+\frac{1}{24}^{2}G\left(\frac{1}{(u_{2}},\frac{1}{u_{1}+u_{3}})+\frac{1}{(u_{1}+u_{3})}\right)+\\ \frac{1}{24}^{2}G\left(\frac{1}{(u_{2}},\frac{1}{u_{2}+u_{3}})+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}\right)+\\ \frac{1}{24}^{2}G\left(\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}\right)+\\ \frac{1}{24}^{2}G\left(\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2}+u_{3})}\right)\\ \frac{1}{(u_{2}+u_{3})}\left(\frac{1}{(u_{2}+u_{3})}+\frac{1}{(u_{2$ $\pi^{2}G\left(\frac{1}{1-u_{3}},\frac{u_{1}-1}{u_{1}+u_{3}-1};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{1}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{2}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{2}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{3}};1\right)+\frac{1}{24}\pi^{2}G\left(\frac{1}{u_{3}},\frac{1}{u_{3}+u_{$ $\left(0, \frac{1}{u_1}, \frac{1}{u_2 + u_3}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right) + \frac{3}{2}G\left(0, 0, \frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right)$ $\frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1$ + $\frac{3}{2}G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_1 + u_3}; 1\right)$ + $\frac{3}{2}G\left(0, 0, \frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right)$ + $\frac{1}{2}G\left(0,\frac{1}{u_{1}},0,\frac{1}{u_{2}};1\right) + G\left(0,\frac{1}{u_{1}},0,\frac{1}{u_{1}+u_{2}};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_{1}},0,\frac{1}{u_{3}};1\right) +$ $G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)$ $\frac{1}{2}G\left(0,\frac{1}{u_{1}},\frac{1}{u_{2}},\frac{1}{u_{1}+u_{2}};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_{1}},\frac{1}{u_{3}},\frac{1}{u_{1}+u_{3}};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_{2}},0,\frac{1}{u_{1}};1\right) + \frac{1}{2}G\left(0,\frac{1}{u_{2}},0,\frac{1}{u_{2}};1\right) + \frac{1}{2}G\left(0,\frac{1}{u_{2}};1\right) + \frac{1}{2}G\left(0,$ $G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-\frac{1}{u_2+u_3};1$ $\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2+u_3};1\right) - \frac{1}{2}G\left(0,\frac{1}{u_2$ $\frac{1}{2}G\left(0,\frac{1}{u_{2}},\frac{1}{u_{3}},\frac{1}{u_{2}+u_{3}};1\right)+\frac{1}{4}G\left(0,\frac{u_{2}-1}{u_{1}+u_{2}-1},0,\frac{1}{1-u_{1}};1\right)+$ $\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$

 $G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, \frac{1}{1-u_1}, 1\right) - \frac{1}{4}G\left(0, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, \frac{1}{1-u_1}, 1\right)$ $G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_1}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_2}; 1\right) + G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_1 + u_3}; 1\right) + G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_1 + u_3}; 1\right)$ $G\left(0, \frac{1}{u_3}, 0, \frac{1}{u_2 + u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_1}, \frac{1}{u_1 + u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_2}, \frac{1}{u_2 + u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3 + u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{$ $\frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_1 + u_3}; 1\right) - \frac{1}{2}G\left(0, \frac{1}{u_3}, \frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right) +$ $\frac{1}{4}G\left(0, \frac{u_1 - 1}{u_1 + u_3 - 1}, 0, \frac{1}{1 - u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_1 - 1}{u_1 + u_3 - 1}, \frac{1}{1 - u_3}, 0; 1\right) \left(0, \frac{u_1 - 1}{u_1 + u_3 - 1}, \frac{1}{1 - u_3}, 1; 1\right) + \frac{1}{4}G\left(0, \frac{u_1 - 1}{u_1 + u_3 - 1}, \frac{1}{1 - u_3}, \frac{1}{1 - u_3}; 1\right)$ $F\left(0, \frac{u_1-1}{u_1+u_3-1}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}; 1\right) + \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, 0, \frac{1}{1-u_2}; 1\right)$ $G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 0; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, 1; 1\right) +$ $\frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}; 1\right) - \frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}; 1\right)$ $\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$

 $\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_2}, v_{321}, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, v_{321}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right)$ $\frac{1}{2}\mathcal{G}\left(v_{123},0,1,\frac{1}{1-u_{1}};1\right) + \frac{1}{2}\mathcal{G}\left(v_{123},0,\frac{1}{1-u_{1}},1;1\right) + \frac{1}{2}\mathcal{G}\left(v_{123},1,0,\frac{1}{1-u_{1}};1\right)$ $\mathcal{G}\left(v_{123}, 1, 1, \frac{1}{1-u_1}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}, 0; 1\right) - \frac{5}{4}\mathcal{G}\left(v_{123}, 1, \frac{1}{1-u_1}, 1; 1\right)$ $i\left(v_{123}, 1, \frac{1}{1-u}, \frac{1}{1-u}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u}, 0, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u}, 1, 0, 1; 1\right)$ $\frac{5}{4}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-u_1}, 1, \frac{1}{1-u_1}; 1\right) +$ $\frac{1}{2}\mathcal{G}\left(v_{123}, \frac{1}{1-w}, \frac{1}{1-w}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{132}, 1, 1, \frac{1}{1-w}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{132}, 1, \frac{1}{1-w}; 1\right)$

 $\frac{1}{4}\mathcal{G}\left(v_{213}, \frac{1}{1-u_2}, 1, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 0, 1, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 0, \frac{1}{1-u_2}, 1; 1\right)$ $\frac{1}{2}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}, 1; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) + \frac{1}{2}\mathcal{G}\left(v_{231}, \frac{1}{1-u_2}, 0, 1; 1\right)$ $\mathcal{G}\left(v_{312}, \frac{1}{1-u_3}, \frac{1}{1-u_3}, 1; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, 1, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1, \frac{1}{1-u_3}; 1, \frac{1}{1-u_3}; 1\right) - \frac{1}{4}\mathcal{G}\left(v_{321}, 1, \frac{1}{1-u_3}; 1$ $\frac{1}{4}\mathcal{G}\left(v_{321}, \frac{1}{1-u_3}, 1, 1; 1\right) - \frac{3}{4}G\left(0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right)H(0; u_1) \frac{3}{4}G\left(0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right) - \frac{1}{4}G\left(0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right) - \frac{1}{4}G\left(0,\frac{1}{u_3},\frac{1}{u_1+u_2};1\right)H\left(0;u_1\right) - \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right) + \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{u_1-1}{u_1+u_3};1\right)H\left(0;u_1\right) + \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{u_1-1}{u_1+u_3};1\right)H\left(0;u_1\right) + \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{u_1-1}{u_1+u_3};1\right)H\left(0;u_1\right) + \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{u_1-1}{u_1+u_3};1\right)H\left(0;u_1\right) + \frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3},\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1\right)H\left(0,\frac{u_1-1}{u_1+u_3};1$ $\frac{1}{4}G\left(0, \frac{u_3-1}{u_2+u_2-1}, \frac{1}{1-u_2}; 1\right)H(0; u_1) - \frac{3}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2+u_2}; 1\right)H(0; u_1)$ $\frac{3}{4}G\left(\frac{1}{u_1}, 0, \frac{1}{u_1 + u_3}; 1\right) H\left(0; u_1\right) + \frac{1}{2}G\left(\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right) H\left(0; u_1\right) +$ $G\left(\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right) H(0; u_1) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_2}, \frac{1}{u_1 + u_2}; 1\right) H(0; u_1) +$ $\frac{1}{4}G\left(\frac{1}{u_{1}},\frac{1}{u_{3}},\frac{1}{u_{1}+u_{3}};1\right)H\left(0;u_{1}\right)-\frac{1}{4}G\left(\frac{1}{1-u_{2}},1,\frac{1}{u_{1}};1\right)H\left(0;u_{1}\right)+$

 $\frac{1}{4}\mathcal{G}\left(v_{132},\frac{1}{1-u_1},1,1;1\right) - \frac{1}{4}\mathcal{G}\left(v_{213},1,1,\frac{1}{1-u_2};1\right) - \frac{1}{4}\mathcal{G}\left(v_{213},1,\frac{1}{1-u_2},1;1\right) - \frac{1}{4}\mathcal{G}\left(v_{213},1,$

 $\frac{3}{4}\mathcal{G}\left(v_{231}, 1, \frac{1}{1-u_{1}}; 1\right)H(0; u_{3}) + \frac{3}{4}\mathcal{G}\left(v_{231}, \frac{1}{1-u_{1}}; 1; 1\right)H(0; u_{3}) +$ $\begin{array}{l} \frac{1}{4}\mathcal{G}\left(v_{312},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{312},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\ \frac{1}{4}\mathcal{G}\left(v_{321},1,\frac{1}{1-u_3};1\right)H\left(0;u_3\right)+\frac{1}{4}\mathcal{G}\left(v_{321},\frac{1}{1-u_3},1;1\right)H\left(0;u_3\right)+\\ \end{array}$ $\frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right)H(0; u_1)H(0; u_3) +$ $G\left(\frac{1}{1}, \frac{u_3 - 1}{1}, 1\right) H(0; u_1) H(0; u_3) \frac{1}{1-u}$; 1) $H(0; u_1) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u}, u_{231}; 1\right) H(0; u_1) H(0; u_3)$ $\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{213}; 1\right) H(0; u_{1}) H(0; u_{3}) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{231}; 1\right) H(0; u_{1}) H(0; u_{3}) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{231}; 1\right) H(0; u_{1}) H(0; u_{3}) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{231}; 1\right) H(0; u_{3}) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{32}; 1\right) H(0$ $\frac{5}{24}\pi^2 H(0; u_1) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}; 1\right) H(0; u_2) H(0; u_3) +$ $G\left(\frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) H(0; u_3) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right) H(0; u_2) H(0; u_3) \left(\frac{1}{1-u_1}, u_{123}; 1\right) H(0; u_2) H(0; u_3) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right) H(0; u_2) H(0; u_3)$ $\begin{array}{l} \frac{1}{4} \mathcal{G} \left(\sum_{i=1}^{i} u_{i}, u_{123}; 1 \right) H \left(0; u_{2} \right) H \left(0; u_{3} \right) + \frac{1}{52} \frac{1}{4^{2}} \mathcal{G} \left(u_{1} \right) H \left(0; u_{3} \right) + \\ \mathcal{H} \left(0; u_{2} \right) H \left(0; 0; u_{1} \right) H \left(0; u_{3} \right) + \mathcal{H} \left(0; u_{1} \right) H \left(0; u_{3} \right) + \\ \frac{1}{4} H \left(0; u_{2} \right) H \left(0, 1; \frac{u_{1} + u_{2} - 1}{u_{2} - 1} \right) H \left(0; u_{3} \right) + \frac{1}{2} H \left(0; u_{1} \right) H \left(0; u_{3} \right) + \\ \end{array} \right)$ $H(0; u_1) H(0, 1; \frac{u_2 + u_3 - 1}{u_1 - 1}) H(0; u_3) + \frac{1}{\alpha} H(0; u_2) H(0, 1; (u_2 + u_3)) H(0; u_3) +$ $H(0; u_2) H(1, 0; u_1) H(0; u_3) + \frac{3}{2} H(0; u_1) H(1, 0; u_2) H(0; u_3) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u}, v_{213}; 1\right) H(0, 0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u}, v_{231}; 1\right) H(0, 0; u_1) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{312}; 1\right)H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{23}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) - \frac{1}{24}\pi^{2}H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(0, 0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{3}}, v_{321}; 1\right)H\left(1, 0; u_{1}\right) + \frac{1}$ $\left(\frac{1}{1-u_1}, v_{123}; 1\right) H(0, 0; u_2) + \frac{1}{4}G\left(\frac{1}{1-u_1}, v_{132}; 1\right) H(0, 0; u_2) +$ $\frac{1}{4}G\left(\frac{1}{1-u_3}, v_{312}; 1\right)H(0, 0; u_2) + \frac{1}{4}G\left(\frac{1}{1-u_3}, v_{321}; 1\right)H(0, 0; u_2) \frac{25}{4}H(0, 0; u_1)H(0, 0; u_2) - \frac{23}{24}\pi^2 H(0, 0; u_2) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{123}; 1\right)H(0, 0; u_3) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-v_{1}}, v_{132}; 1\right)H(0, 0; u_{3}) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-v_{2}}, v_{213}; 1\right)H(0, 0; u_{3}) +$ $\left(\frac{1}{1-u_{0}}, v_{231}; 1\right) H(0, 0; u_{3}) + 3H(0; u_{1}) H(0; u_{2}) H(0, 0; u_{3}) (0, 0; u_1) H (0, 0; u_3) - \frac{25}{4} H (0, 0; u_2) H (0, 0; u_3) - \frac{23}{24} \pi^2 H (0, 0; u_3) + \frac{1}{12} \pi^2 H (0, 1; u_1) +$ $\frac{1}{2}\pi^{2}H(0, 1; u_{2}) - \frac{1}{24}\pi^{2}H\left(0, 1; \frac{u_{1} + u_{2} - 1}{u_{2} - 1}\right) + \frac{1}{2}H(0; u_{1})H(0; u_{2})H(0, 1; (u_{1} + u_{2})) +$ $\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_1+u_2-1}{u_1+u_2-1},1,0;1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right) + \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{u_1+u_2-1},\frac{1}{u_1+u$ $\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right) - \frac{1}{4}G\left(\frac{1}{1-u_1};1\right) - \frac{1}{4$ $\frac{1}{4}G\left(\frac{1}{1-u_1}, \frac{u_2-1}{u_1+u_2-1}, \frac{u_2-1}{u_1+u_2-1}, 1; 1\right) +$ $G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_2};1\right)$ $\frac{1}{2}G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1 + u_2}; 1\right) - G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_3}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_1}, 0, 0, \frac{1}{u_1 + u_3}; 1\right) \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) - \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right) - \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) - \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right) - \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right) - \frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right) - \frac{1}{4}G\left(\frac{$ $\begin{array}{c} 1 \\ 4 \\ 4 \\ 4 \\ 4 \\ 6 \\ \left(\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{u_1}, 2 \\ 1 \\ -\frac{1}{4} \\ G \\ \left(\frac{1}{1-u_2}, \frac{1}{u_1}, \frac{1}{u_1}, \frac{1}{1-u_2}, 1 \\ 1 \\ -\frac{1}{1-u_2}, \frac{1}{u_1}, \frac{1}{1-u_2}, 1 \\ 1 \\ -\frac{1}{u_1}, \frac{1}{u_1}, \frac{1}$ $\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right) + \frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},1,0;1\right) - \frac{1}{4$ $\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right) \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}, \frac{1}{1-u_2}; 1\right) \frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{u_3-1}{u_2+u_3-1}, 1; 1\right)$ $\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2};1\right)-G\left(\frac{1}{u_2},0,0,\frac{1}{u_1};1\right)+$ $\begin{array}{l} 4 & (1-u_2,u_2+u_3-1), u_2+u_3-1, 1-u_2 \end{pmatrix} (u_2, \dots, u_1, 1) \\ \frac{1}{2}G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_1+u_2}; 1\right) - G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_3}; 1\right) + \frac{1}{2}G\left(\frac{1}{u_2}, 0, 0, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_2+u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{u_2}, 1\right) - \frac{1}{4}G\left(\frac{1}{$ $\frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right) - \frac{1}{4}G\left(\frac{1}{1 - u_3}, 1, \frac{1}{u_2}, 0; 1\right) +$ $\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{u_1-1}{1-u_3},1,\frac{1}{1-u_3};1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right) - \frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right) - \frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},1,0;1\right) G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 0; 1\right) + \frac{1}{4}G\left(\frac{1}{1-u_3}, \frac{u_1-1}{u_1+u_3-1}, \frac{1}{1-u_3}, 1; 1\right)$ $\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1+u_3}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right) \frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$ $\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$

 $\frac{1}{u_3 - 1}$, $\frac{u_3 - 1}{u_3 - 1}$, 1; 1 $H(0; u_1)$ $\frac{1}{4}G\left(\frac{1}{1-u_2}, \frac{u_3-1}{u_2+u_3-1}, \frac{1}{1-u_2}; 1\right)H(0; u_1) + \frac{1}{2}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1}; 1\right)H(0; u_1) - \frac{1}{2}G\left(\frac{1}{u_1}, 0$ $\frac{1}{4}G\left(\frac{1}{u_2}, 0, \frac{1}{u_1 + u_2}; 1\right)H\left(0; u_1\right) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1}, \frac{1}{u_1 + u_2}; 1\right)H\left(0; u_1\right) +$ $\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0;1\right)H(0;u_1)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)H(0;u_1)$ $-\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1};1\right)H\left(0;u_1\right)+\frac{1}{2}G\left(\frac{1}{u_3},0,\frac{1}{u_1};1\right)H\left(0;u_1\right) \frac{1}{4}G\left(\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+\frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)H\left(0;u_1\right)+$ $\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_{1}},v_{123};1\right)H\left(0;u_{1}\right)+\frac{1}{4}\mathcal{G}\left(0,\frac{1}{1-u_{1}},v_{132};1\right)H\left(0;u_{1}\right)+$ $\frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_{2}}, v_{213}; 1\right) H\left(0; u_{1}\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_{2}}, v_{231}; 1\right) H\left(0; u_{1}\right) +$ $\frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_{3}}, v_{312}; 1\right) H\left(0; u_{1}\right) - \frac{1}{4}\mathcal{G}\left(0, \frac{1}{1-u_{3}}, v_{321}; 1\right) H\left(0; u_{1}\right) \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{u_{1}}; 1\right)H\left(0; u_{1}\right) - \frac{1}{4}\mathcal{G}\left(0, u_{231}, \frac{1}{1-u_{2}}; 1\right)H\left(0; u_{1}\right) +$ $\begin{array}{c} 4\\ \frac{4}{4}G\left(0,u_{312},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,u_{312},\frac{u_{1}-1}{u_{1}+u_{3}-1};1\right)H\left(0;u_{1}\right)\\ \frac{4}{4}G\left(0,v_{123},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)+\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0;u_{1}\right)-\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)+\frac{4}{4}G\left(0,v_{132},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-u_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(0,v_{1},\frac{1}{1-v_{1}};1\right)H\left(1,v_{1},\frac{1}{1-v_{1}};1\right)H\left(1,v_{1},\frac{1}{1-v_{1}};1\right)H\left(1$ $\frac{1}{2}\mathcal{G}\left(0, v_{231}, \frac{1}{1-u_2}; 1\right) H\left(0; u_1\right) + \frac{1}{2}\mathcal{G}\left(0, v_{312}, \frac{1}{1-u_3}; 1\right) H\left(0; u_1\right) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-w}, 0, v_{123}; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-w}, 0, v_{132}; 1\right) H(0; u_1) +$ $\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}}, v_{123}; 1\right)H(0; u_{1}) + \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}}, v_{132}; 1\right)H(0; u_{1})$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{1}}, v_{123}, 1; 1\right) H\left(0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{1}}, v_{123}, \frac{1}{1-u_{1}}; 1\right) H\left(0; u_{1}\right) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, 1; 1\right)H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1}, v_{132}, \frac{1}{1-u_1}; 1\right)H(0; u_1) +$ $\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{213}; 1\right) H(0; u_1) - \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_2}, 0, v_{231}; 1\right) H(0; u_1) +$ $\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_{2}},\frac{1}{1-u_{2}},v_{213};1\right)H\left(0;u_{1}\right)-\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_{2}},\frac{1}{1-u_{2}},v_{231};1\right)H\left(0;u_{1},\frac{1}{1-u_{2}},v_{231};1\right)$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{\nu}}, u_{231}, 1; 1\right) H(0; u_1) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{\nu}}, u_{231}, \frac{1}{u_{\nu}}; 1\right) H(0; u_1) +$ $\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, u_{231}, \frac{1}{1-u_{2}}; 1\right) H\left(0; u_{1}\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{2}}, v_{213}, 0; 1\right) H\left(0; u_{1}\right) +$ $\begin{array}{l} \begin{array}{c} & & \\$

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 $\frac{1}{2}\pi^{2}H(0, 1; (u_{1} + u_{2})) + \frac{1}{19}\pi^{2}H(0, 1; u_{3}) + \frac{1}{4}H(0; u_{1})H(0; u_{2})H(0, 1; \frac{u_{1} + u_{3} - 1}{2}) \frac{1}{24}\pi^2 H\left(0,1;\frac{u_1+u_3-1}{u_1-1}\right)+\frac{1}{12}\pi^2 H\left(0,1;(u_1+u_3)\right)-\frac{1}{24}\pi^2 H\left(0,1;\frac{u_2+u_3-1}{u_3-1}\right)+$ $\frac{1}{12}\pi^{2}H(0, 1; (u_{2} + u_{3})) - \frac{1}{2}G(0, \frac{1}{u_{1} + u_{2}}; 1)H(1, 0; u_{1}) \frac{1}{2}G\left(0, \frac{1}{u_1+u_3}; 1\right)H(1, 0; u_1) + \frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right)H(1, 0; u_1) +$ $\frac{1}{4}G\left(\frac{1}{u_{1}}, \frac{1}{u_{2}+u_{3}}; 1\right)H(1, 0; u_{1}) + \frac{1}{4}G\left(\frac{1}{u_{0}}, \frac{1}{u_{2}+u_{3}}; 1\right)H(1, 0; u_{1}) +$ $\frac{u_1 - 1}{u_2}$, $\frac{u_1 - 1}{u_1 + u_2 - 1}$; 1) $H(1, 0; u_1) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1 + u_3}; 1\right) H(1, 0; u_2)$ $\frac{1}{1-u_2}$, u_{312} ; 1) $H(1, 0; u_1) - \frac{3}{4}H(0, 0; u_2)H(1, 0; u_1) - \frac{3}{4}H(0, 0; u_3)H(1, 0; u_1) +$ $\frac{1}{4}H\left(0,1;\frac{u_{1}+u_{3}-1}{u_{1}-1}\right)H\left(1,0;u_{1}\right)-\frac{1}{2}\pi^{2}H\left(1,0;u_{1}\right)-\frac{1}{2}G\left(0,\frac{1}{u_{1}-u_{2}};1\right)H\left(1,0;u_{2}\right) \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}; 1\right)H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{1 - u_1}, \frac{u_2 - 1}{u_1 + u_2 - 1}; 1\right)H(1, 0; u_2) +$ $\frac{1}{4}G\left(\frac{1}{u_1}, \frac{1}{u_1+u_2}; 1\right)H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_1+u_2}; 1\right)H(1, 0; u_2) +$ $\frac{1}{4}G\left(\frac{1}{u_2}, \frac{1}{u_2+u_3}; 1\right)H(1, 0; u_2) + \frac{1}{4}G\left(\frac{1}{u_3}, \frac{1}{u_2+u_3}; 1\right)H(1, 0; u_2) \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_{u}}, u_{123}; 1\right)H(1, 0; u_{2}) - \frac{3}{4}H(0, 0; u_{1})H(1, 0; u_{2}) - \frac{3}{4}H(0, 0; u_{3})H(1, 0; u_{2})$ $\frac{1}{4}H\left(0,1;\frac{u_{1}+u_{2}-1}{u_{2}-1}\right)H\left(1,0;u_{2}\right)-\frac{1}{4}H\left(1,0;u_{1}\right)H\left(1,0;u_{2}\right)-\frac{1}{3}\pi^{2}H\left(1,0;u_{2}\right) \frac{1}{2}G\left(0, \frac{1}{u_1 + u_3}; 1\right)H(1, 0; u_3) - \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}; 1\right)H(1, 0; u_3) + \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}; 1\right)H(1, 0; u_3) + \frac{1}{2}G\left(0, \frac{1}{u_1 + u_3}; 1\right)H(1, 0; u_3) + \frac{1}{2}G\left(0, \frac{1}{u_2 + u_3}$ $\frac{1}{4}G\left(\frac{1}{u_{*}},\frac{1}{u_{1}+u_{2}};1\right)H\left(1,0;u_{3}\right)+\frac{1}{4}G\left(\frac{1}{1-u_{2}},\frac{u_{3}-1}{u_{2}+u_{3}-1};1\right)H\left(1,0;u_{3}\right)+$ $\frac{1}{4}G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)H(1,0;u_3) - \frac{1}{3}\pi^2H(1,0;u_3) + \frac{1}{4}G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)H(1,0;u_3) + \frac{1}{3}H(1,0;u_3) + \frac$ $G\left(\frac{1}{u_3}, \frac{1}{u_2 + u_3}; 1\right) H(1, 0; u_3) - \frac{1}{4}G\left(\frac{1}{1 - u_2}, u_{231}; 1\right) H(1, 0; u_3)$ $_{1}H(0; u_{2})H(1, 0; u_{3}) - \frac{3}{4}H(0, 0; u_{1})H(1, 0; u_{3}) - \frac{3}{4}H(0, 0; u_{2})H(1, 0; u_{3})$ $\frac{1}{2}H\left(0, 1; \frac{u_2 + u_3 - 1}{u_2}\right)H(1, 0; u_3) - \frac{1}{2}H(1, 0; u_1)H(1, 0; u_3) - \frac{1}{2}H(1, 0; u_2)H(1, 0; u_3) - \frac{1}{2}H(1, 0; u_2)H(1, 0; u_3)$ $\frac{1}{\alpha_4}\pi^2 H(1, 1; u_1) + \frac{1}{\alpha_4}\pi^2 H(1, 1; u_2) + \frac{1}{\alpha_4}\pi^2 H(1, 1; u_3) + \frac{1}{\alpha_4}H(0; u_2) H(0, 0, 0; u_1) +$ $\frac{1}{2}H(0; u_3)H(0, 0, 0; u_2) + \frac{1}{2}H(0; u_1)H(0, 0, 0; u_3) - \frac{1}{2}H(0; u_2)H(0, 0, 1; \frac{u_1 + u_2 - 1}{u_1 - u_1})$ $\frac{1}{2}H(0; u_3)H(0, 0, 1; \frac{u_1 + u_2 - 1}{1}) - H(0; u_1)H(0, 0, 1; (u_1 + u_2)) H(0; u_2) H(0, 0, 1; (u_1 + u_2)) - \frac{1}{2} H(0; u_1) H(0, 0, 1; \frac{u_1 + u_3 - 1}{2})$



$ \begin{array}{l} G\left(\frac{1}{w_{0}},0,\frac{1}{w_{1}};1\right)+\frac{1}{2}G\left(\frac{1}{w_{1}},0,0,\frac{1}{w_{1}}+\frac{1}{w_{1}};1\right)+\frac{1}{2}G\left(\frac{1}{w_{0}},0,0,\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)-\frac{1}{4}G\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)-\frac{1}{4}G\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)-\frac{1}{4}G\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}},\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}},\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}},\frac{1}{w_{0}}+\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}},\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}},\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0}},0,\frac{1}{w_{0}};1\right)+\frac{1}{2}g\left(\frac{1}{w_{0$	$\begin{split} &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{1-u_{2}}, 0, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{1-u_{2}}, 1, 1\right) - \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{1-u_{2}}, \frac{1}{1-u_{2}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{u_{22}}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{22}}, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{22}}, 1\right) - \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{1-u_{2}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{21}}, 1\right) - \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{22}}, 0, 1\right) + \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{u_{1}-u_{1}}{u_{21}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{1}}, 1\right) - \frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{1}{u_{12}}, 0, 1\right) + \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, \frac{u_{1}-u_{1}}{u_{21}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, \frac{1}{u_{1}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, 0, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, 0, \frac{1}{u_{21}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, \frac{1}{u_{11}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{21}, 0, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, 0, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, \frac{1}{u_{21}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 0, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, 0, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 0, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{21}, 1, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) + \\ &\frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) + \frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(0, u_{22}, 1, \frac{1}{u_{22}}, 1\right) - \frac{1}{4} \mathcal{G}\left(1, \frac{1}{u_{22}}, 0, u_{22}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 0, \frac{1}{u_{22}}, 1\right) - \frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 0, u_{22}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 1, \frac{1}{u_{22}}, 1\right) - \frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 1, \frac{1}{u_{22}}}, 1\right) - \frac{1}{4} \mathcal{G}\left(\frac{1}{u_{22}}, 1\right) - \\ &\frac{1}{4} \mathcal{G}$	$\begin{split} &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{1}{1-u_1},\frac{1}{1-u_1},1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_2-1}{u_1+u_2-1},1;1\right) - \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{u_1-1}{u_1+u_2-1},\frac{1}{1-u_1},1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},\frac{0}{0},0;1\right) + \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,0;1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},0,\frac{1}{1-u_1},1\right) - \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,0;1\right) - \frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) - \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1\right) - \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1,1,1,1\right) + \frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1,1,1\right) - \\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_1},u_{123},1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1$	$ \begin{split} &\frac{1}{4} \mathcal{G} \left(\frac{1}{1-w_1}, \frac{y_{233}, \frac{1}{1-w_2}}{1-w_1}, \frac{1}{1-w_1}, 1$	
$\begin{split} &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},0,v_{231};1\right)H\left(0,u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{232};1\right)H\left(0,u_1\right)-\\ &\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},\frac{1}{1-u_3},v_{232};1\right)H\left(0,u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},u_1\right)H\left(0,u_1\right)-\\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},u_{231};1\right)H\left(0,u_1\right)+\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},\frac{u_1}{u_1},\frac{u_1}{u_2},1\right)H\left(0,u_1\right)+\\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},u_{231};1\right)H\left(0,u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},\frac{u_1}{u_1},\frac{u_1}{u_2},1\right)H\left(0,u_1\right)+\\ &\frac{1}{4}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},u_{231};1\right)H\left(0,u_1\right)+\frac{1}{2}\mathcal{G}\left(\frac{1}{1-u_3},v_{232},\frac{u_1}{u_1},\frac{u_1}{u_2},1\right)H\left(0,u_1\right)+\\ &\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_1},1\right)H\left(0,u_1\right)+\frac{1}{2}\mathcal{G}\left(v_{232},\frac{1}{1-u_3},v_{231};1\right)H\left(0,u_1\right)+\\ &\frac{1}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1\right)H\left(0,u_1\right)+\frac{1}{4}\mathcal{G}\left(v_{232},\frac{1}{1-u_3},v_{231};1\right)H\left(0,u_1\right)-\\ &\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1\right)H\left(0,u_1\right)-\frac{3}{4}\mathcal{G}\left(v_{232},\frac{1}{1-u_3},v_{231};1\right)H\left(0,u_1\right)-\\ &\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1\right)H\left(0,u_1\right)-\frac{3}{4}\mathcal{G}\left(v_{232},\frac{1}{1-u_3},v_{231};1\right)H\left(0,u_1\right)-\\ &\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1\right)H\left(0,u_1\right)-\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_3},v_{231};1\right)H\left(0,u_1\right)-\\ &\frac{3}{4}\mathcal{G}\left(v_{231},1,\frac{1}{1-u_2},1\right)H\left(0,u_2\right)-\frac{3}{4}\mathcal{G}\left(0,\frac{1}{u_1},\frac{1}{u_1},v_{231},1\right)H\left(0,u_2\right)-\\ &\frac{3}{4}\mathcal{G}\left(v_{231},\frac{1}{1-u_2},1\right)H\left(0,u_2\right)-\frac{3}{4}\mathcal{G}\left(0,\frac{1}{u_1},\frac{1}{u_1},v_{231},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_2},1\right)H\left(v_1,u_2\right)+\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_2}{u_1},\frac{u_2}{u_1},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_2},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)+\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_2}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)+\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_2},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)+\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)+\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0,u_2\right)-\\ &\frac{4}{4}\mathcal{G}\left(\frac{1}{u_1},\frac{u_1}{u_1},\frac{u_1}{u_1},1\right)H\left(0$	$ \begin{array}{l} \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_1}, v_{123}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_1}, v_{123}; 1 \right) \mathcal{H} \left(0, u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_2}, v_{231}; 1 \right) \mathcal{H} \left(0, u_2 \right) + \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_2}, v_{231}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_2}, v_{231}; 1 \right) \mathcal{H} \left(0, u_2 \right) + \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_2}, v_{231}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(0, u_{232}, \frac{1}{1-u_3}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \frac{1}{4} \mathcal{G} \left(0, u_{233}, \frac{u_2-1}{1-u_2-1}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(0, u_{232}, \frac{1}{u_2}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \frac{1}{4} \mathcal{G} \left(0, u_{233}, \frac{u_2-1}{1-u_2}; 1 \right) \mathcal{H} \left(0, u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(0, v_{233}, \frac{1}{1-u_1}; 1 \right) \mathcal{H} \left(0, u_2 \right) - \frac{1}{4} \mathcal{G} \left(0, v_{233}, \frac{1}{1-u_2}; 1 \right) \mathcal{H} \left(0, u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(0, v_{233}; \frac{1}{1-u_1}; 2; 1 \right) \mathcal{H} \left(0, v_2 \right) - \frac{1}{4} \mathcal{G} \left(0, v_{332}; \frac{1}{1-u_1}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, 0, v_{123}; 1 \right) \mathcal{H} \left(0, v_2 \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, 0, v_{123}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, u_{233}; 0; 1 \right) \mathcal{H} \left(0, v_2 \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, \frac{1}{1-u_1}; v_{233}; 0; 1 \right) \mathcal{H} \left(0, v_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, u_{233}; 0; 1 \right) \mathcal{H} \left(0, v_2 \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_1}, \frac{1}{1-u_1}; v_{233}; 0; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1-u_1}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 0; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1-u_1}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{u_2}, \frac{1}{u_2}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{v_{233}; 1 \right) \mathcal{H} \left(0, v_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_2}, \frac{1}{u_2}, \frac{1}{u_2}; 1 \right) \mathcal{H} \left(0, v_2 \right) - \frac{1}{4$	$ \begin{array}{l} \frac{1}{4} \mathcal{G} \left(v_{122}, 1, \frac{1}{1-u_1}, 1 \right) \mathcal{H} \left(0; u_2 \right) - \frac{1}{4} \mathcal{G} \left(v_{123}, \frac{1}{1-u_2}, 1; 1 \right) \mathcal{H} \left(0; u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(v_{23}, 1, \frac{1}{1-u_2}, 1 \right) \mathcal{H} \left(0; u_2 \right) + \frac{1}{4} \mathcal{G} \left(v_{233}, \frac{1}{1-u_2}, 1; 1 \right) \mathcal{H} \left(0; u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(v_{23}, 1, \frac{1}{1-u_2}, 1 \right) \mathcal{H} \left(0; u_2 \right) + \frac{1}{4} \mathcal{G} \left(v_{233}, \frac{1}{1-u_2}, 1; 1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(v_{233}, 1, \frac{1}{1-u_2}, 1 \right) \mathcal{H} \left(0; u_2 \right) - \frac{1}{4} \mathcal{G} \left(v_{233}, \frac{1}{1-u_2}, 1; 1 \right) \mathcal{H} \left(0; u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(v_{233}, 1, \frac{1}{1-u_3}, 1 \right) \mathcal{H} \left(0; u_2 \right) + \frac{1}{4} \mathcal{G} \left(v_{233}, \frac{1}{1-u_2}, 1; 1 \right) \mathcal{H} \left(0; u_2 \right) + \\ \frac{1}{4} \mathcal{G} \left(v_{233}, 1, \frac{1}{1-u_3}, 1 \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_1 \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{u_3}, \frac{1}{u_1+u_2}, 1 \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \\ \frac{1}{4} \mathcal{G} \left(v_{233}, \frac{1}{u_1+u_2}, 1 \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_1 \right) \mathcal{H} \left(0; u_2 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_3}, \frac{u_{233}}{u_{233}, 1} \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{u_1}, \frac{u_{233}}{u_2}, \frac{u_{233}}{u_1}, 1 \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left(\frac{1}{u_1}, \frac{u_{233}}{u_2}, \frac{u_{233}}{u_1}, 1 \right) \mathcal{H} \left(0; u_3 \right) - \\ \frac{1}{4} \mathcal{G} \left($	$ \begin{array}{l} \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_{0}}, v_{312:1} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(0, \frac{1}{1-u_{0}}, v_{321:1} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(0, u_{23,1}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{1-u_{1}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{u_{23}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{u_{23}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{4} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{u_{23}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{u_{23}} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{1}}, \frac{1}{u_{23}} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{2}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(0, u_{0} \right) + \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{2}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(0, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(1, u_{0} \right) \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(1, u_{0} \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}}, \frac{1}{u_{23}} \left(u_{23} \right) \mathcal{H} \left(1, u_{0} \right) \right) - \frac{1}{2} \mathcal{G} \left(\frac{1}{1-u_{23}$	
$\begin{split} &\frac{1}{2}H\left(0;u_2\right)H\left(0,0,1;\frac{u_1+u_2-1}{u_1-1}\right)-H\left(0;u_1\right)H\left(0,0,1;\left(u_1+u_3\right)\right)-\\ &H\left(0;u_3\right)H\left(0,0,1;\frac{u_2+u_2-1}{u_1-1}\right)-\frac{1}{2}H\left(0;u_1\right)H\left(0,0,1;\frac{u_2+u_2-1}{u_2-1}\right)-\\ &\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_2+u_3-1}{u_1-1}\right)-H\left(0;u_2\right)H\left(0,0,1;\left(u_2+u_3\right)\right)-\\ &H\left(0;u_3\right)H\left(0,0,1;\frac{u_2+u_3-1}{u_2-1}\right)-\frac{1}{2}H\left(0;u_3\right)H\left(0,1,1;u_1+u_2-1\right)-\\ &\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_2+u_3-1}{u_3-1}\right)-\frac{1}{2}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_2-1}\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)+\frac{1}{4}H\left(0;u_1\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)+\frac{1}{2}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)+\frac{1}{2}H\left(0;u_3\right)H\left(0,0,1;\frac{u_3+u_3-1}{u_3-1}\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(0,1,1;\frac{u_3+u_3-1}{u_3-1}\right)-\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\\ &\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\\ &\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{2}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)+\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0;u_3\right)+\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0;u_3\right)H\left(1,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{3}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{3}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,1,0;u_3\right)+\\ &\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{1}{4}H\left(0,0,0;u_3\right)-\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{1}{4}H\left(0,0,0;u_3\right)-\\ &\frac{1}{4}H\left(0,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+H\left(1,0,0;u_3\right)+\\ &\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\\ &\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,0;u_3\right)+\frac{1}{4}H\left(1,0,$	$ \begin{split} \mathbf{S} & \frac{1}{24} \pi^2 H\left(0;u_3\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) - \frac{1}{24} \pi^2 H\left(0;u_1\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) + \frac{1}{4} \pi^2 H\left(0;u_3\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) - \frac{1}{3} \pi^2 H\left(0;u_1\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) + \frac{1}{3} \pi^2 H\left(0;u_1\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(1;\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_2\right) \mathcal{H}\left(0;\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(0;u_1\right) \mathcal{H}\left(0;u_1,\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(0;u_1\right) \mathcal{H}\left(0;u_1,\frac{1}{u_{332}}\right) + \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(0;u_1\right) \mathcal{H}\left(0;u_1,\frac{1}{u_{332}}\right) - \frac{1}{4} H\left(0;u_1\right) \mathcal{H}\left(0;u_1,\frac{1}$	$\begin{split} &\frac{1}{4}H\left(0,u_{2}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{2}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{v_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(0,1,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)+\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,0,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}\right)\mathcal{H}\left(1,1,1;\frac{1}{u_{22}}\right)-\frac{1}{4}H\left(0,u_{3}$	 ¹/₄ H (1,1,0,1; ¹/_{1,20}) + ²/₂ H (1,1,1; ¹/_{1,22}) + ²/₂ H (1,1,1; ¹/_{1,21}) + ²/₂ H (1,1,1; ¹/_{1,22}) 1 Arastasion, J. Bern, I. J. Daron and D. A. Kosover, "Planar amplitudes in maximally subsymmetric Yang-Mills theory," Phys. Rev. Lett. 91 (2003) 251002 (2007) (20	

Sut what about after regularization and *loop integration*? What is the *mathematical form* of the predictions made by QFT?



Classical Polylogarithms for Amplitudes and Wilson Loops

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We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions Li_k with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R(u_1, u_2, u_3) = \sum_{i=1}^{3} \left(L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \\ - \frac{1}{8} \left(\sum_{i=1}^{3} \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 \left(J^2 + \zeta_2 \right)$$



But what about after regularization and loop integration? What is the *mathematical form* of the predictions made by QFT?





<u>State of the art:</u> 6-point (N)MHV @ (6) 7 loops(!!!) 7-point (N)MHV @ 4 loops (symbol-level)

[Dixon, et al (2019);...]

Amplitudes: a Virtuous Cycle

Compute Something beyond the reach of recent imagination

Exploit Simplicity to build more powerful computational technology

study it, understand it, & explore consequences

Discover Simplicity **beyond** expectations

Understand Why



What Form do Ob• In a general (say, 4d) QFT, it was by "experts" that observables
$$\mathcal{A} = \mathcal{A}^{\text{tree}} + \hbar \mathcal{A}^{(L=1)} + \hbar^2 \mathcal{A}$$
rational + $\begin{pmatrix} \text{weight-2} \\ \text{polylogs} \end{pmatrix} + \begin{pmatrix} \text{weight-1} \\ \text{polylogs} \end{pmatrix} + \begin{pmatrix} \text{wei$

servables Take?

- would have *recently been* expected s took the following general form: (general dimension $d: 2 \mapsto \lfloor d/2 \rfloor$) $\mathcal{A}^{(L=2)} + \ldots + \hbar^L \mathcal{A}^{(L)} + \ldots$
- $\left(\begin{array}{c} \text{ight-4} \\ \text{ylogs} \end{array} \right) + \cdots + \left(\begin{array}{c} \text{weight-2L} \\ \text{polylogs} \end{array} \right) + \cdots$

- + ••• + ght-1 ylogs
- onal







Unfortunately, many pesky counterexamples were to be found: [Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]







contributes to electron (g-2)

. . .

Doran, Harder, Thompson (2019)









an 8-loop vacuum graph evaluating to a K3 period

Unfortunately, many pesky counterexamples were to be found: [Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]

. . .



Doran, Harder, Thompson (2019)

2d, masive

>4d, massless

Brown, Schnetz (2011)









Outside a *small* list of **extremely** limited/simple cases are expected to be non-polylogarithmic

CY_{2(L-1)}









Why is Perturbation Theory so Hard? • Feynman diagrams (esp. with *scalar* numerators) are *horrible* difficult to integrate, explosive in number, non-physical,...

• Regularization obscures symmetries (+is technically difficult)

 Most familiar mater integrand bases are unnecessarily bad: don't satisfy nice / canonical differential equations contain multiple elliptic(+worse(!)) geometries, ner Tracts in Mindeen Physics 35 Andimir A: Smirnov

 $\mathbf{I}_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} = e^{2\gamma_E \epsilon} \iint \frac{d^D k_1 d^D}{(i\pi^{d/2})^2}$

$$\frac{P_{k_{2}}}{P_{2}^{a_{1}}P_{2}^{a_{2}}P_{3}^{a_{3}}P_{4}^{a_{4}}P_{5}^{a_{5}}P_{6}^{a_{6}}P_{7}^{a_{7}}}$$

Analytic Tools for Feynman Integrals



- Use unitarity to choose the *nicest / easiest* integrals to integrate (*of course*, integration "ease" changes with time and new methods)
 - search for as many *pure* integrals as you can

 —those which satisfy nice (canonical) differential equations
- **Definition:** a function *f*(*s*) is called *pure* if:
 there exists a grading of functions by "transcendental" *weight*any derivative of *f*(*s*) *is strictly lower in weight*



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e.g.
$$g(s) \log(f(s))$$
 with $\frac{\partial}{\partial s} \left[g(s) \log(f(s)) \right] = g'(s)$

ould be *impure*





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 $\frac{s_{13}}{s_{12}s_{23}s_{56}} \left(\begin{array}{c} \mathrm{tr}_{+}[p_{3}, p_{12}, p_{6}, p_{1}](\mathrm{Li}_{4}(\cdots)'\mathrm{s} + \dots) \\ +\mathrm{tr}_{+}[p_{12}, p_{6}, p_{1}, p_{3}](\mathrm{Li}_{4}(\cdots)'\mathrm{s} + \dots) \end{array} \right)$



- + Use unitarity to choose the *nicest/easiest* integrals to integrate (of course, integration "ease" changes with time and new methods)
 - search for as many *pure* integrals as you can —those which satisfy nice (canonical) differential equations

Avoid regularization whenever possible:



- can all(?) finite quantities be computed without regularization? -without expanding them in terms of divergent integrals? (Answer: sometimes)
 - [**JB**, Langer, Patatoukos (2021); ...]
 - with numerator `1', this integrates to a sum of (an *impure* combination of) polylogarithms & elliptic-polylogarithms involving 4 elliptic curves



Unitarity-Based Strategies: a modern perspective

Generalized Unitarity: a modern take

The basic idea behind unitarity-based methods is that any *Feynman integrand* is a *rational differential form on loop momenta*as such, it can be expanded into a basis 3 of such forms:

For any fixed QFT (spacetime dimension, particle content), the space of all amplitude integrands is finite-dimensional *all-multiplicity amplitudes* can be expressed in a *finite basis*!
Key observation: viewed as a potential element of *some* basis, *every* Feynman integrand can be interesting!
Why not try to find the *best/easiest* integrands—and use these?

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$



Stratifying Quantum Field Theories

• QFTs can be partially ordered by the scope of the basis required to represent their amplitudes

 $[Standard Model] > [(Standard Model \Higgs)] > [QCD] > [Yang-Mills]$ $[Yang-Mills] > [\mathcal{N} = 2 \text{ super-Yang-Mills}] > [\mathcal{N} = 4 \text{ super-Yang-Mills}]$ $[\mathcal{N}=4 \text{ Yang-Mills}] > [\text{planar } \mathcal{N}=4 \text{ super-Yang-Mills}] > \cdots > [\text{fishnet theory}]$

This reflects UV behavior ("power-counting") of theories; it suggests a possible stratification of integrand bases



$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

 $\mathfrak{B}^{\mathrm{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$

 Suppose that a basis could be carved up into subspaces (by any arbitrary means):



Is it possible to stratify integrand bases by physical structure? $\{\text{finite}\} \oplus \{\text{divergent}\}$

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

with
$$\mathcal{A}_p \coloneqq \mathcal{A} \bigcap \mathfrak{B}_p \coloneqq \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \mathfrak{b}_p^i$$

JB, Langer, Zhang, (2021) [JB, Herrmann, Langer, Patatoukos, et al (2021)]



 Suppose that a basis could be carved up into subspaces (by any arbitrary means):



 $\{\text{finite}\} \oplus \{(\text{UV-divergent})\} \oplus \{(\text{IR-divergent})\}$

$$\sum_{i \in \mathfrak{B}} a_i \mathfrak{b}^{a}$$

 $\mathcal{A} =$

with
$$\mathcal{A}_p \coloneqq \mathcal{A} \bigcap \mathfrak{B}_p \coloneqq \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \mathfrak{b}_p^i$$

Is it possible to stratify integrand bases by physical structure?

JB, Langer, Zhang, (2021) [JB, Herrmann, Langer, Patatoukos, et al (2021)]



 Suppose that a basis could be carved up into subspaces (by any arbitrary means):

 $\mathfrak{B} \eqqcolon \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \qquad \mathcal{A} = \sum_p \mathcal{A}_p \quad \mathbf{w}$

Is it possible to stratify integrand bases by physical structure? $\Big\{\text{finite}\Big\} \oplus \Big\{ \Big(\mathcal{O}(1/\epsilon^{2L}) \text{-divergent}\Big) \oplus \Big(\mathcal{O}(1/\epsilon^{2L-1}) \text{-divergent}\Big) \oplus \dots \oplus \Big(\mathcal{O}(1/\epsilon) \text{-divergent}\Big) \Big\}$ $\oplus \left\{ \left(\log(m)^{2L} \text{-divergent} \right) \oplus \left(\log(m)^{2L-1} \text{-divergent} \right) \oplus \dots \oplus \left(\log(m) \text{-divergent} \right) \right\}$ JB, Langer, Zhang, (2021) [JB, Herrmann, Langer, Patatoukos, et al (2021)]

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

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 Suppose that a basis could be carved up into subspaces (by any arbitrary means):

 $\mathfrak{B} \rightleftharpoons \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \qquad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p \coloneqq \mathcal{A} \bigcap \mathfrak{B}_p \coloneqq \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \mathfrak{b}_p^i$ Can we further stratify each part by transcendental structure? finite max-weight $\} \oplus \{$ next-to-max-weig ${\text{polylogs}} \oplus {\text{elliptic-polylogs}} \oplus {\text{K3-p}}$

$$\sum_{i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

 $\mathcal{A} =$



Prescriptive Integrand Bases $\mathcal{A} = \sum c_i \mathcal{I}_i^0$

with coefficients C_i determined by cuts: a spanning set of cycles $\{\Omega_i\}$

 $\oint_{\Omega_i} \mathcal{I}_i^0 \eqqcolon \mathbf{M}_{i,j}$

 $a_j \coloneqq \oint_{\Omega_j} \mathcal{A} = \sum_i c_i \mathbf{M}_{i,j} \implies c_j = \sum_i a_i (\mathbf{M}^{-1})_{i,j}$



How generalized unitarity has been used to match amplitudes:



Prescriptive Integrand Bases $\mathcal{A} = \sum c_i \mathcal{I}_i^0$

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Prescriptive Integrand Bases $\mathcal{A} = \sum c_i \mathcal{I}_i^0$





How generalized unitarity has been used to match amplitudes:

- with coefficients C_i determined by cuts: a spanning set of cycles $\{\Omega_j\}$
 - $\left\{ \sum_{i=1}^{n} -\sum_{i=1}^{n} \left\{ i \right\} \right\}$





A basis is called *prescriptive* if it is the **cohomological dual** of a spanning set of **cycles** $\{\Omega_j\}$



Strategies for Building Bases

 Given some integrand basis (or strata thereof), one should diagonalize the space of integrands according to a homological/cohomological pairing:

• choose a *spunning-set* of compute, • normalize and diagonalize the basis by the requirement $\int \mathfrak{b}_i = \delta_{ij}$ $\Omega_j: 4L$ -dimensional compact contours $\begin{cases} \text{"residues"} \\ \text{elliptic periods} \\ \text{K3 periods, etc.} \end{cases}$

This trivializes the representation of amplitudes:

• the coefficient of any amplitude in this basis will simply be the on-shell function evaluated on the contour (a leading singularity) Choosing a maximal set of IR/UV-divergence-probing contours



- choose a spanning-set of compact, max-dimensional contours Ω_i

ensures(?) that the basis is split into finite / divergent subspaces



Amplitudes: a Virtuous Cycle



Compute Something beyond the reach of recent imagination

Exploit Simplicity to build more powerful computational technology



study it, understand it, & explore consequences



Discover Simplicity **beyond** expectations

Understand Why



Today's Revolution in QFT Defining an ongoing revolution in science: when all the textbooks of a field become obsolete Experts who need QFT no longer use textbook tools













Diagonalization of Rigidity Consider the following sets of pentabox integrands

#ints	<pre># polylogs</pre>	# elliptics	#impure/mixed
1	0	0	1
6	0	4 (4)	2
36	10	12 (12)	14
120	76	44 (22)	0



pure

[JB, Kalyanapuram (2022)]



