



PennState

# *Adventures in Perturbation Theory*

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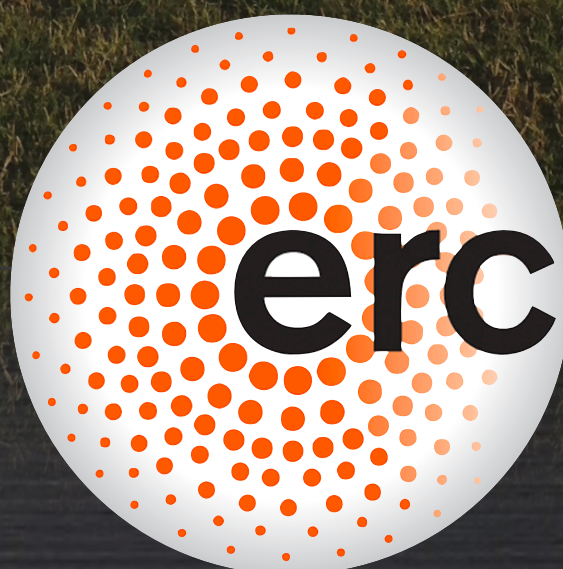
Niels Bohr International Academy, University of Københavns

*Current Themes in High Energy Physics and Cosmology*



The Niels Bohr  
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# What Form do Observables Take?

- ◆ What is the *mathematical form* of the predictions made by QFT?  
(perturbatively, say?)

$$g_e = 2 + \frac{\alpha}{\pi} (1) = \text{tree-level} + \text{one-loop} + \text{two-loop} + \dots$$

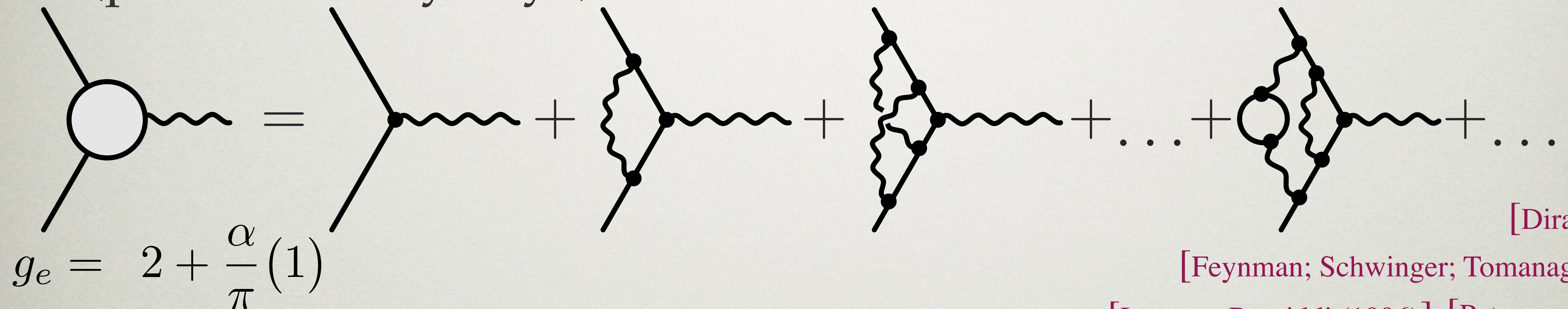
[Dirac (1933)]

[Feynman; Schwinger; Tomanaga (1947)]

# What Form do Observables Take?



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[Dirac (1933)]

[Feynman; Schwinger; Tomonaga (1947)]

[Laporta, Remiddi (1996)] [Petermann (1957)]

[Laporta (2017)] [Kinoshita (1990)]

$$+ \frac{\alpha^2}{\pi^2} \left( \frac{3}{2} \zeta_3 - \pi^2 \log(2) + \zeta_2 + \frac{197}{72} \right)$$

\* ( $u_{3,1}$  is related to  $\text{Li}_4(1/2)$  and  $\log(2)$ )

$$- \frac{\alpha^3}{\pi^3} \left( \frac{215}{12} \zeta_5 + \frac{100}{3} u_{3,1} + \frac{13}{4} \zeta_4 - \frac{139}{9} \zeta_3 + \frac{1192}{3} \zeta_2 \log(2) - \frac{34202}{135} \zeta_2 - \frac{28259}{2592} \right)$$

beyond three loops, no analytic expression is known  
(CY periods appear)

# What Form do Observables Take?



- ◆ What is the *mathematical form* of the predictions made by QFT?  
(perturbatively, say?)

*e.g.* maximally supersymmetric ( $\mathcal{N}=4$ ) Yang-Mills theory (planar limit)

$$\begin{aligned}\gamma_{\text{cusp}} = & a \times 1 \\ & + a^2 \times 2 \zeta_2 \\ & + a^3 \times 22 \zeta_4 \\ & + a^4 \times 2(24 \zeta_2^3 + 4 \zeta_3^2 + 2 \zeta_2 \zeta_4 + \zeta_6) \\ & + a^5 \times 8(252 \zeta_4^2 + 20 \zeta_3 \zeta_5 + 4 \zeta_2 \zeta_3^2 + \zeta_2 \zeta_6) \\ & + a^6 \times 8(282 \zeta_2^5 + \zeta_2^3 \zeta_4 + 4 \zeta_2 \zeta_4^2 + 80 \zeta_2 \zeta_3 \zeta_5 + 5 \zeta_2^2 \zeta_6 + 48 \zeta_3^2 \zeta_4 + 102 \zeta_5^2 + 210 \zeta_3 \zeta_7 + 3 \zeta_{10}) \\ & + \dots\end{aligned}$$

[Beisert, Eden, Staudacher (2007);...]

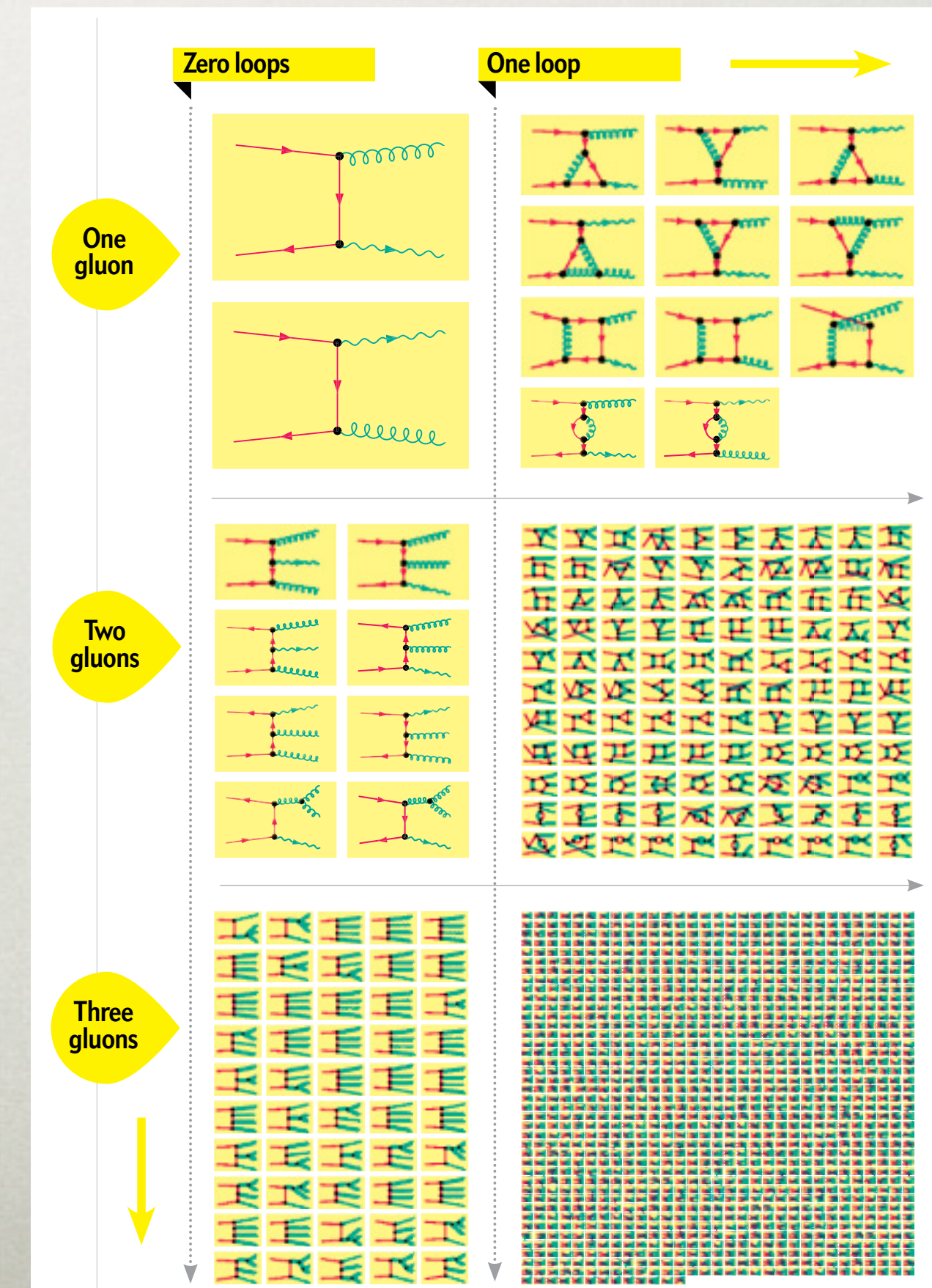
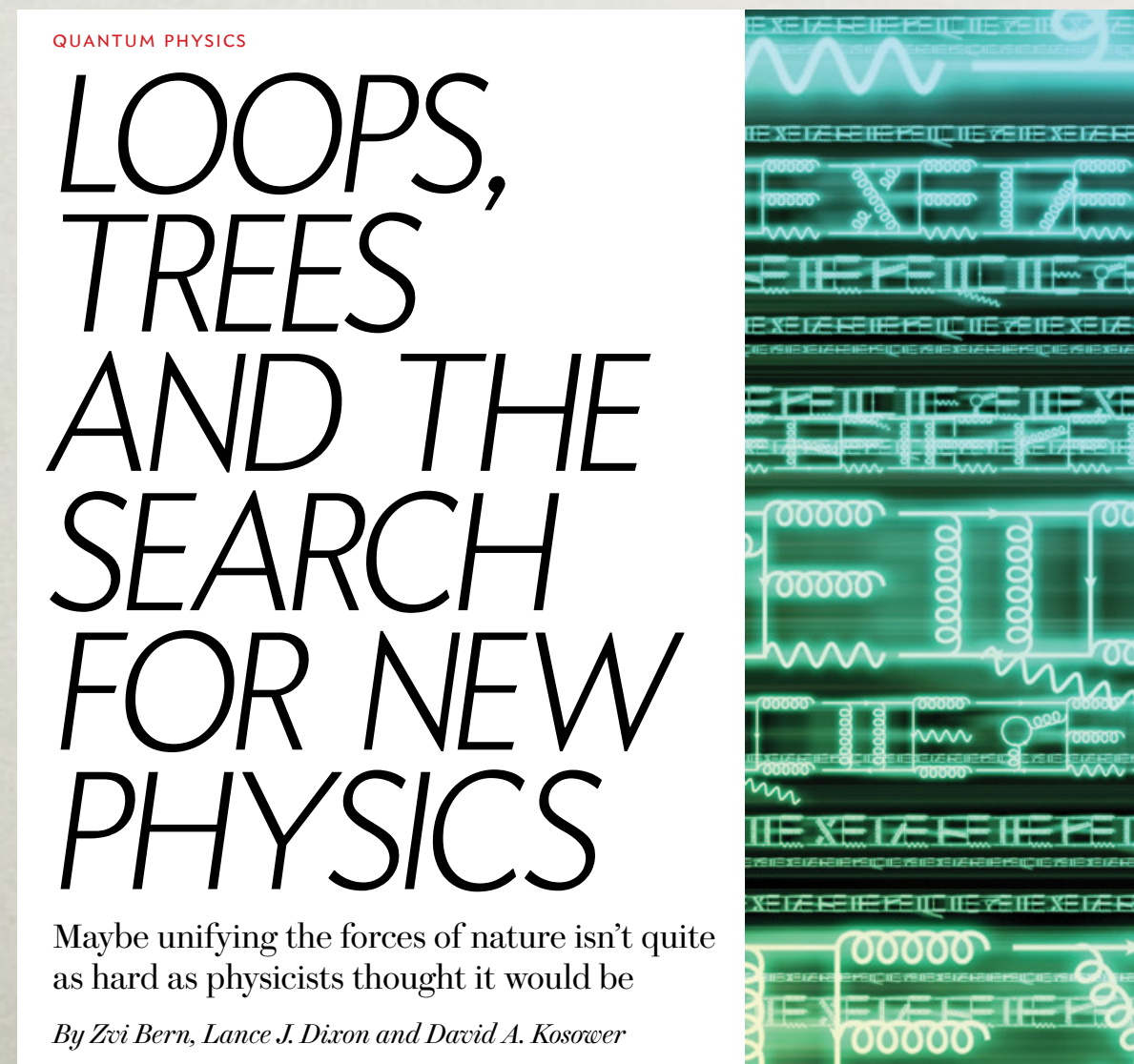
$\Rightarrow$  “maximal *transcendentality*” of planar  $\mathcal{N}=4$  super Yang-Mills (?)



# Explosions of Complexity

- ◆ While ultimately correct, the Feynman expansion renders *all but the most trivial* predictions—  
involving the **fewest particles**, at the **lowest orders** of perturbation—

or **computationally intractable**  
**theoretically inscrutable**



[Bern, Dixon, Kosower, *Scientific American* (2012)]



# Needs (Once) Beyond Our Reach

- ◆ Background amplitudes crucial for *e.g.* colliders

## Supercollider physics [Rev.Mod.Phys. 56 (1984)]

E. Eichten

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

I. Hinchliffe

Lawrence Berkeley Laboratory, Berkeley, California 94720

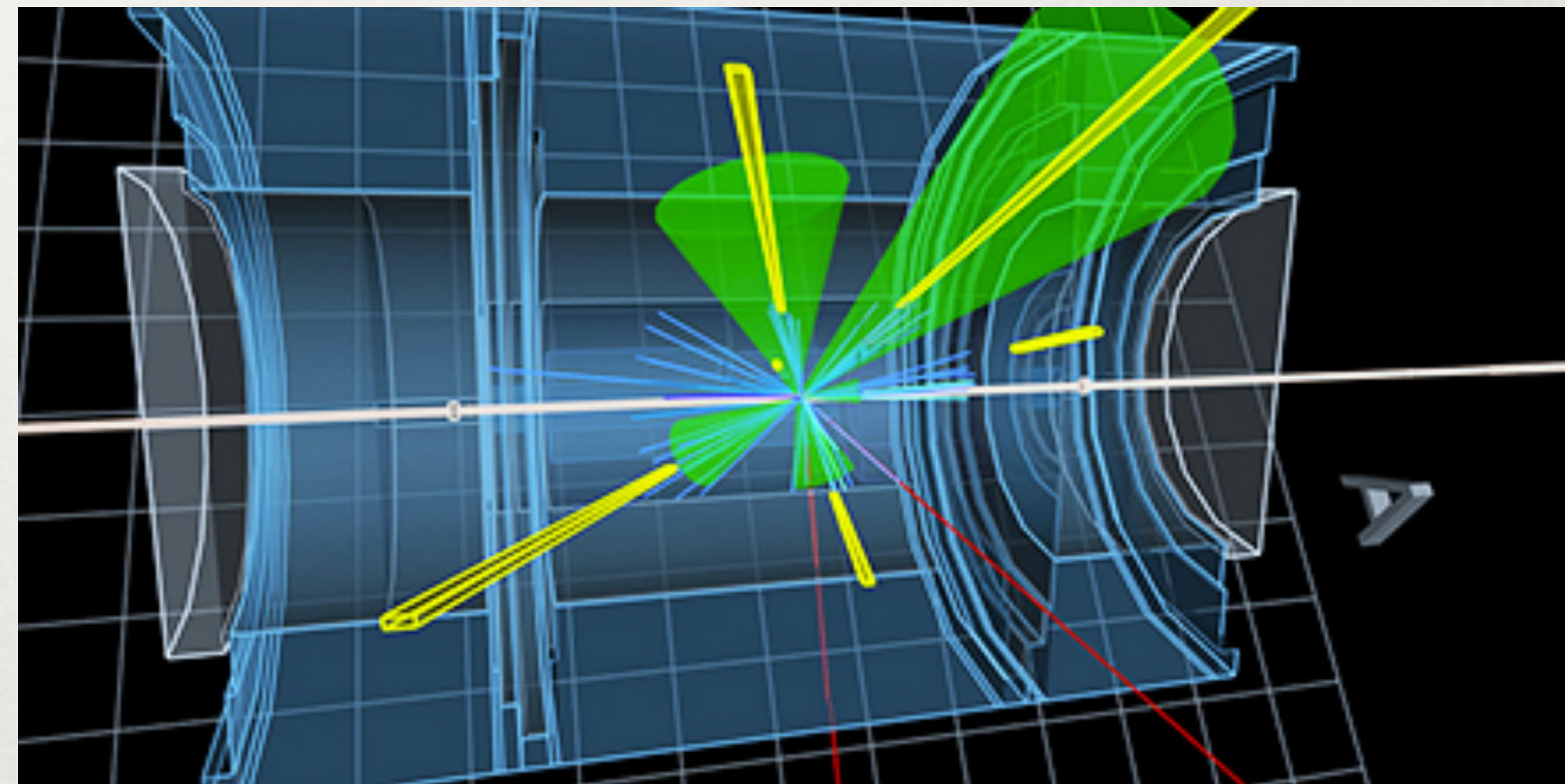
K. Lane

The Ohio State University, Columbus, Ohio 43210

C. Quigg

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510

Eichten *et al.* summarize the motivation for exploring the 1-TeV ( $=10^{12}$  eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



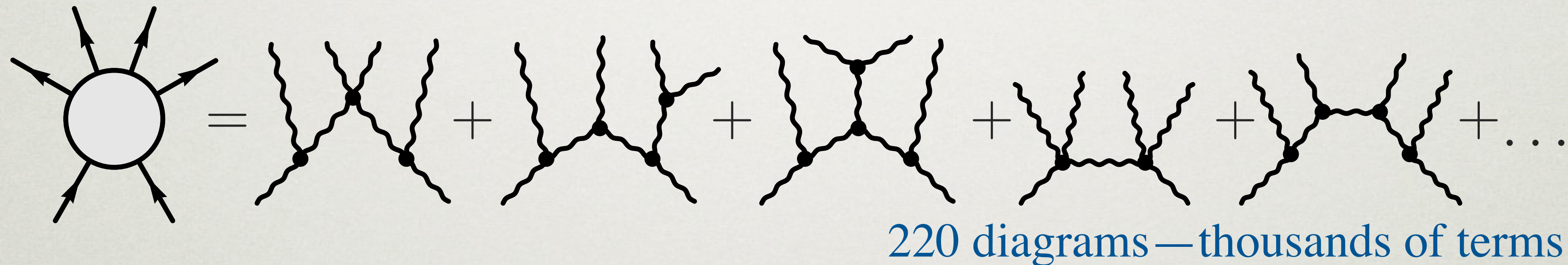
- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of  $W^+W^-$  pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



# *Needs (Once) Beyond Our Reach*

- ◆ Background amplitudes **crucial** for *e.g.* colliders



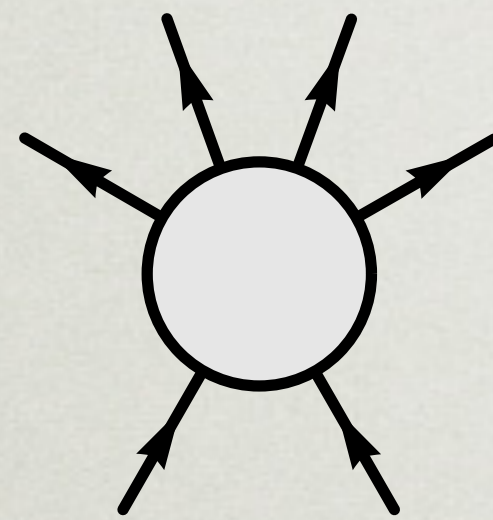
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# Needs (*Once*) Beyond Our Reach

- ◆ Background amplitudes crucial for *e.g.* colliders



## THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON-GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

*Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA*

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys.* B269 (1985)]

- ◆ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of  $W^+W^-$  pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



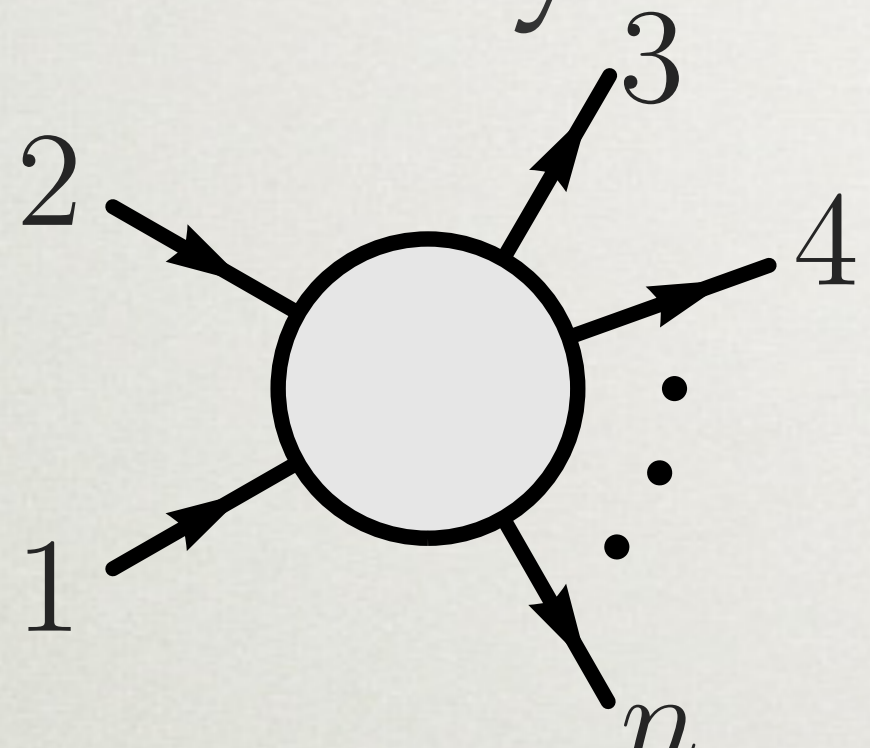






# Discovery of Shocking Simplicity

- ◆ Within six months, Parke-Taylor stumbled on a simple guess —unquestionably a *theorist's delight*:



$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

**Amplitude for  $n$ -Gluon Scattering [PRL 56 (1986)]**

Stephen J. Parke and T. R. Taylor  
*Fermi National Accelerator Laboratory, Batavia, Illinois 60510*  
 (Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^\mu \equiv \sigma_{\alpha\dot{\alpha}}^\mu \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$$

$$\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$$

$$[a b] \equiv \det(\tilde{\lambda}_a, \tilde{\lambda}_b)$$

[van der Waerden (1929)]

# *Perturbations of Parke/Taylor's Guess*



- ◆ What about beyond the leading order of approximation?

[Bern, Dixon, Dunbar, Kosower (1994)]

$$\begin{aligned}
 & \left( \text{Diagram: Circle with } n \text{ external legs } 1, 2, 3, 4, \dots, n \right) = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \dots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \left( \text{Diagram: Box with legs } a, b \text{ and a wavy line} \right) + \dots \right\}
 \end{aligned}$$

# Perturbations of Parke/Taylor's Guess



## ◆ What about beyond the leading order of approximation?

[Vergu (2009)]

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complexity of the computations. It has also been useful to use the results for the cuts already computed when computing the coefficients of integrals detected by new cuts. In this way, one can linearize the consistency of results from different cuts and reduce the number of unknowns at the same time.

Let us make a further comment about our computation procedure. The conformal integrals with pentagon loops have numerators containing the loop momenta in combinations like  $(k+l)^2$ , where  $l$  is the loop momentum and  $k$  is an external on-shell momentum. If the propagator with momentum  $l$  is cut then, on that cut, one cannot distinguish between  $(k+l)^2$  and  $2k \cdot l$ . However, it is easy to see that one can choose to cut another propagator and in that case this ambiguity does not arise and the numerator factor is uniquely defined.

**IV. RESULTS**

We use dual variable notation (see Ref. [48]) for the integrals. The external dual variables are listed in clockwise direction. To the left loop we associate the dual variable  $x_p$  and to the right loop we associate the dual variable  $x_q$ . We use the notation  $x_{ij} = x_i - x_j$ . We introduce the following notation which will be useful in the following

$$\left[ \begin{matrix} a & b & c & \dots \\ a' & b' & c' & \dots \end{matrix} \right] = x_{a'b'} x_{b'c'} \dots \pm (\text{permutations of } \{a', b', c', \dots\}). \quad (6)$$

The sign  $\pm$  above takes into account the signature of the permutation of  $\{a', b', c', \dots\}$ . It is easy to show that

$$\left[ \begin{matrix} a & b & c & \dots \\ a' & b' & c' & \dots \end{matrix} \right] = \frac{\det(x_{ij}^{(a,b,c,\dots)}; x_{i'j'}^{(a',b',c',\dots)})}{x_{i'j'}^{(a,b,c,\dots)}}. \quad (7)$$

For some topologies, the expansion of the  $\left[ \begin{matrix} a & b & c & \dots \\ a' & b' & c' & \dots \end{matrix} \right]$  symbol yields terms that would cancel propagators. For those cases we make the convention that all the terms that would cancel propagators are absent. In fact, as we will see, terms that would cancel propagators of the double pentagon topologies naturally yield coefficients for some of the topologies with a smaller number of propagators.

**A. Double box topologies**

In the case of the double box topologies the massive legs attached to the vertices incident with the common edge have to be a sum of at least three massless momenta. The cases where these massive legs are the sum of two massless momenta are treated separately in the subsection. IV A 7. This distinction only arises for the double box topologies.

**1. No legs attached**

$$\frac{1}{2} (x_{2,3}^2)^2 x_{2,3}^2 \quad (8)$$

$$\frac{1}{2} (x_{2,3}^2)^2 x_{2,3}^2 \quad (9)$$

$$-\frac{1}{2} x_{2,3}^2 (x_{2,3}^2 - x_{2,3}^2 - x_{2,3}^2)^2 \quad (10)$$

**2. One massless leg attached**

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) x_{2,3}^2 \quad (11)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (12)$$

$$-\frac{1}{4} x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 \quad (13)$$

$$-\frac{1}{4} x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 \quad (14)$$

**3. Two massless legs attached**

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (15)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (16)$$

$$\frac{1}{4} (-x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (17)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - 2x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (18)$$

**4. One massive leg attached**

$$\frac{1}{4} x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 \quad (19)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (20)$$

$$0 \quad (21)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (22)$$

**5. One massless leg and one massive leg attached**

$$0 \quad (23)$$

$$0 \quad (24)$$

$$-\frac{1}{4} x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 \quad (25)$$

$$0 \quad (26)$$

$$\frac{1}{4} x_{2,3}^2 (x_{2,3}^2 - x_{2,3}^2 - x_{2,3}^2) \quad (27)$$

$$\frac{1}{4} (-x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (28)$$

$$0 \quad (29)$$

$$0 \quad (30)$$

$$0 \quad (31)$$

**7. Extra double boxes**

$$0 \quad (32)$$

$$0 \quad (33)$$

$$0 \quad (34)$$

$$\frac{1}{4} (-x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (35)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & b-1 & b \\ a+2 & b-1 & a-1 \end{matrix} \right] \quad (36)$$

$$0 \quad (37)$$

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & a+2 \\ a+2 & a+3 & a-2 \end{matrix} \right] \quad (38)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) x_{2,3}^2 \quad (39)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & b-1 & b \\ b+1 & b+2 & a-1 \end{matrix} \right] \quad (40)$$

$$0 \quad (41)$$

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & a+2 \\ a+3 & a+4 & a-2 \end{matrix} \right] \quad (42)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) x_{2,3}^2 \quad (43)$$

$$-\frac{1}{4} \left[ \begin{matrix} a-1 & a & a+1 \\ a+3 & a+4 & a-2 \end{matrix} \right] \quad (44)$$

$$0 \quad (45)$$

$$0 \quad (46)$$

$$-\frac{1}{2} \left[ \begin{matrix} 2 & 3 & 4 \\ 6 & 7 & 8 \end{matrix} \right] \quad (47)$$

$$0 \quad (48)$$

$$-\frac{1}{4} \left[ \begin{matrix} a-2 & a-1 & a \\ a+2 & b-1 & b \end{matrix} \right] \quad (49)$$

$$-\frac{1}{4} \left[ \begin{matrix} a-3 & a-2 & a-1 \\ a+1 & a+2 & a+3 \end{matrix} \right] \quad (50)$$

$$0 \quad (51)$$

$$0 \quad (52)$$

**B. Kissing double-box topologies**

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 \\ b+1 & c-1 & c \end{matrix} \right] \quad (53)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & b-1 & b \\ b+1 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} b & b+1 \\ a-1 & a \end{matrix} \right] - \frac{1}{4} (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2)^2 + x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 x_{2,3}^2 \quad (54)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a-1 & a \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} a+1 & a+2 \\ b-1 & b \end{matrix} \right] \quad (55)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a+1 & a+2 \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} b+1 & b+2 \\ a-1 & a \end{matrix} \right] \quad (56)$$

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} b & b+1 \\ c-1 & c \end{matrix} \right] \quad (57)$$

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & c-1 & c \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} b+1 & b+2 \\ c-1 & c \end{matrix} \right] \quad (58)$$

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & b-1 & b \\ c+1 & d-1 & d \end{matrix} \right] + \frac{1}{4} \left[ \begin{matrix} a & a+1 \\ b-1 & b \end{matrix} \right] \left[ \begin{matrix} c & c+1 \\ d-1 & d \end{matrix} \right] \quad (59)$$

**C. Box-Pentagon topologies**

**1. No legs attached**

$$\frac{1}{2} x_{2,3}^2 x_{2,3}^2 (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (60)$$

$$\frac{1}{2} x_{2,3}^2 x_{2,3}^2 (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (61)$$

**2. One massless leg attached**

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (62)$$

$$\frac{1}{4} x_{2,3}^2 (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (63)$$

**3. One massive leg attached**

$$0 \quad (65)$$

$$\frac{1}{4} (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) (x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (66)$$

$$\frac{1}{4} x_{2,3}^2 (x_{2,3}^2 x_{2,3}^2 + x_{2,3}^2 x_{2,3}^2 - x_{2,3}^2 x_{2,3}^2) \quad (67)$$

**4. One massless, one massive leg attached**

$$0 \quad (68)$$

$$\frac{1}{4} \left[ \begin{matrix} a & a+1 & b & b+1 \\ b+2 & c-1 & c & q \end{matrix} \right] \quad (69)$$

Note that in the previous formula we suppress the terms containing  $x_{2,3}^2$  which would otherwise cancel a propagator of the underlying topology. When expanded out, the expression above has 12 terms.

$$-\frac{1}{4} \left[ \begin{matrix} a-2 & a-1 & a+1 \\ a+2 & b-1 & b & q \end{matrix} \right] \quad (70)$$

In the previous formula we suppress the terms containing  $x_{2,3}^2$  which would otherwise cancel a propagator of the underlying topology.

**5. Two massless legs attached**

$$\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & a-1 & q \end{matrix} \right] \quad (71)$$

In the previous formula we suppress the terms containing  $x_{2,3}^2$  which would otherwise cancel a propagator of the underlying topology.

**D. Double pentagon topologies**

**1. No legs attached**

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & a-1 & q \end{matrix} \right] \quad (73)$$

In the expansion of the above formula we drop terms that would cancel propagators (in this case, the terms containing  $x_{2,3}^2$ ,  $x_{2,3}^2$ , or  $x_{2,3}^2$ ). This expression has 6 terms when expanded.

**2. One massless leg attached**

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & a+2 & b-1 & b \\ b & b+1 & a-1 & a & q \end{matrix} \right] \quad (74)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms are  $x_{2,3}^2$ ,  $x_{2,3}^2$ , or  $x_{2,3}^2$ ). This expression has 15 terms when expanded.

**3. One massive leg attached**

$$\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b & b+1 & c-1 & c & q \end{matrix} \right] \quad (75)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing  $x_{2,3}^2$ ,  $x_{2,3}^2$ , or  $x_{2,3}^2$ ). This expression has 16 terms when expanded.

**4. Two massless legs attached**

$$-\frac{1}{4} \left[ \begin{matrix} a+1 & a+2 & b-1 & b \\ b+1 & b+2 & a-1 & a & q \end{matrix} \right] \quad (76)$$

In the formula we drop terms that would cancel propagators (in this case, the terms containing  $x_{2,3}^2$ ). This expression has 64 terms when expanded.

**5. One massless, one massive leg attached**

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ b+1 & b+2 & a-1 & c & q \end{matrix} \right] \quad (77)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing  $x_{2,3}^2$ ). This expression has 78 terms when expanded.

**6. Two massive legs attached**

$$-\frac{1}{4} \left[ \begin{matrix} a & a+1 & b-1 & b \\ c & c+1 & d-1 & d & q \end{matrix} \right] \quad (78)$$

In the formula above we drop terms that would cancel propagators (in this case, the terms containing  $x_{2,3}^2$ ). When expanded, the above expression contains 96 terms. The number of conformal drawings is 160 (the number of coefficients unrelated by symmetries is lower).

**E. Assembly of the result**

As explained in Sec. II, for the MHV amplitudes the ratio between the  $l$ -loop amplitude and the tree-level amplitude can be written as a sum between parity even and parity odd contributions

$$M_n^{(l)} = M_n^{(l, \text{even})} + M_n^{(l, \text{odd})}. \quad (79)$$

Then, the even part can be written

$$M_n^{(l, \text{even})} = -\pi^{-D} \int d^D x \mu^2 \sum_{a \in \text{topologies}} \sum_{c_i \in I_i} a_i c_i. \quad (80)$$

where the first sum runs over cyclic and anti-cyclic permutations of the external legs, the second sum runs over all the topologies,  $a_i$  is a symmetry factor associated to topology  $i$ ,  $c_i$  is the numerator of the topology  $i$ , as listed in Sec. IV and  $I_i$  is the denominator or the product of propagators in the topology  $i$ .

Apart from the parity odd part which we have not computed, there is also a contribution which is not detectable from four-dimensional cuts, denoted by  $M_n^{(l, \text{odd})}$ . This part of the result is such that its integral vanishes in four dimensions, but the integral itself can give contributions to the divergent and finite parts. In Ref. [52], for  $n=6$  case, this part of the result was found to be closely related to  $\text{Cr}(c)$  contributions at one loop,  $M_n^{(l, \text{odd})}$ . Based on previous computations we expect that the odd part and the  $\mu$  integrals will not be needed in order to compare with the Wilson loop results. The odd parts could be

# *Perturbations of Parke/Taylor's Guess*



- ◆ What about beyond the leading order of approximation?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2010)]

$$= \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times$$

$$\left\{ 1 + \sum_{a < b} \text{[Diagram with wavy line]} + \sum_{a < b < c < d} \text{[Diagram with two wavy lines]} + \dots \right\}$$

# Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Arkani-Hamed, **JB**, Cachazo, Trnka (2011)]

$$\begin{aligned}
 & \text{Diagram: A circle with } n \text{ external legs labeled } 1, 2, 3, 4, \dots, n. \\
 & = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \dots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \text{Diagram 1} + \sum_{a < b < c < d} \text{Diagram 2} \right. \\
 & \quad + \sum_{\substack{a < b \leq c < \\ < d \leq e < f}} \text{Diagram 3} + \left. \sum_{\substack{a \leq b < c < \\ < d \leq e < f}} \text{Diagram 4} + \dots \right\}
 \end{aligned}$$

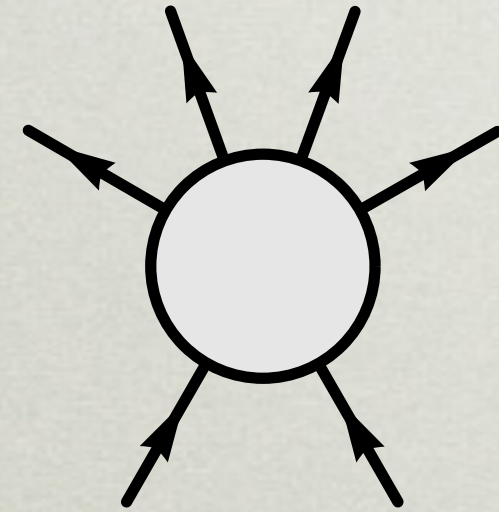
The diagrams are:
 

- Diagram 1: A square with vertices \$a\$ (top-left, white) and \$b\$ (bottom-right, black). A wavy line connects \$a\$ and \$b\$.
- Diagram 2: A hexagon with vertices \$a, b, c, d\$ (white) and two black vertices. A wavy line connects \$a\$ and \$b\$.
- Diagram 3: A more complex diagram with vertices \$a, b, c, d, e, f\$ (white) and black vertices. A wavy line connects \$a\$ and \$b\$.
- Diagram 4: A similar diagram to Diagram 3, with a wavy line connecting \$a\$ and \$b\$.

# What Form do Observables Take?



- ◆ But what about after **regularization** and *loop integration*?  
What is the *mathematical form* of the predictions made by QFT?



## The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

---

### Vittorio Del Duca

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[Del Duca, Duhr, Smirnov (2010)]

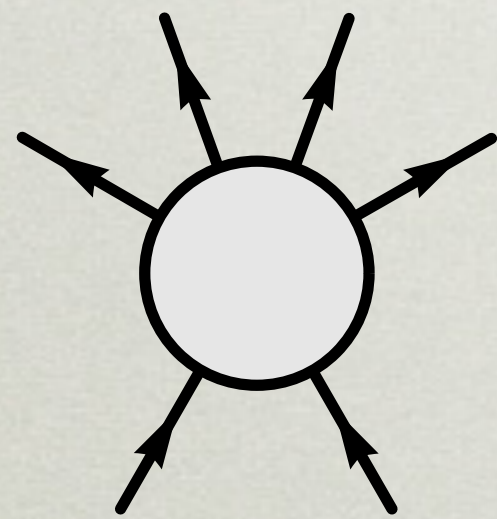




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What is the *mathematical form* of the predictions made by QFT?



## Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,<sup>1</sup> M. Spradlin,<sup>2</sup> C. Vergu,<sup>2</sup> and A. Volovich<sup>2</sup>

<sup>1</sup>Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA

<sup>2</sup>Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA

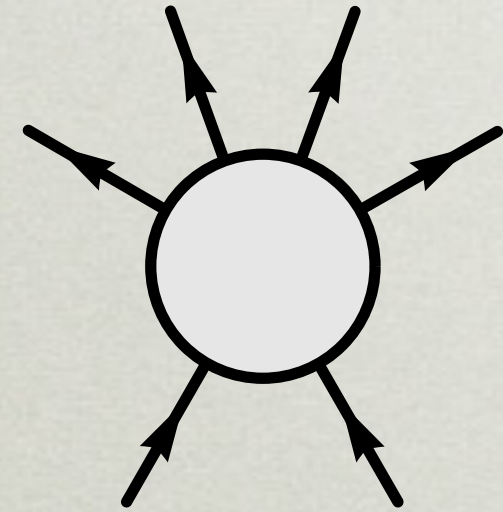
We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions  $\text{Li}_k$  with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$R(u_1, u_2, u_3) = \sum_{i=1}^3 \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) - \frac{1}{8} \left( \sum_{i=1}^3 \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 (J^2 + \zeta_2)$$

# What Form do Observables Take?



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What is the *mathematical form* of the predictions made by QFT?



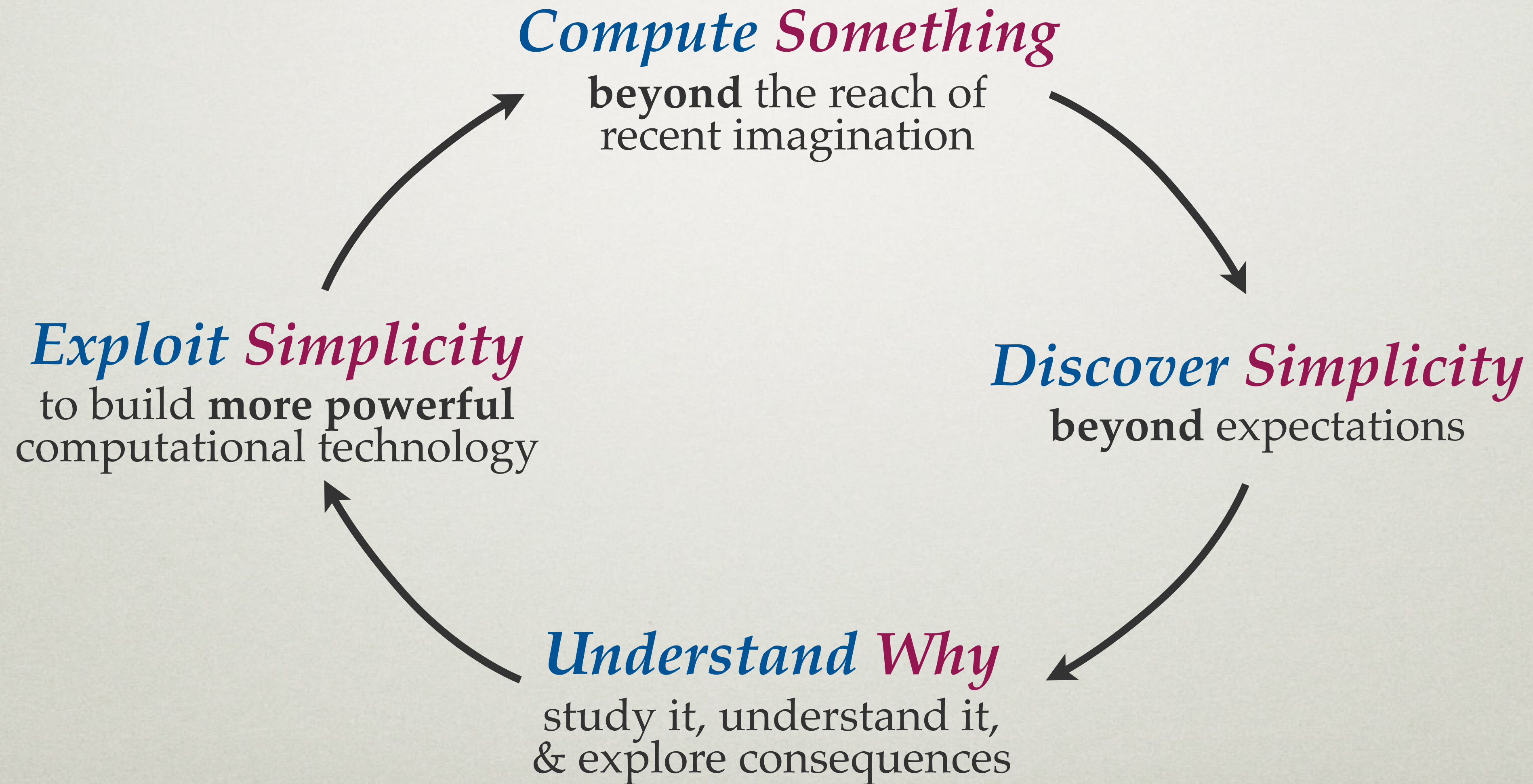
## State of the art:

6-point (N)MHV @ (6) 7 loops(!!!)

7-point (N)MHV @ 4 loops (symbol-level)

[Dixon, *et al* (2019);...]

# *Amplitudes: a Virtuous Cycle*





# What Form do Observables Take?

- ◆ In a general (say, 4d) QFT, it would have *recently been expected* by “experts” that observables took the following general form:

$$\begin{aligned}
 \mathcal{A} = & \mathcal{A}^{\text{tree}} + \hbar \mathcal{A}^{(L=1)} + \hbar^2 \mathcal{A}^{(L=2)} + \dots + \hbar^L \mathcal{A}^{(L)} + \dots \\
 & \text{rational} + \binom{\text{weight-2}}{\text{polylogs}} + \binom{\text{weight-4}}{\text{polylogs}} + \dots + \binom{\text{weight-2L}}{\text{polylogs}} + \dots \\
 & + \binom{\text{weight-1}}{\text{polylogs}} + \vdots + \dots + \vdots \\
 & + \text{rational} + \binom{\text{weight-1}}{\text{polylogs}} + \dots + \vdots \\
 & + \text{rational}
 \end{aligned}$$

(general dimension  $d$ :  $2 \mapsto \lfloor d/2 \rfloor$ )

coefficients:

*leading singularities*



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$$\text{rational} + \binom{\text{weight-2}}{\text{polylogs}} + \binom{\text{weight-4}}{\text{polylogs}} + \dots + \binom{\text{weight-2L}}{\text{polylogs}} + \dots$$

$$+ \binom{\text{weight-1}}{\text{polylogs}} + \vdots + \dots + \vdots$$

planar  $\mathcal{N}=4$ ?

$$+ \text{rational} + \binom{\text{weight-1}}{\text{polylogs}} + \dots + \vdots$$

**coefficients:**

*leading singularities*

$$+ \text{rational}$$



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rational	+	$\left( \begin{array}{c} \text{weight-2} \\ \text{polylogs} \end{array} \right)$	+	<del><math>\left( \begin{array}{c} \text{weight-4} \\ \text{polylogs} \end{array} \right)</math></del>	+	...	+	<del><math>\left( \begin{array}{c} \text{weight-2L} \\ \text{polylogs} \end{array} \right)</math></del>	+	...
----------	---	--	---	--	---	-----	---	---	---	-----

+	$\left( \begin{array}{c} \text{weight-1} \\ \text{polylogs} \end{array} \right)$	+	⋮	+	...	+	⋮	planar $\mathcal{N}=4$ ?
---	--	---	---	---	-----	---	---	--------------------------

+	rational	+	$\left( \begin{array}{c} \text{weight-1} \\ \text{polylogs} \end{array} \right)$	+	...	+	⋮
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**coefficients:**

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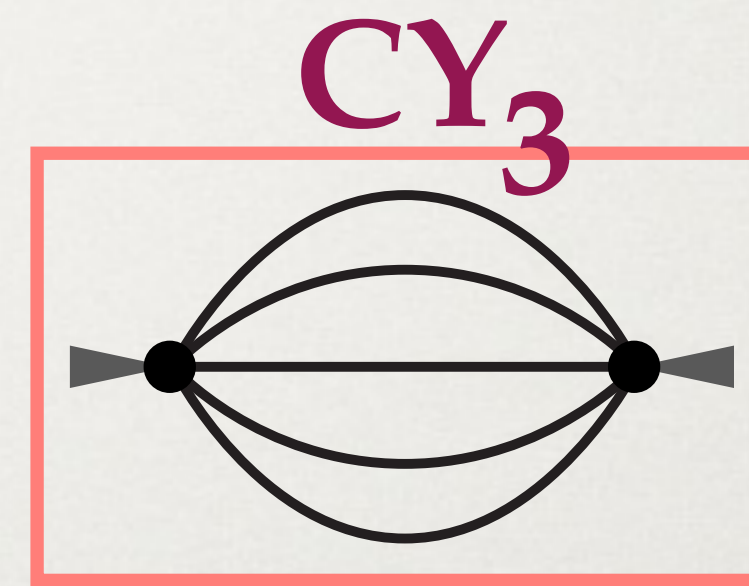
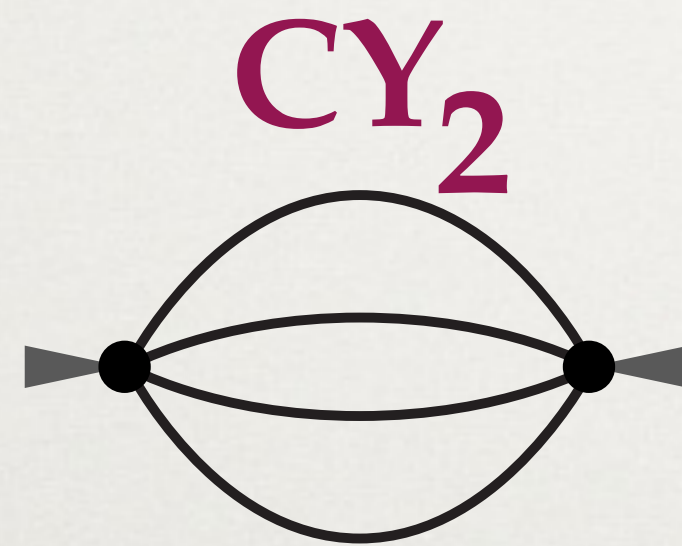
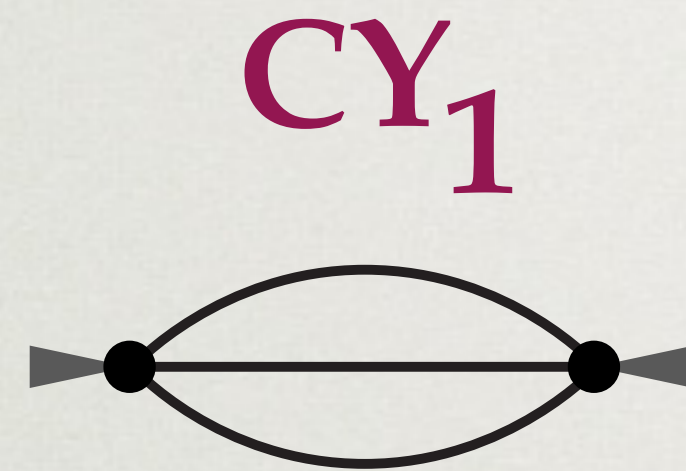


◆ Unfortunately, many pesky counterexamples were to be found:

[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]

[Doran, Harder, Thompson (2019)]

sunrises:



...

} 2d, masive

contributes to electron ( $g-2$ )



# What Form do Observables Take?

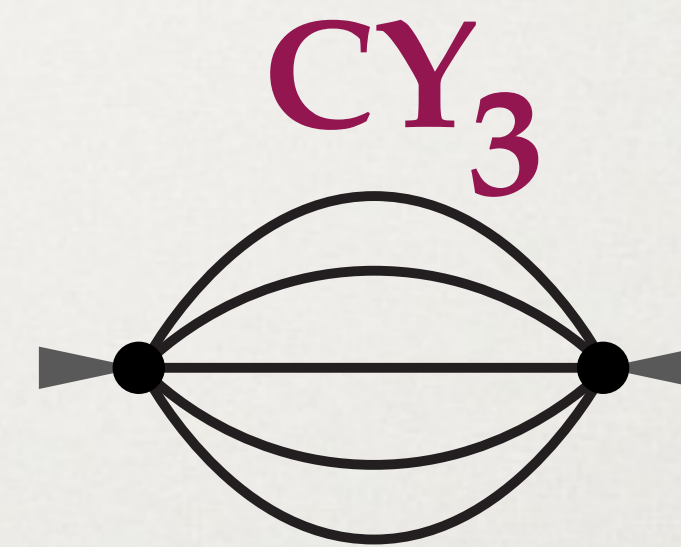
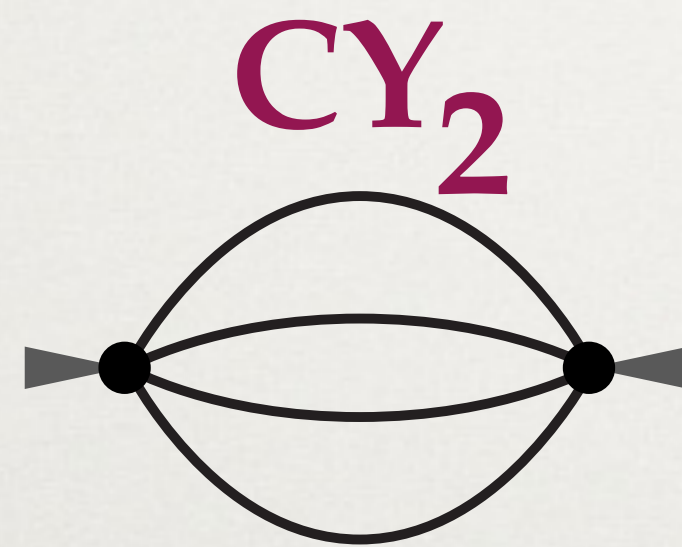
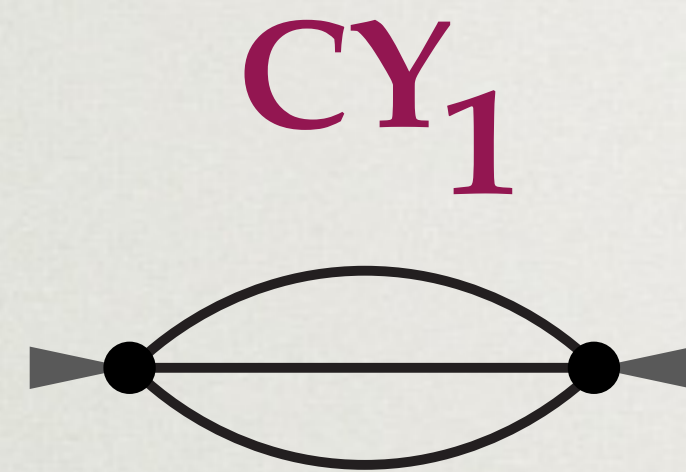


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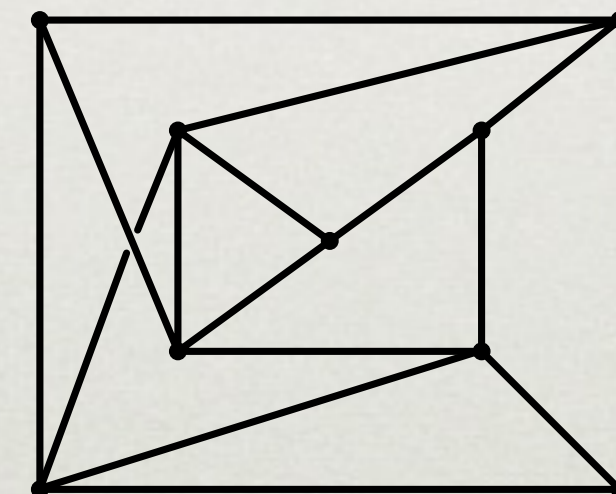
[Doran, Harder, Thompson (2019)]

sunrises:



...

} 2d, massive



an 8-loop vacuum graph  
evaluating to a **K3 period**

} 4d, massless

[Brown, Schnetz (2011)]

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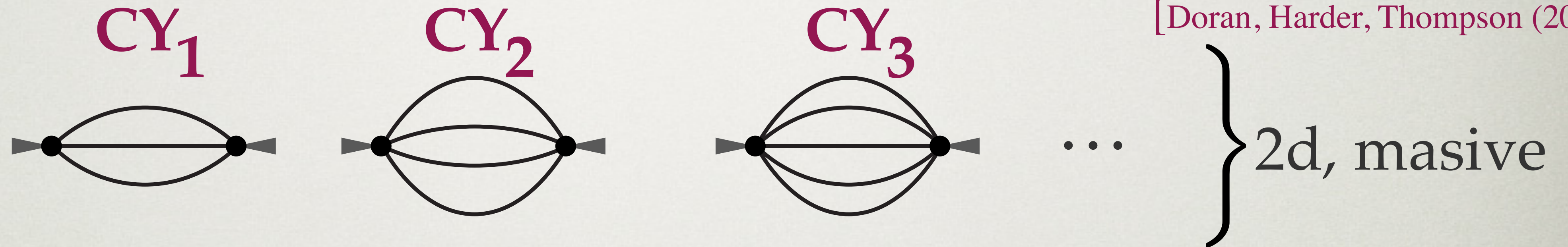


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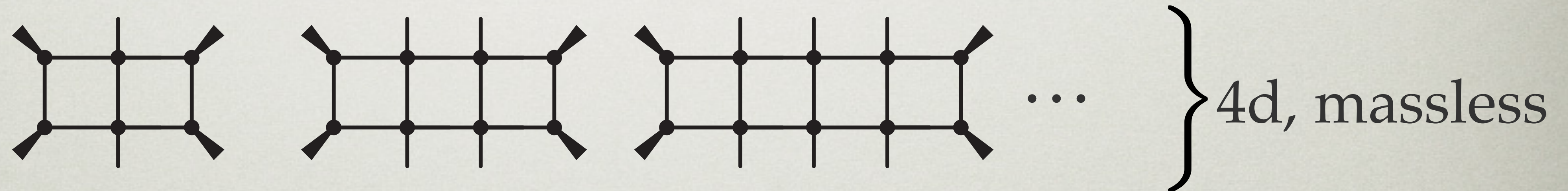
[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]

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sunrises:

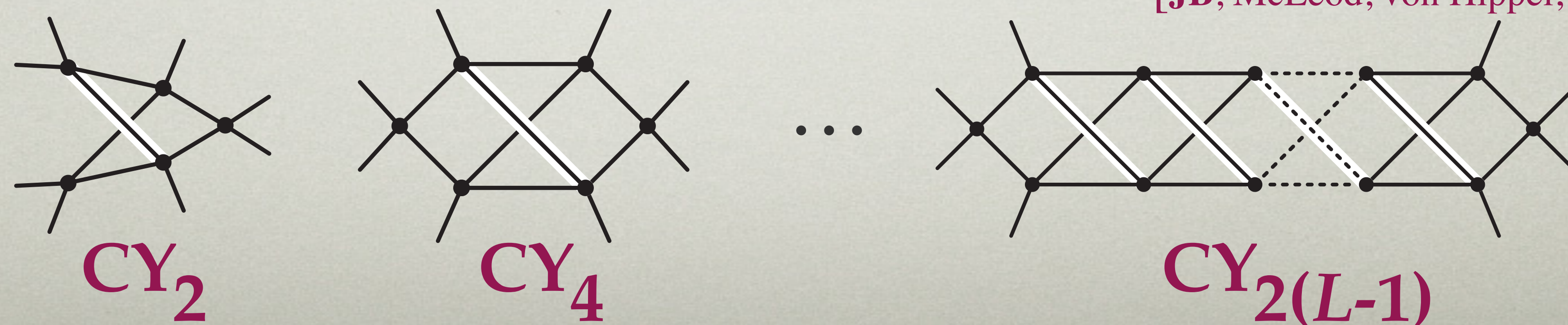


traintracks:



[JB, McLeod, von Hippel, Wilhelm (2018)]

tardigrades:





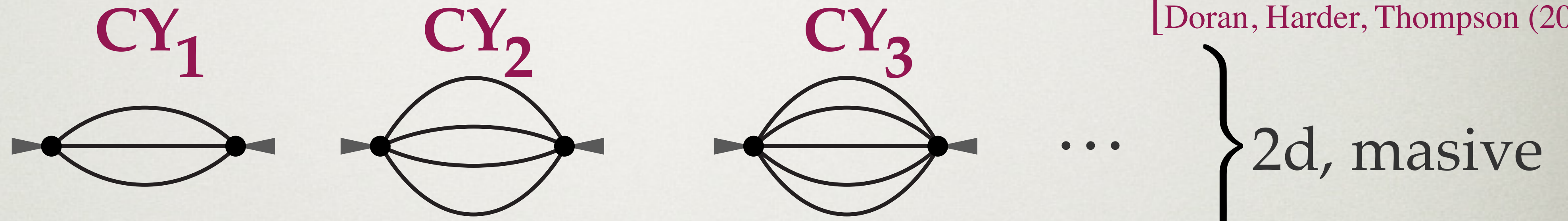
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sunrises:



traintra

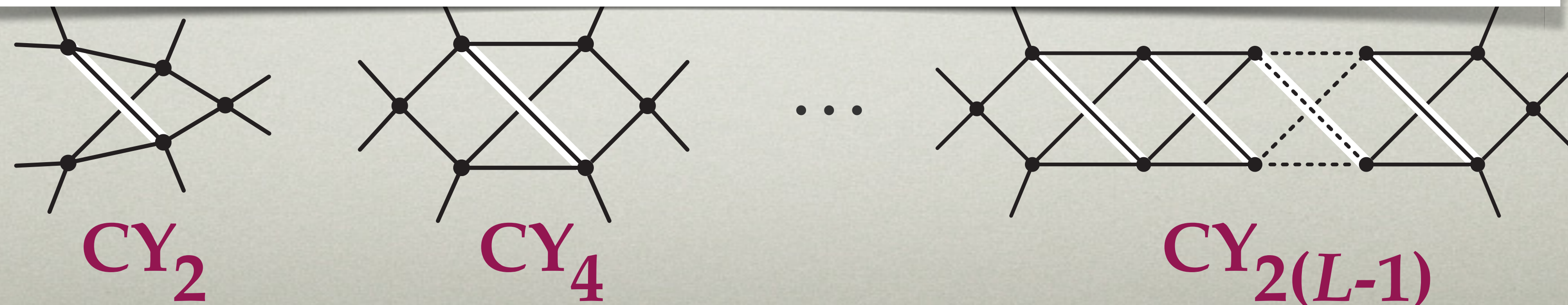
Outside a *small* list of **extremely** limited / simple cases (which are widely expected\* to be exceptions):

*almost all* observables in *almost all* QFTs are expected to be non-polylogarithmic

massless

[Wilhelm (2018)]

tardigrades:





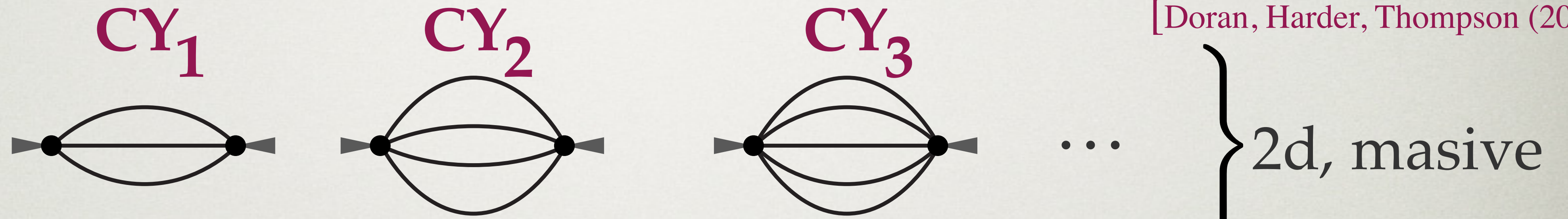
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[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]

[Doran, Harder, Thompson (2019)]

sunrises:



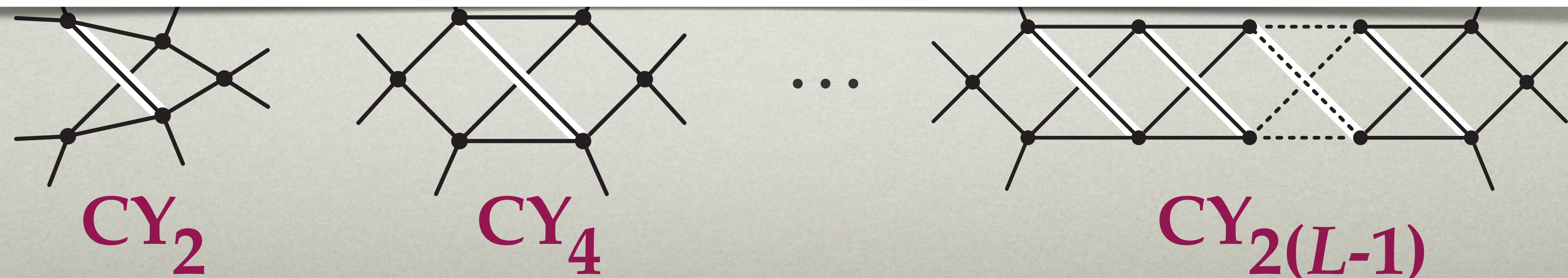
◆ In every instance known, the Calabi-Yau itself is **very simple**:

▶ degree- $(d+1)$  hypersurface in  $\mathbb{P}^d$  (or multiple-cover thereof)

◆ kinematic data (momenta / masses) control the moduli

▶ often **extremely singular**

tardigrades:



SS

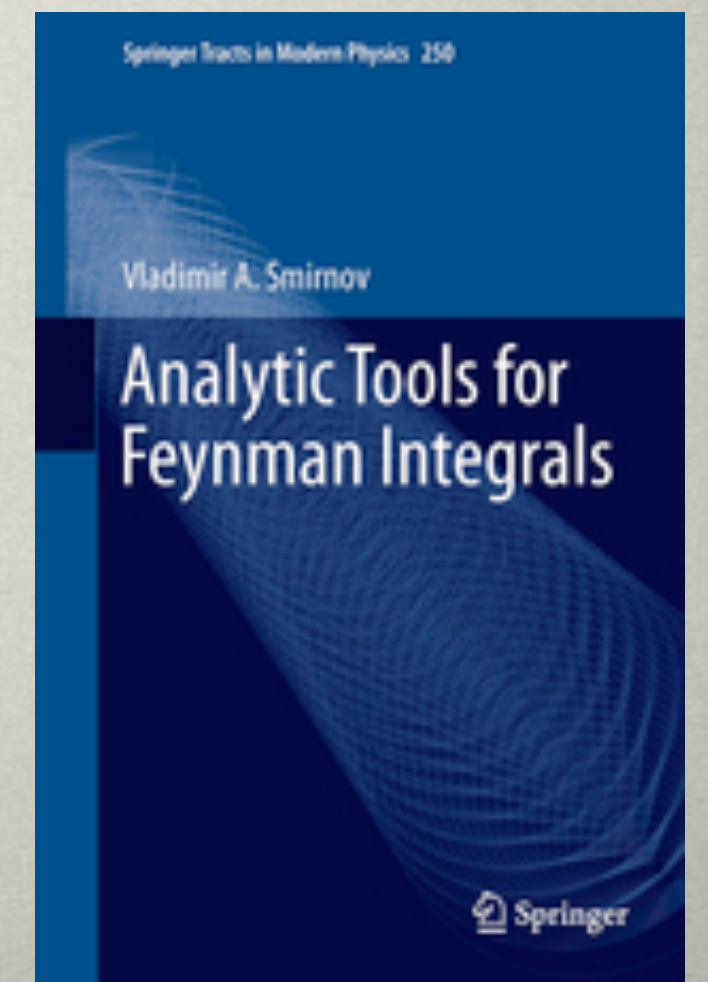
(2018)]

# Why is Perturbation Theory so Hard?



- ◆ **Feynman diagrams** (esp. with *scalar* numerators) are *horrible*
  - ▶ difficult to integrate, explosive in number, non-physical,...
- ◆ **Regularization** obscures symmetries (+is technically difficult)
- ◆ **Most** familiar **master integrand bases** are *unnecessarily bad*:
  - ▶ don't satisfy nice / canonical differential equations
  - ▶ contain *multiple elliptic*(+worse(!)) geometries,
  - ▶ ...

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} = e^{2\gamma_E \epsilon} \iint \frac{d^D k_1 d^D k_2}{(i\pi^{d/2})^2} \frac{P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$





# *How can We Make it Easier?*

- ◆ **Use unitarity** to choose the *nicest/easiest* integrals to integrate (*of course*, integration “ease” changes with time and new methods)
  - ▶ search for as many *pure* integrals as you can
    - those which satisfy nice (canonical) differential equations

**Definition:** a function  $f(s)$  is called *pure* if:

- ▶ there exists a grading of functions by “transcendental” *weight*
- ▶ any **derivative** of  $f(s)$  is *strictly lower in weight*



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e.g.  $g(s) \log(f(s))$  would be *impure*

$$\frac{\partial}{\partial s} [g(s) \log(f(s))] = g'(s) \log(f(s)) + g(s) f'(s) / f(s)$$

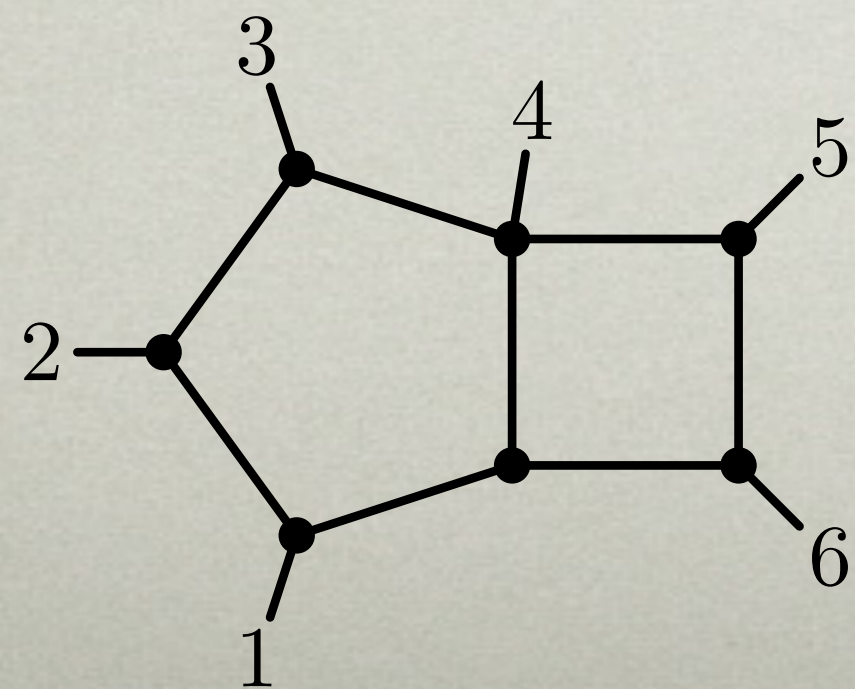


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$$\Rightarrow \frac{s_{13}}{s_{12}s_{23}s_{56}} \left( \begin{array}{l} \text{tr}_+ [p_3, p_{12}, p_6, p_1] (\text{Li}_4(\dots)'s + \dots) \\ + \text{tr}_+ [p_{12}, p_6, p_1, p_3] (\text{Li}_4(\dots)'s + \dots) \end{array} \right)$$





# How can We Make it Easier?

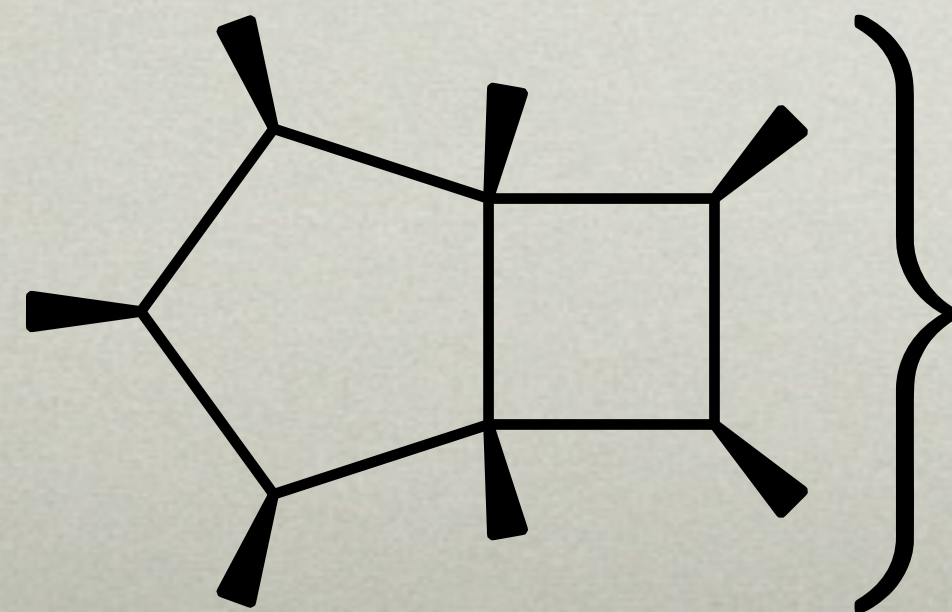
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- ◆ **Avoid regularization** whenever possible:

- ▶ can all(?) *finite* quantities be computed *without regularization*?
  - without expanding them in terms of divergent integrals?

(Answer: sometimes)

[JB, Langer, Patatoukos (2021); ...]



with numerator ‘1’, this integrates to a sum of (an *impure* combination of) **polylogarithms** & **elliptic-polylogarithms** involving *4 elliptic curves*

*Unitarity-Based Strategies:*  
*a modern perspective*

# *Generalized Unitarity: a modern take*



- ◆ The basic idea behind **unitarity**-based methods is that any *Feynman integrand* is a *rational differential form on loop momenta*
  - ▶ as such, it can be expanded into a **basis**  $\mathfrak{B}$  of such forms:

$$A = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$

- ◆ For any fixed QFT (spacetime dimension, particle content), the space of **all amplitude integrands** is **finite-dimensional**
  - ▶ *all-multiplicity amplitudes* can be expressed in a *finite basis*!
- ◆ **Key observation:** viewed as a potential element of *some* basis, *every* Feynman integrand can be interesting!
  - ▶ Why not try to find the *best/easiest* integrands—and use these?

# Stratifying Quantum Field Theories



- ◆ QFTs can be *partially ordered* by the scope of the basis required to represent their amplitudes

$$A = \sum_{\mathbf{b}^i \in \mathfrak{B}} a_i \mathbf{b}^i$$

[Standard Model]  $\succ$  [(Standard Model \ Higgs)]  $\succ$  [QCD]  $\succ$  [Yang-Mills]

[Yang-Mills]  $\succ$  [ $\mathcal{N} = 2$  super-Yang-Mills]  $\succ$  [ $\mathcal{N} = 4$  super-Yang-Mills]

[ $\mathcal{N} = 4$  Yang-Mills]  $\succ$  [planar  $\mathcal{N} = 4$  super-Yang-Mills]  $\succ \dots \succ$  [fishnet theory]

This reflects **UV behavior** (“*power-counting*”) of theories;  
it suggests a possible *stratification of integrand bases*

$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$



# Stratifying Integrand Bases

- ◆ Suppose that a basis could be carved up into subspaces (by any arbitrary means):

$$\mathcal{A} = \sum_{\mathfrak{b}^i \in \mathfrak{B}} a_i \mathfrak{b}^i$$
$$\mathfrak{B} =: \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \quad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p := \mathcal{A} \cap \mathfrak{B}_p := \sum_{\mathfrak{b}_p^i \in \mathfrak{B}_p} a_i \mathfrak{b}_p^i$$

- ◆ ¿Is it possible to stratify integrand bases by *physical structure*?
- $\{\text{finite}\} \oplus \{\text{divergent}\}$

[JB, Langer, Zhang, (2021)]

[JB, Herrmann, Langer, Patatoukos, *et al* (2021)]



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- ◆ ¿Is it possible to stratify integrand bases by *physical structure*?

$$\{\text{finite}\} \oplus \left\{ \left( \text{UV-divergent} \right) \right\} \oplus \left\{ \left( \text{IR-divergent} \right) \right\}$$

[JB, Langer, Zhang, (2021)]

[JB, Herrmann, Langer, Patatoukos, *et al* (2021)]

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- ◆ ¿Is it possible to stratify integrand bases by *physical structure*?

$$\left\{ \text{finite} \right\} \oplus \left\{ \left( \mathcal{O}(1/\epsilon^{2L})\text{-divergent} \right) \oplus \left( \mathcal{O}(1/\epsilon^{2L-1})\text{-divergent} \right) \oplus \dots \oplus \left( \mathcal{O}(1/\epsilon)\text{-divergent} \right) \right\}$$

$$\oplus \left\{ \left( \log(m)^{2L}\text{-divergent} \right) \oplus \left( \log(m)^{2L-1}\text{-divergent} \right) \oplus \dots \oplus \left( \log(m)\text{-divergent} \right) \right\}$$

[JB, Langer, Zhang, (2021)]

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- Can we further stratify each part by *transcendental structure*?

{ finite }

{ max-weight } ⊕ { next-to-max-weight } ⊕ ⋯ ⊕ { rational }

{ polylogs } ⊕ { elliptic-polylogs } ⊕ { K3-polylogs } ⊕ ⋯

[JB, Langer, Zhang, (2021)]

[JB, Herrmann, Langer, Patatoukos, et al (2021)]





# *Prescriptive Integrand Bases*

- ◆ How *generalized unitarity* has been used to match amplitudes:

$$\mathcal{A} = \sum_i c_i \mathcal{I}_i^0$$

with coefficients  $c_i$  determined by **cuts**: a spanning set of cycles  $\{\Omega_j\}$

$$\oint_{\Omega_j} \mathcal{I}_i^0 =: \mathbf{M}_{i,j}$$

$$a_j := \oint_{\Omega_j} \mathcal{A} = \sum_i c_i \mathbf{M}_{i,j} \implies c_j = \sum_i a_i (\mathbf{M}^{-1})_{i,j}$$

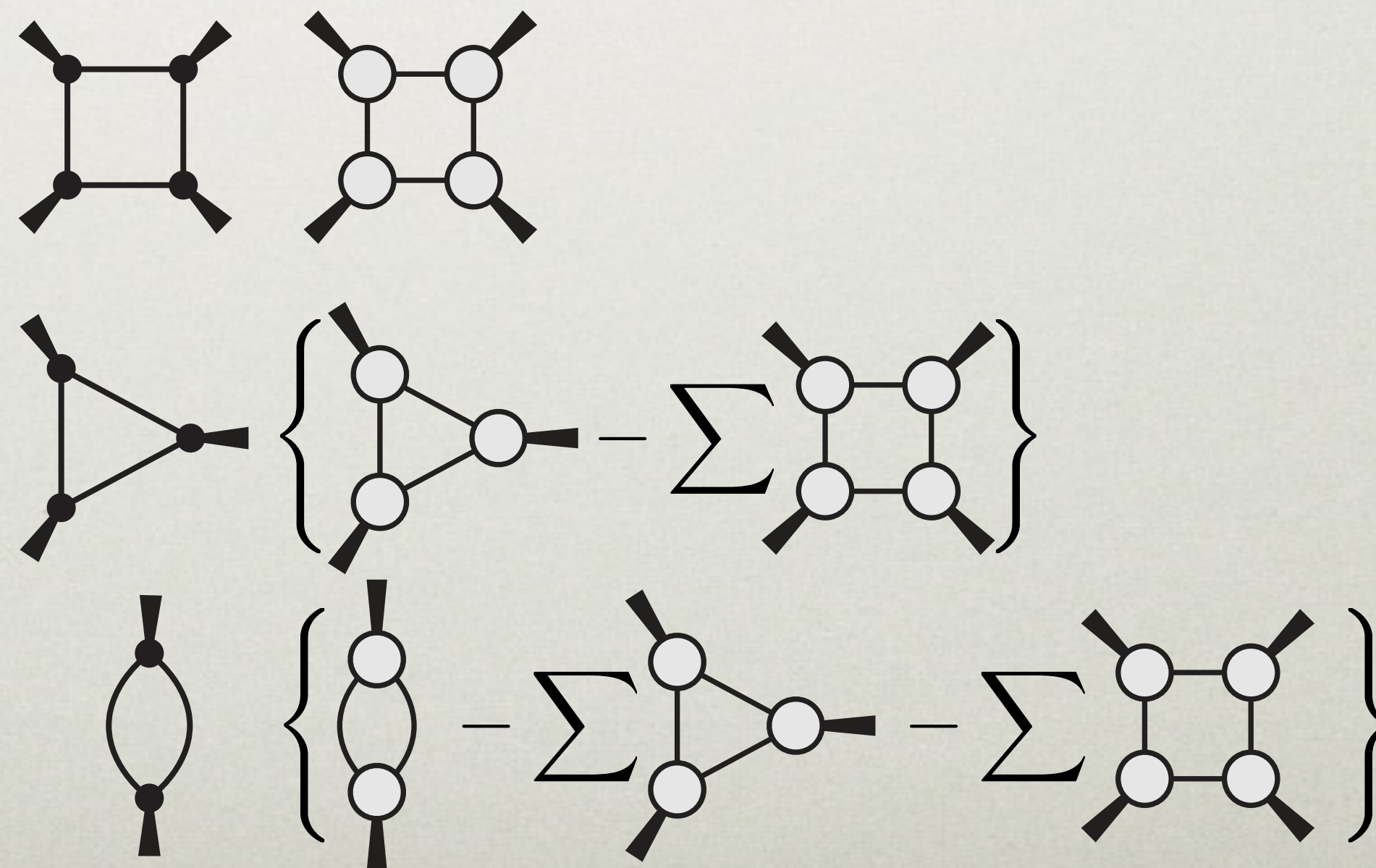


# Prescriptive Integrand Bases

- ◆ How *generalized unitarity* has been used to match amplitudes:

$$A = \sum_i c_i \mathcal{I}_i^0$$

with coefficients  $c_i$  determined by **cuts**: a spanning set of cycles  $\{\Omega_j\}$



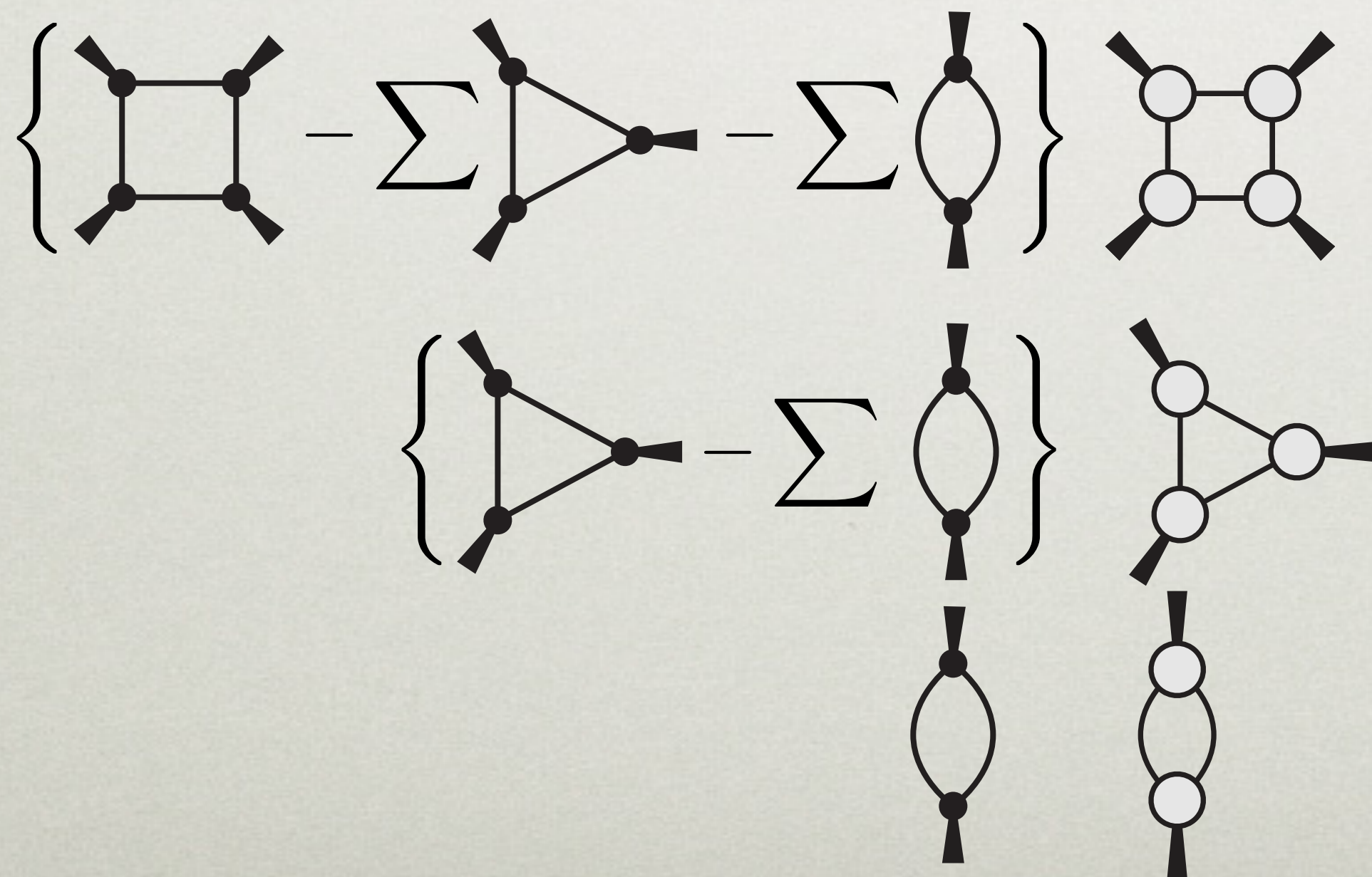


# Prescriptive Integrand Bases

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# *Prescriptive Integrand Bases*

- ◆ How *generalized unitarity* has been used to match amplitudes:

$$A = \sum_i a_i \mathcal{I}_i$$

with coefficients  $C_i$  determined by **cuts**: a spanning set of cycles  $\{\Omega_j\}$

$$\mathcal{I}_j := \sum_i \mathcal{I}_i^0 (\mathbf{M}^{-1})_{i,j} \quad \oint_{\Omega_j} \mathcal{I}_i = \delta_{i,j}$$

- ◆ A basis is called *prescriptive* if it is the **cohomological dual** of a spanning set of **cycles**  $\{\Omega_j\}$

# Strategies for Building Bases



- ◆ Given *some* integrand basis (or strata thereof), one should *diagonalize* the space of integrands according to a

## homological/cohomological pairing:

- ▶ choose a *spanning-set* of **compact, max-dimensional** contours  $\Omega_j$
- ▶ normalize and diagonalize the basis by the requirement

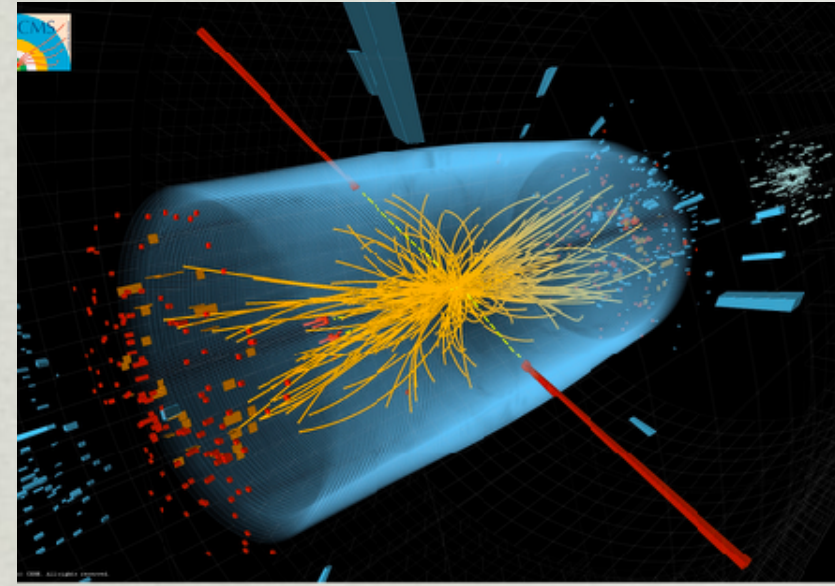
$$\int_{\Omega_j} \mathbf{b}_i = \delta_{ij}$$

$\Omega_j$  :  $4L$ -dimensional compact contours

{	“residues”
	elliptic periods
	K3 periods, etc.

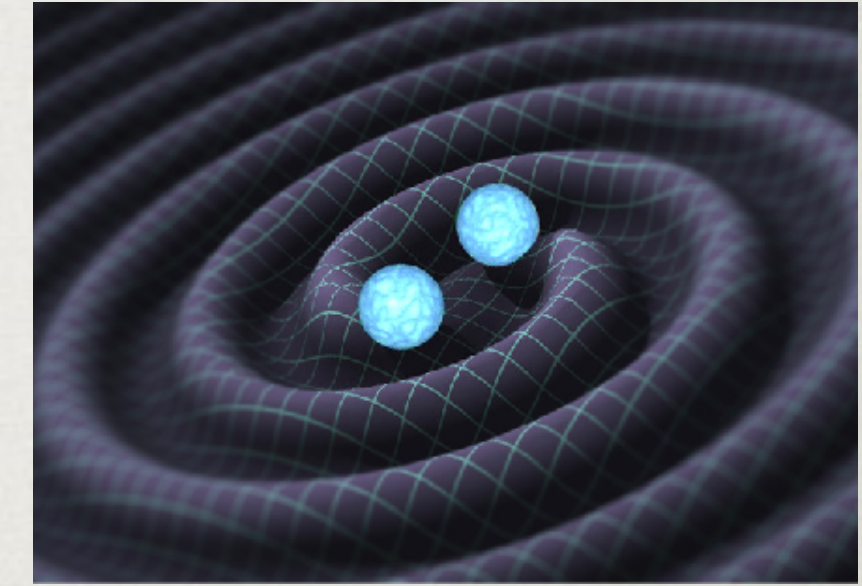
- ◆ This *trivializes* the representation of amplitudes:
  - ▶ the **coefficient** of any amplitude in this basis will simply be the *on-shell function* evaluated on the contour (a **leading singularity**)
- ◆ Choosing a **maximal** set of IR/UV-**divergence-probing contours** ensures(?) that the basis is split into finite / divergent subspaces

# *Amplitudes: a Virtuous Cycle*



*Compute Something*

beyond the reach of  
recent imagination



*Exploit Simplicity*  
to build more powerful  
computational technology

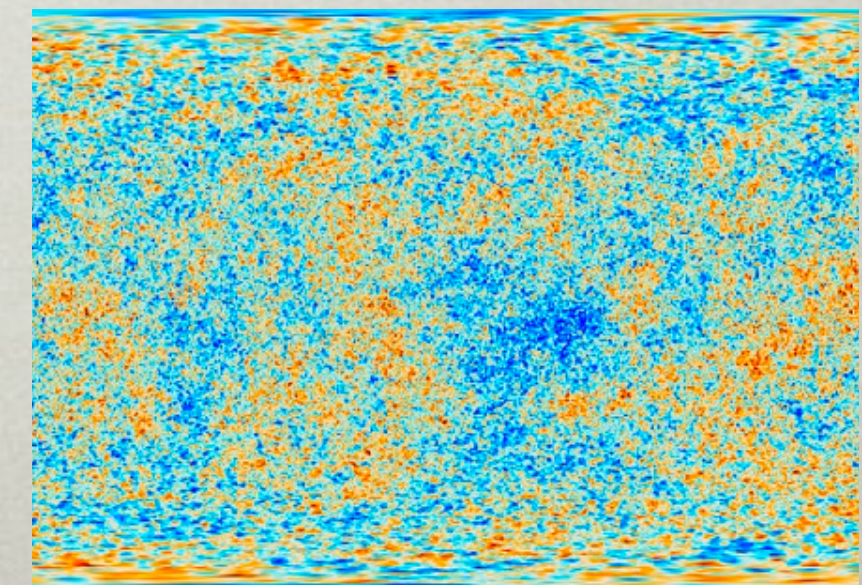
*Discover Simplicity*

beyond expectations



*Understand Why*

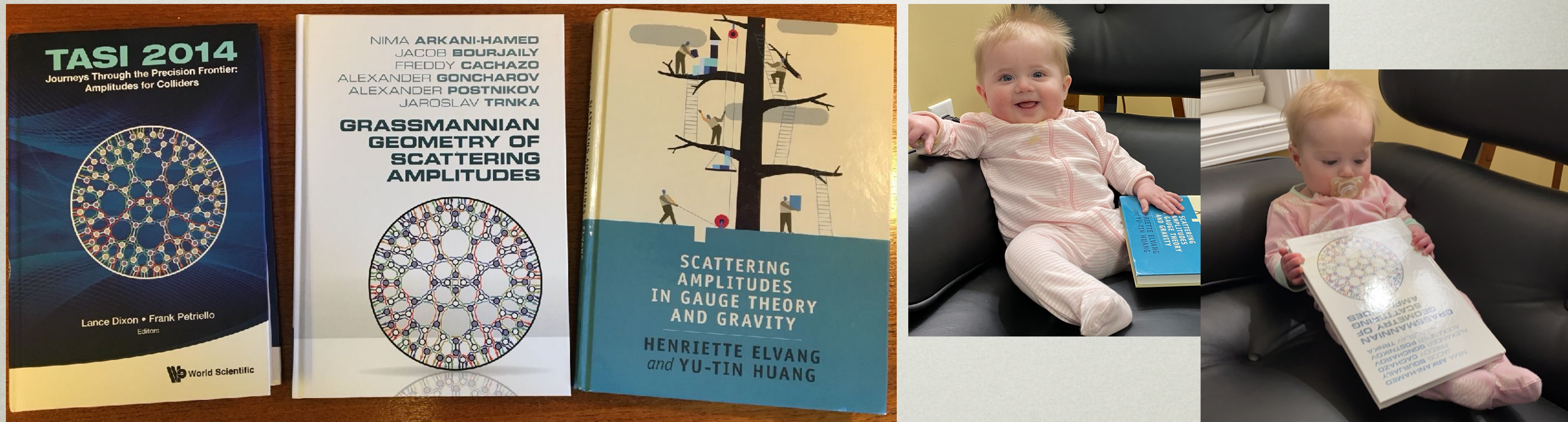
study it, understand it,  
& explore consequences





# Today's *Revolution* in QFT

- ◆ Defining an *ongoing revolution* in science:
  - when *all the textbooks* of a field become *obsolete*
- ◆ Experts who *need* QFT *no longer use textbook tools*



- ◆ The foundations for *the future's textbooks* are *still being discovered* every day (*it's exciting!*)

*Thank you!*



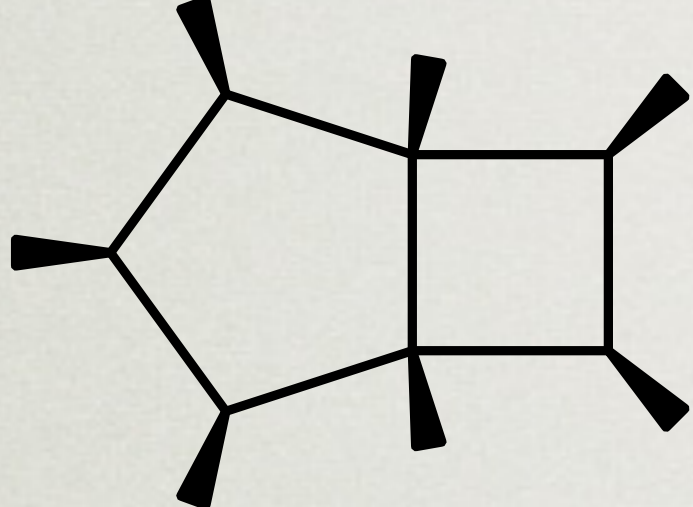
# *Stratifying Rigidity*



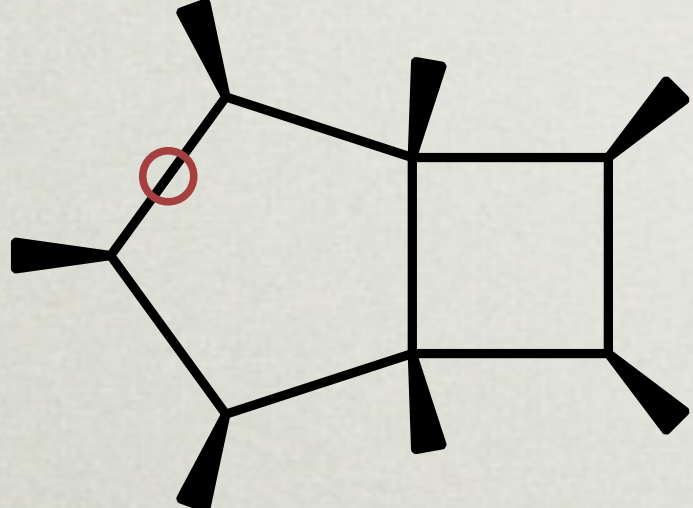
# Diagonalization of Rigidity

◆ Consider the following sets of pentabox integrands

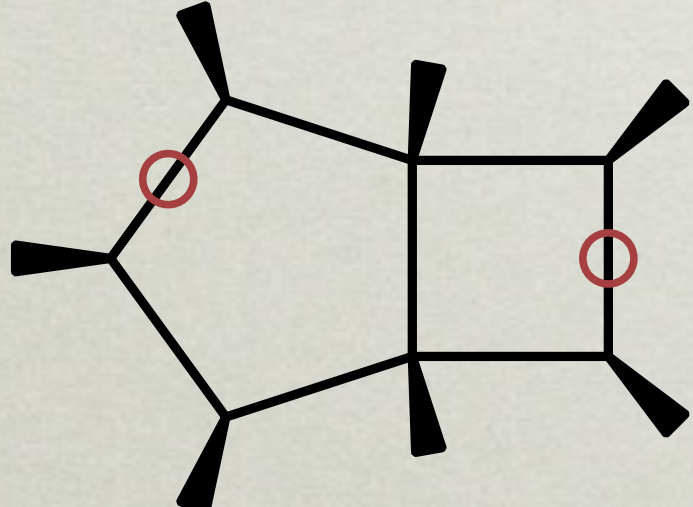
[JB, Kalyanapuram (2022)]



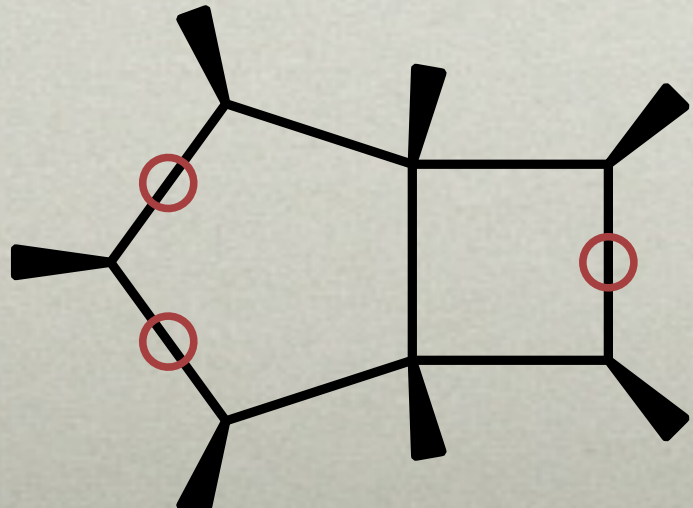
$$\Leftrightarrow \mathcal{I}_0 := \frac{(l_1|X)}{(l_1|a)(l_1|b)(l_1|c)(l_1|d)(l_1|l_2)(l_2|e)(l_2|f)(l_2|g)}$$



$$\Leftrightarrow \mathcal{I}_1(N) := \frac{(l_1|N)}{(l_1|a)(l_1|b)(l_1|c)(l_1|d)(l_1|l_2)(l_2|e)(l_2|f)(l_2|g)}$$



$$\Leftrightarrow \mathcal{I}_2(N, M) := \frac{(l_1|N)(l_2|M)}{(l_1|a)(l_1|b)(l_1|c)(l_1|d)(l_2|X)(l_1|l_2)(l_2|e)(l_2|f)(l_2|g)}$$



$$\Leftrightarrow \mathcal{I}_3(\vec{N}, M) := \frac{(l_1|N_1)(l_1|N_2)(l_2|M)}{(l_1|a)(l_1|b)(l_1|c)(l_1|d)(l_1|X)(l_2|X)(l_1|l_2)(l_2|e)(l_2|f)(l_2|g)}$$

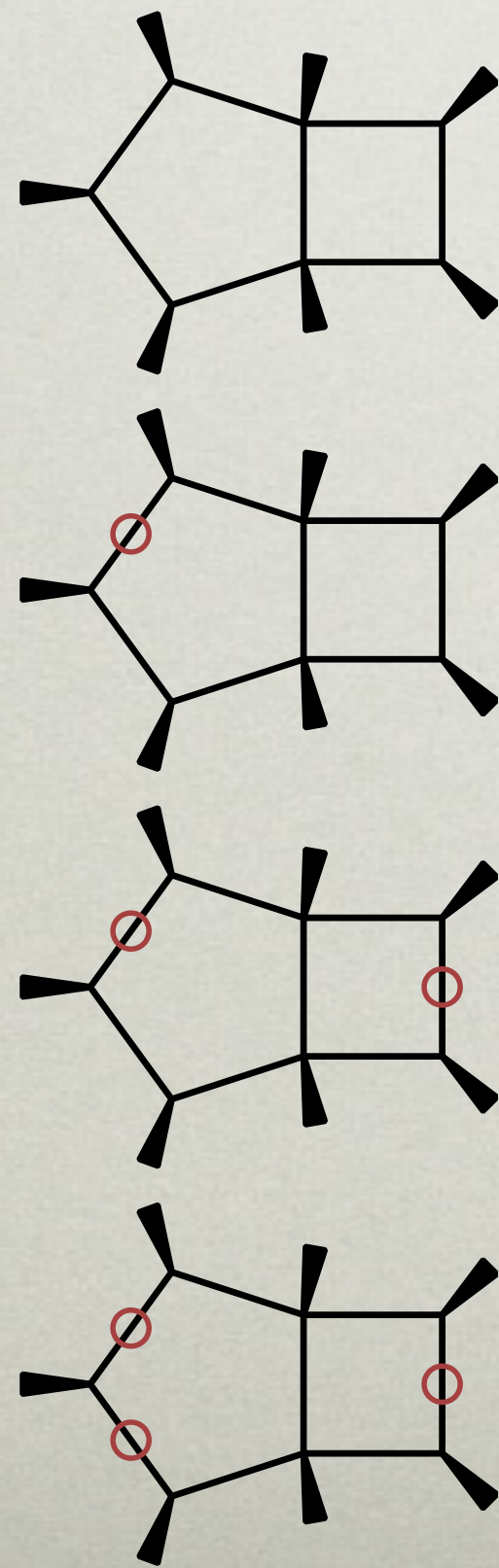


# Diagonalization of Rigidity

◆ Consider the following sets of pentabox integrands

[JB, Kalyanapuram (2022)]

pure



# ints	# polylogs	# elliptics	# impure/mixed
1	0	0	1
6	0	4 (4)	2
36	10	12 (12)	14
120	76	44 (22)	0