



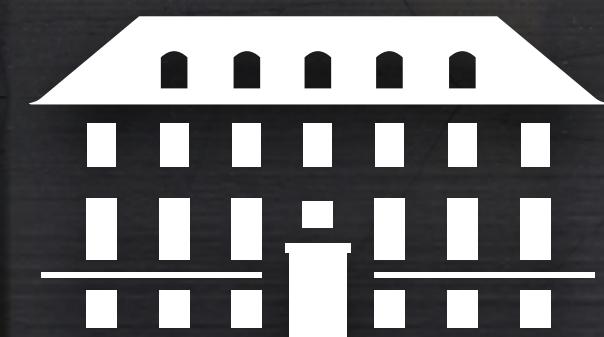
PennState

# *Adventures in Perturbation Theory*

Jacob Bourjaily

Penn State University &  
Niels Bohr International Academy, University of Københavns

*Current Themes in High Energy Physics and Cosmology*



The Niels Bohr  
International Academy

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# What Form do Observables Take?

- ♦ What is the *mathematical form* of the predictions made by QFT?  
(perturbatively, say?)

$$\text{Diagram: } \text{A bare vertex (circle)} = \text{a bare vertex (dot)} + \text{loop correction} + \text{higher-order loop correction} + \dots$$
$$g_e = 2 + \frac{\alpha}{\pi}(1)$$

[Dirac (1933)]

[Feynman; Schwinger; Tomanaga (1947)]



# What Form do Observables Take?

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$$\text{Diagram: } \text{Feynman diagram for the electron self-energy} = \text{One-loop diagram} + \text{Two-loop diagram} + \text{Three-loop diagram} + \dots + \text{Four-loop diagram} + \dots$$

[Dirac (1933)]

$$g_e = 2 + \frac{\alpha}{\pi}(1) + \frac{\alpha^2}{\pi^2} \left( \frac{3}{2}\zeta_3 - \pi^2 \log(2) + \zeta_2 + \frac{197}{72} \right) - \frac{\alpha^3}{\pi^3} \left( \frac{215}{12}\zeta_5 + \frac{100}{3}u_{3,1} + \frac{13}{4}\zeta_4 - \frac{139}{9}\zeta_3 + \frac{1192}{3}\zeta_2 \log(2) - \frac{34202}{135}\zeta_2 - \frac{28259}{2592} \right)$$

[Feynman; Schwinger; Tomanaga (1947)]

[Laporta, Remiddi (1996)] [Petermann (1957)]

[Laporta (2017)] [Kinoshita (1990)]

<sup>\*</sup>( $u_{3,1}$  is related to  $\text{Li}_4(1/2)$  and  $\log(2)$ )

beyond three loops, no analytic expression is known  
(CY periods appear)



# What Form do Observables Take?

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e.g. maximally supersymmetric ( $\mathcal{N}=4$ ) Yang-Mills theory (planar limit)

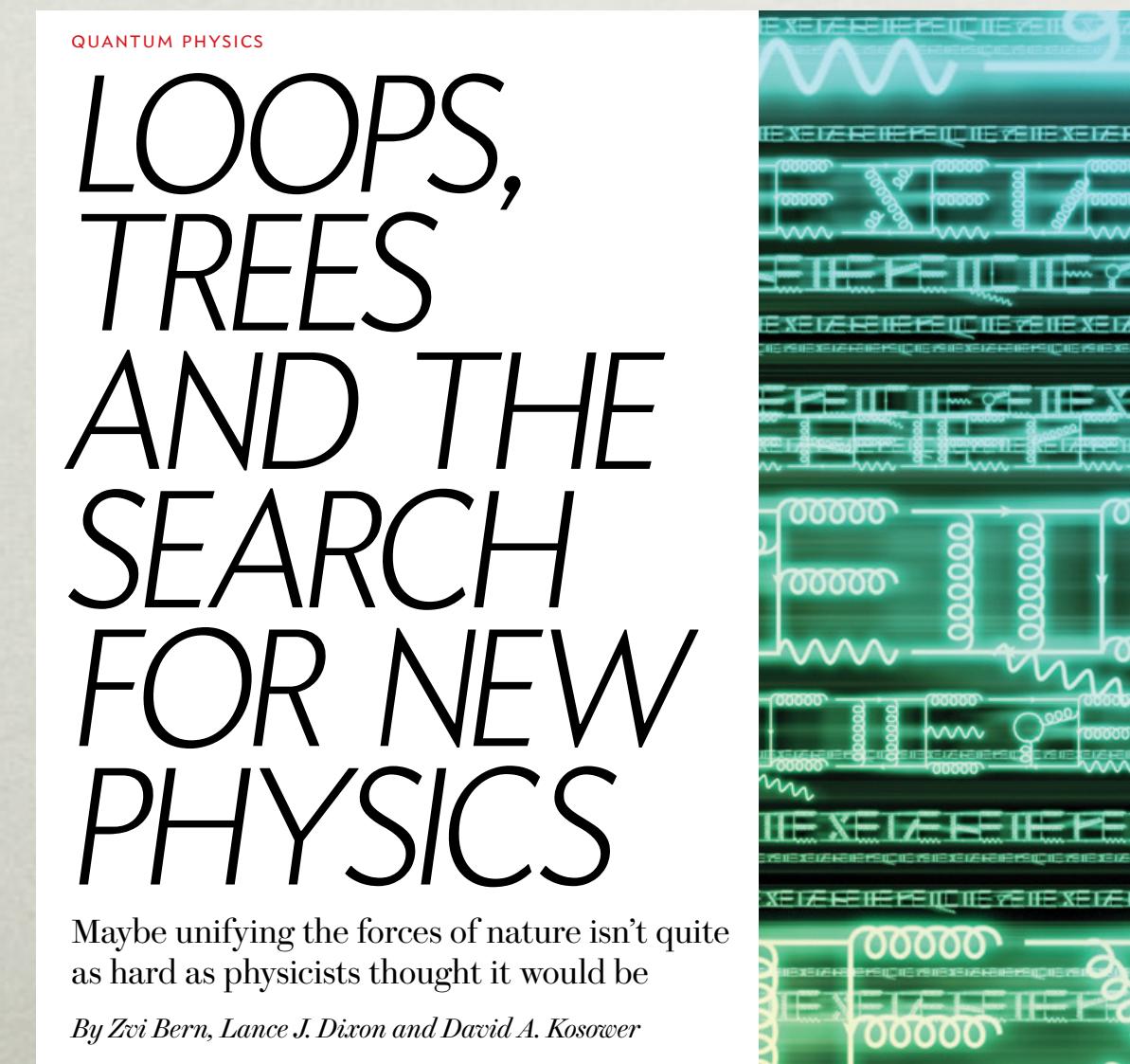
$$\begin{aligned}\gamma_{\text{cusp}} = & \quad a \times 1 && [\text{Beisert, Eden, Staudacher (2007);...}] \\ & + a^2 \times 2 \zeta_2 \\ & + a^3 \times 22 \zeta_4 \\ & + a^4 \times 2(24 \zeta_2^3 + 4 \zeta_3^2 + 2 \zeta_2 \zeta_4 + \zeta_6) \\ & + a^5 \times 8(252 \zeta_4^2 + 20 \zeta_3 \zeta_5 + 4 \zeta_2 \zeta_3^2 + \zeta_2 \zeta_6) \\ & + a^6 \times 8(282 \zeta_2^5 + \zeta_2^3 \zeta_4 + 4 \zeta_2 \zeta_4^2 + 80 \zeta_2 \zeta_3 \zeta_5 + 5 \zeta_2^2 \zeta_6 + 48 \zeta_3^2 \zeta_4 + 102 \zeta_5^2 + 210 \zeta_3 \zeta_7 + 3 \zeta_{10}) \\ & + \dots\end{aligned}$$

$\Rightarrow$  “maximal *transcendentality*” of planar  $\mathcal{N}=4$  super Yang-Mills (?)

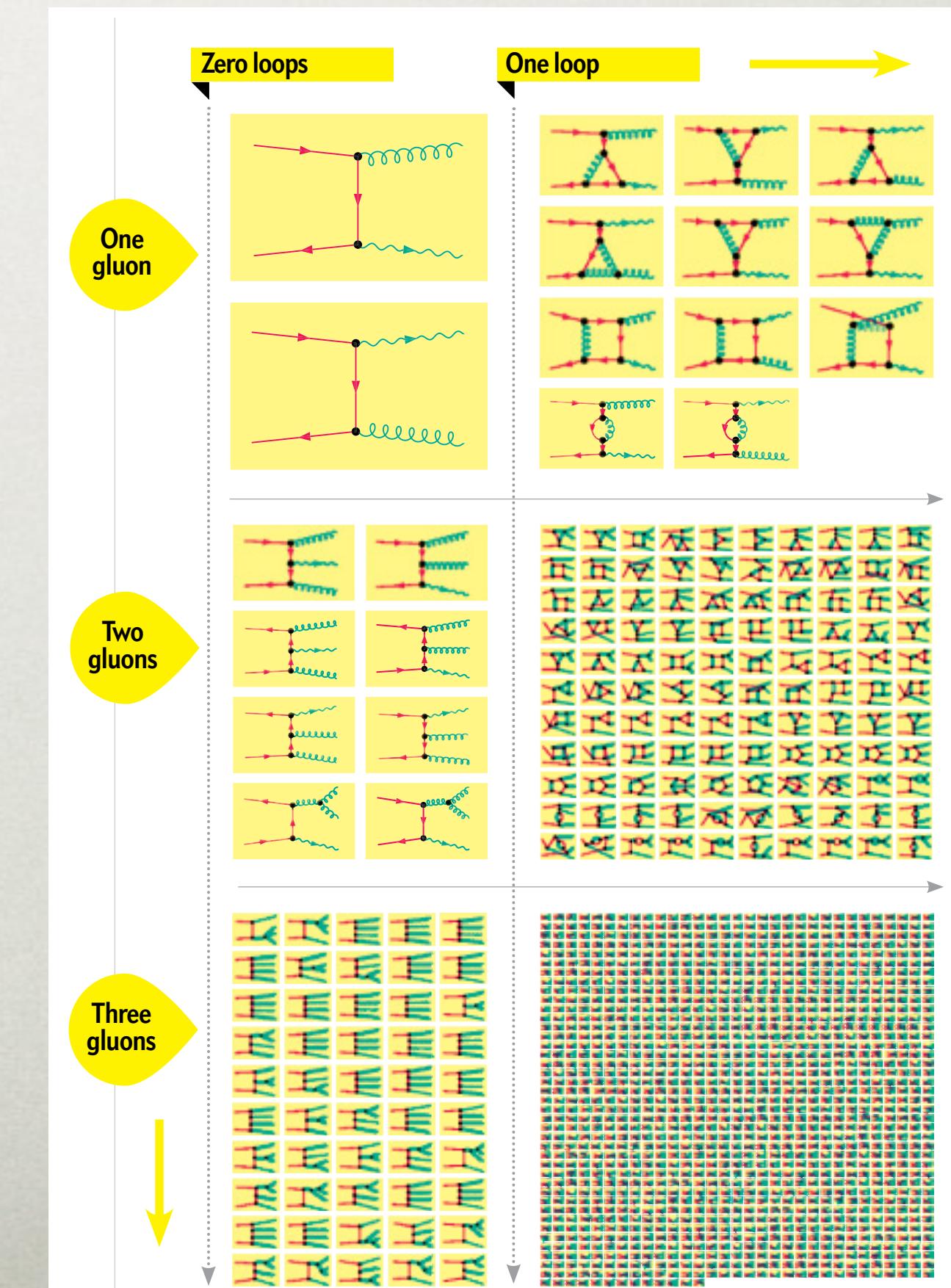


# Explosions of Complexity

- ♦ While ultimately correct, the Feynman expansion renders *all but the most trivial predictions—  
involving the fewest particles, at the  
lowest orders of perturbation—  
computationally *intractable*  
or theoretically *inscrutable**



[Bern, Dixon, Kosower, *Scientific American* (2012)]





# Needs (Once) Beyond Our Reach

- ♦ Background amplitudes crucial for e.g. colliders

**Supercollider physics** [*Rev.Mod.Phys.* **56** (1984)]

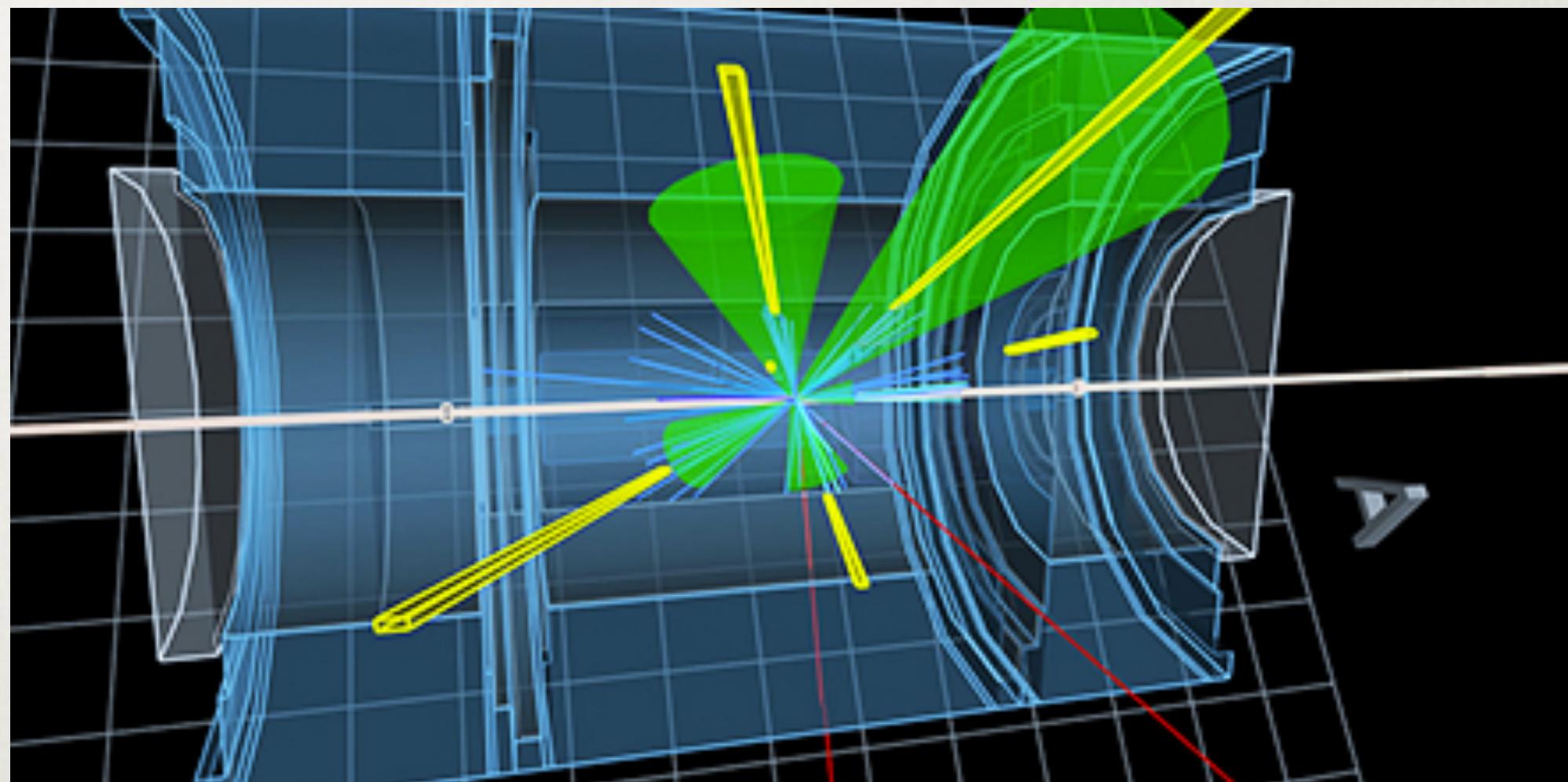
E. Eichten  
*Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510*

I. Hinchliffe  
*Lawrence Berkeley Laboratory, Berkeley, California 94720*

K. Lane  
*The Ohio State University, Columbus, Ohio 43210*

C. Quigg  
*Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, Illinois 60510*

Eichten *et al.* summarize the motivation for exploring the 1-TeV ( $=10^{12}$  eV) energy scale in elementary particle interactions and explore the capabilities of proton-(anti)proton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.



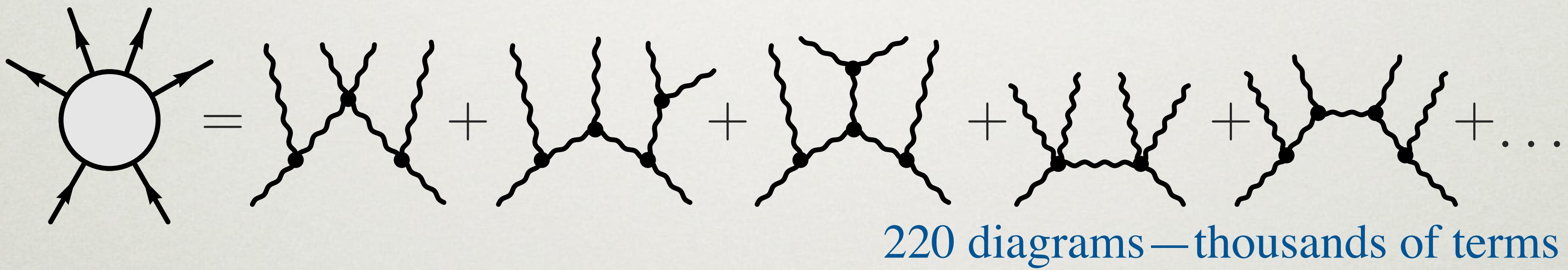
- ♦ Once considered *computationally intractable*

For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of  $W^+W^-$  pairs in their nonleptonic decays. The cross sections for the elementary two $\rightarrow$ four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



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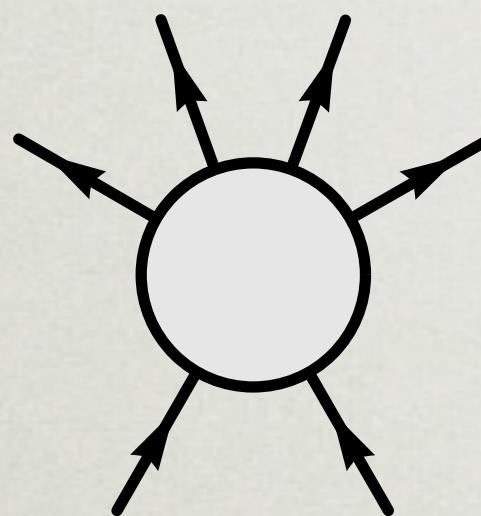
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# Needs (Once) Beyond Our Reach

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## THE CROSS SECTION FOR FOUR-GLUON PRODUCTION BY GLUON-GLUON FUSION

Stephen J. PARKE and T.R. TAYLOR

Fermi National Accelerator Laboratory, P.O. Box 500, Batavia, IL 60510 USA

Received 13 September 1985

The cross section for two-gluon to four-gluon scattering is given in a form suitable for fast numerical calculations.

[*Nucl.Phys. B269* (1985)]

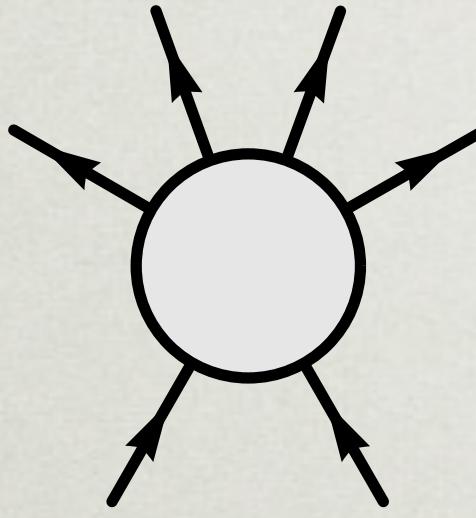
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For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of  $W^+W^-$  pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

# *Needs (Once) Beyond Our Reach*



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gluons. The cross section for the scattering of two gluons with momenta  $p_1, p_2$  into four gluons with momenta  $p_3, p_4, p_5, p_6$  is obtained from eq. (5) by setting  $I = 2$  and replacing the momenta  $p_3, p_4, p_5, p_6$  by  $-p_3, -p_4, -p_5, -p_6$ .

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$\begin{aligned} A_{\binom{2}{2}}(p_1, p_2, p_3, p_4, p_5, p_6) \\ = (\mathcal{D}^+, \mathcal{D}_\rho^+, \mathcal{D}_\sigma^+, \mathcal{D}_\tau^+) \binom{2}{2} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathcal{D} \\ \mathcal{D}_\rho \\ \mathcal{D}_\sigma \\ \mathcal{D}_\tau \end{pmatrix} \binom{2}{2}, \end{aligned} \quad (6)$$

where  $\mathcal{D}, \mathcal{D}_\rho, \mathcal{D}_\sigma$  and  $\mathcal{D}_\tau$  are 11-component complex vector functions of the momenta  $p_1, p_2, p_3, p_4, p_5$  and  $p_6$ , and  $K, K_\rho, K_\sigma$  and  $K_\tau$  are constant  $11 \times 11$  symmetric matrices. The vectors  $\mathcal{D}_\rho, \mathcal{D}_\sigma$  and  $\mathcal{D}_\tau$  are obtained from the vector  $\mathcal{D}$  by the permutations  $(p_3 \leftrightarrow p_4), (p_3 \leftrightarrow p_6)$  and  $(p_4 \leftrightarrow p_5, p_4 \leftrightarrow p_6)$ , respectively, of the momentum variables in  $\mathcal{D}$ . The individual components of the vector  $\mathcal{D}$  represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices  $K$ , which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to  $N^4(N^2-1)$  and  $N^2(N^2-1)$ , respectively ( $N$  is the number of colors,  $N=3$  for QCD):

$$K = \frac{1}{8}g^8 N^4 (N^2-1)^{K^{(4)}} + \frac{1}{8}g^8 N^2 (N^2-1)^{K^{(2)}}. \quad (7)$$

Here  $g$  denotes the gauge coupling constant. The matrices  $K^{(4)}$  and  $K^{(2)}$  are given in table 1. The vector  $\mathcal{D}$  is related to the thirty-three diagrams  $D^G(I=1-33)$  for two-gluon to four-scalar scattering, eleven diagrams  $D^F(I=1-11)$  for two-fermion to four-scalar scattering and sixteen diagrams  $D^S(I=1-16)$  for two-scalar to four-scalar scattering, in the following way:

$$\begin{aligned} \mathcal{D}_0 &= \frac{2s_{14}}{\sqrt{|s_{13}s_{45}s_{16}s_{46}|s_{23}s_{56}}} \{ t_{123}^2 C^G \cdot D_0^G - 4s_{14}t_{123}E(p_5+p_6, p_6)C^F \cdot D_0^F \\ &\quad - 2s_{14}G(p_5+p_6, p_5+p_6)C^S \cdot D_0^S \}, \\ \mathcal{D}_2 &= \frac{s_{56}}{s_{23}} C^G \cdot D_2^G, \end{aligned} \quad (8)$$

where the constant matrices  $C^G(11 \times 33)$ ,  $C^F(11 \times 11)$  and  $C^S(11 \times 16)$  are given in table 2. The Lorentz invariants  $s_y$  and  $t_{ykl}$  are defined as  $s_y = (p_i + p_j)^2$ ,  $t_{ykl} = (p_i + p_j + p_k)^2$  and the complex functions  $E$  and  $G$  are given by

$$\begin{aligned} E(p_n, p_i) &= \frac{1}{2} \{ (p_1 p_4)(p_2 p_3) - (p_1 p_3)(p_2 p_4) - (p_1 p_2)(p_3 p_4) + ie_{\mu\nu\rho\lambda} p_1^\mu p_2^\nu p_3^\rho p_4^\lambda \} / (p_1 p_4), \\ G(p_n, p_j) &= E(p_n, p_5)E(p_j, p_6), \end{aligned} \quad (9)$$

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TABLE 1

Matrices  $K(I, J)$  [ $I = 1-11, J = 1-11$ ].

Matrix $K^{(4)}$											Matrix $K^{(2)}$										
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
8	4	-2	2	-1	2	0	1	0	0	-1	0	0	0	0	0	0	0	0	0	3	3
4	8	-1	-1	0	2	1	0	1	-1	0	0	0	0	0	0	0	0	0	0	3	3
-2	-1	8	4	4	1	1	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0
2	1	4	8	2	-1	-1	4	1	1	1	0	0	0	0	0	0	0	0	0	0	0
-1	-1	4	2	8	1	2	4	-2	1	4	0	0	0	0	0	0	0	0	0	0	0
2	0	1	-1	1	8	4	-1	0	1	0	0	0	0	0	0	0	0	0	0	3	3
0	2	1	-1	2	4	8	-2	0	0	0	0	0	0	0	0	0	0	0	0	3	3
1	1	2	4	2	4	-1	-2	8	-1	-1	2	0	0	0	0	0	0	0	0	0	0
0	0	2	1	-2	0	0	1	8	4	-2	0	3	3	0	0	0	0	3	3	0	0
0	1	1	1	-1	1	0	-1	4	8	-1	3	3	0	0	0	0	3	3	0	0	0
-1	-1	2	1	4	0	0	2	-2	1	8	-3	0	0	0	0	0	0	0	0	0	0
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
0	0	0	0	1	1	0	1	1	0	-1	3	3	0	3	0	0	0	3	0	0	0
0	0	0	0	2	0	1	1	2	1	-2	3	3	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	0	1	1	1	0	3	0	3	0	0	3	0	0	0	0
1	2	0	1	0	1	2	2	0	0	2	0	3	0	0	0	0	0	0	0	3	3
1	0	1	0	1	4	2	0	0	0	-1	0	0	3	3	0	0	0	0	3	3	0
0	1	1	0	2	2	4	0	0	-2	0	0	0	0	0	0	0	0	0	0	3	0
1	1	2	2	2	4	0	0	0	0	0	3	0	0	0	0	0	0	0	0	3	0
1	2	0	0	0	0	0	0	2	-1	0	0	3	0	3	0	3	3	0	0	0	0
0	1	1	1	1	0	0	0	0	2	4	0	0	0	3	3	3	3	0	0	0	0
-1	-2	1	0	2	-1	-2	0	-1	0	4	0	0	0	3	3	0	0	0	0	0	0
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
4	2	0	2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	3	0
2	4	0	1	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	3	0
0	0	4	2	2	1	1	1	2	1	0	0	0	0	0	0	0	0	0	0	0	0
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1	0	1	2	0	0	0	0	0	1	2	0	0	0	0	0	0	3	3	0	0	0
0	1	1	1	0	0	0	0	0	2	4	0	0	0	3	3	3	3	0	0	0	0
0	0	2	1	4	1	2	2	0	0	-4	0	3	0	0	0	0	0	0	0	0	0
0	1	1	0	2	2	4	0	0	-2	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	2	0	0	1	-4	-2	4	-3	0	0	0	0	0	-3	0	0	0	0
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
0	1	-1	-1	1	1	0	1	2	0	0	3	3	0	0	0	3	3	0	0	0	0
1	0	2	-1	2	0	1	1	4	-2	0	3	3	0	0	0	3	3	0	0	0	0
-1	-2	0	0	0	1	1	1	-1	1	0	0	3	3	3	3	0	0	3	0	0	0
-1	-1	0	1	0	2	1	0	1	-1	0	0	3	3	3	3	0	0	3	0	0	0
1	2	0	0	1	-1	-1	2	0	-2	1	0	0	3	3	3	3	0	0	3	0	0
1	0	1	2	-1	0	1	-2	2	4	-1	3	3	0	0	0	3	3	0	0	0	0
0	1	1	-1	1	-1	0	1	4	8	-1	3	3	0	0	0	3	3	0	0	0	0
1	1	1	0	0	-2	-1	0	2	-2	0	0	0	3	3	3	3	0	0	3	0	0
2	4	-1	1	-2	2	4	2	1	0	-2	0	0	0	0	0	0	0	0	0	3	3
0	2	1	-1	2	4	8	-2	0	0	0	0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	1	-1	-1	0	-2	0	2	0	0	0	3	3	0	-3	0	0	0	0

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where  $\varepsilon$  is the totally antisymmetric tensor,  $\varepsilon_{xyz} = 1$ . For the future use, we define one more function,

$$F(p_1, p_2) = \{(p_1 p_4)(p_1 p_2) + (p_1 p_2)(p_1 p_4) - (p_1 p_2)(p_1 p_4)\}/(p_1 p_4). \quad (10)$$

Note that when evaluating  $A_0$  and  $A_2$  at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions  $E$ ,  $F$  and  $G$  on the momenta  $p_1, p_4, p_5, p_6$ .

The diagrams  $D_2^G$  are listed below:

$$\begin{aligned} D_2^G(1) &= \frac{\delta_2}{s_{14}s_{25}t_{36}} \{[(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] - [(p_2 - p_5)(p_3 + p_6)] \\ &\quad \times [(p_1 - p_4)(p_3 - p_6)] + [(p_2 + p_3)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_3)]\}, \\ D_2^G(2) &= \frac{1}{s_{25}s_{36}} \{2E(p_2 - p_5, p_3 - p_6) - 2E(p_3 - p_6, p_2 - p_5) + \delta_2[(p_2 - p_5)(p_3 - p_6)]\}, \\ D_2^G(3) &= \frac{4}{s_{25}s_{36}t_{125}} \{[(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)]E(p_2, p_3) \\ &\quad - [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6)]E(p_2, p_6) \\ &\quad - [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6)]E(p_3, p_3) \\ &\quad + [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6)]E(p_3, p_6) \\ &\quad \pm [p_1(p_2 - p_5)]E(p_3 - p_6, p_3 + p_6) - [p_4(p_3 - p_6)]E(p_2 + p_5, p_2 - p_3) \\ &\quad + \delta_2[p_1(p_2 - p_5)][p_4(p_3 - p_6)]\}, \\ D_2^G(4) &= \frac{-2}{s_{36}t_{125}} \{E(p_3 - p_6, p_3 + p_6) - \delta_2[p_4(p_3 - p_6)]\}, \\ D_2^G(5) &= \frac{-2}{s_{25}t_{125}} \{E(p_2 + p_5, p_2 - p_5) - \delta_2[p_1(p_2 - p_5)]\}, \\ D_2^G(6) &= \frac{\delta_2}{t_{125}}, \\ D_2^G(7) &= \frac{4}{s_{12}s_{36}t_{125}} \{[(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)]E(p_2, p_3) \\ &\quad - [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6)]E(p_2, p_6) - [p_4(p_3 - p_6)]E(p_2, p_2 - p_5)\}, \\ D_2^G(8) &= \frac{4}{s_{34}s_{25}t_{125}} \{[(p_1 + p_2 - p_5)(p_4 + p_3 - p_6)]E(p_2, p_3) \\ &\quad - [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6)]E(p_3, p_3) - [p_1(p_2 - p_5)]E(p_3 - p_6, p_3)\}, \end{aligned}$$

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$$\begin{aligned}
D_2^G(9) &= \frac{4}{s_{15}s_{36}f_{123}} \{ [(p_1 - p_2 + p_3)(p_4 + p_3 - p_6)]E(p_5, p_3) \\
&\quad - [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6)]E(p_5, p_6) + [p_4(p_3 - p_6)]E(p_5, p_2 - p_5) \}, \\
D_2^G(10) &= \frac{4}{s_{25}s_{46}f_{123}} \{ [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6)]E(p_2, p_6) \\
&\quad - [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6)]E(p_5, p_6) + [p_1(p_2 - p_5)]E(p_3 - p_6, p_6) \}, \\
D_2^G(11) &= \frac{\delta_2}{s_{36}f_{124}} [s_{35} - s_{56} + s_{36}], \\
D_2^G(12) &= \frac{-\delta_2}{s_{36}f_{145}} [s_{23} - s_{26} - s_{36}], \\
D_2^G(13) &= \frac{\delta_2}{s_{14}s_{36}f_{124}} [s_{12} - s_{24}][s_{35} - s_{56} + s_{36}], \\
D_2^G(14) &= \frac{\delta_2}{s_{14}s_{36}f_{145}} [s_{15} - s_{45}][s_{23} - s_{26} - s_{36}], \\
D_2^G(15) &= \frac{\delta_2}{s_{14}s_{36}} (p_1 - p_4)(p_3 - p_6), \\
D_2^G(16) &= \frac{-4}{s_{12}s_{36}f_{124}} [s_{35} - s_{56} + s_{36}]E(p_2, p_2), \\
D_2^G(17) &= \frac{4}{s_{36}s_{45}f_{145}} [s_{23} - s_{26} - s_{36}]E(p_5, p_5), \\
D_2^G(18) &= \frac{-4}{s_{12}s_{36}f_{45}} [2(p_1 + p_2)(p_3 - p_6) - s_{36}]E(p_2, p_5), \\
D_2^G(19) &= \frac{-2}{s_{12}s_{36}} E(p_2, p_3 - p_6), \\
D_2^G(20) &= \frac{2}{s_{36}s_{45}} E(p_3 - p_6, p_5), \\
D_2^G(21) &= \frac{-4}{s_{25}s_{34}f_{134}} [s_{26} - s_{56} + s_{25}]E(p_3, p_5), \\
D_2^G(22) &= \frac{4}{s_{16}s_{25}f_{146}} [s_{23} - s_{35} - s_{25}]E(p_6, p_6), \\
D_2^G(23) &= \frac{4}{s_{16}s_{25}s_{34}} [2(p_1 + p_6)(p_2 - p_5) + s_{25}]E(p_6, p_3),
\end{aligned}$$

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$$\begin{aligned}
D_2^G(24) &= \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3), \\
D_2^G(25) &= \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5), \\
D_2^G(26) &= \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5), \\
D_2^G(27) &= \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6), \\
D_2^G(28) &= \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5), \\
D_2^G(29) &= \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3), \\
D_2^G(30) &= \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125}] E(p_2, p_3), \\
D_2^G(31) &= \frac{4}{s_{15}s_{46}t_{125}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}] E(p_2, p_6), \\
D_2^G(32) &= \frac{4}{s_{15}s_{34}t_{125}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3), \\
D_2^G(33) &= \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_3)(p_4 - p_3 + p_6) - t_{125}] E(p_5, p_6),
\end{aligned}
\quad (6)$$

where  $\delta_2 = 1$ .

The diagrams  $D_0^G$  are obtained from  $D_2^G$  by replacing  $\delta_2$  by  $\delta_0 = 0$  and the function  $E(p_a, p_b)$  by  $G(p_a, p_b)$ .

The diagrams  $D_0^F$  are listed below:

$$\begin{aligned}
D_0^F(1) &= \frac{4}{s_{15}s_{34}t_{125}} \{ F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) \\
&\quad + [F(p_6, p_3) + s_{34}]E(p_5, p_3) \}, \\
D_0^F(2) &= \frac{-4}{s_{16}s_{25}s_{34}} \{ [F(p_6, p_2) + \frac{1}{2}s_{16}]E(p_3, p_5) \\
&\quad + [F(p_2, p_3) + \frac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5) \}, \\
D_0^F(3) &= \frac{4}{s_{15}s_{34}t_{125}} \{ F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5) \\
&\quad - [F(p_3, p_6) - \frac{1}{2}s_{36} - \frac{1}{2}s_{34} + \frac{1}{2}s_{46}]E(p_5, p_3) \},
\end{aligned}$$

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$$\begin{aligned}
D_0^F(4) &= \frac{4}{s_{25}s_{34}t_{125}} \{ F(p_2, p_3)E(p_5, p_5) - F(p_5, p_3)E(p_2, p_5) \\
&\quad + [F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_3, p_5) \}, \\
D_0^F(5) &= \frac{2}{s_{16}s_{25}t_{146}} [s_{35} - s_{23} + s_{25}]E(p_6, p_5), \\
D_0^F(6) &= \frac{2}{s_{25}s_{34}t_{134}} [s_{56} - s_{26} - s_{25}]E(p_3, p_5), \\
D_0^F(7) &= \frac{4}{s_{25}s_{36}t_{125}} \{ [F(p_5, p_2) - \frac{1}{2}s_{25} - \frac{1}{2}s_{12} + \frac{1}{2}s_{15}]E(p_3, p_5) \\
&\quad + [F(p_2, p_3) + \frac{1}{4}t_{125}]E(p_5, p_5) - [F(p_5, p_3) + \frac{1}{4}t_{125}]E(p_2, p_5) \}, \\
D_0^F(8) &= \frac{1}{s_{14}s_{36}} E(p_3 - p_6, p_5), \\
D_0^F(9) &= \frac{2}{s_{14}s_{36}t_{124}} [s_{35} - s_{56} + s_{36}]E(p_2, p_5), \\
D_0^F(10) &= \frac{2}{s_{14}s_{36}t_{145}} [s_{23} - s_{26} - s_{36}]E(p_5, p_5), \\
D_0^F(11) &= \frac{1}{2s_{14}s_{25}s_{36}} \{ [s_{23} + s_{35} - s_{26} - s_{36}]E(p_2 - p_5, p_5) \\
&\quad - [s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + 
\end{aligned}$$

The diagrams  $D_0^S$  are listed below:

$$\begin{aligned}
D_0^S(1) &= \frac{1}{s_{25}s_{36}t_{125}} [s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}], \\
D_0^S(2) &= \frac{1}{s_{14}s_{36}t_{124}} [s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}], \\
D_0^S(3) &= \frac{1}{s_{14}s_{36}t_{145}} [s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}], \\
D_0^S(4) &= \frac{1}{s_{15}s_{36}t_{125}} [s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}], \\
D_0^S(5) &= \frac{1}{s_{15}s_{34}t_{156}} [s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}], \\
D_0^S(6) &= \frac{1}{s_{15}s_{34}t_{125}} [s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}],
\end{aligned}$$

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$$\begin{aligned}
D_0^S(7) &= \frac{1}{s_{25}s_{34}t_{125}} [s_{36} - s_{46} + s_{34}][s_{12} - s_{15} - s_{23}], \\
D_0^S(8) &= \frac{1}{s_{16}s_{25}t_{146}} [s_{25} + s_{35} - s_{23}][s_{14} - s_{46} + s_{16}], \\
D_0^S(9) &= \frac{1}{s_{25}s_{34}t_{134}} [s_{14} + s_{34} - s_{13}][s_{26} - s_{56} + s_{23}], \\
D_0^S(10) &= \frac{1}{s_{25}s_{36}} (p_2 - p_5)(p_3 - p_6), \\
D_0^S(11) &= \frac{1}{s_{14}s_{36}} (p_1 - p_4)(p_3 - p_6), \\
D_0^S(12) &= \frac{1}{s_{16}s_{25}} (p_6 - p_1)(p_2 - p_5), \\
D_0^S(13) &= \frac{1}{s_{15}s_{34}} (p_5 - p_1)(p_3 - p_4), \\
D_0^S(14) &= \frac{1}{s_{25}s_{34}} (p_2 - p_5)(p_3 - p_4), \\
D_0^S(15) &= \frac{1}{s_{14}s_{25}s_{36}} \{ [(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)] \\
&\quad + [(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] \\
&\quad + [(p_1 + p_4)(p_2 - p_3)][(p_1 - p_4)(p_3 - p_6)] \}, \\
D_0^S(16) &= \frac{2}{s_{16}s_{34}s_{25}} \{ [(p_2 - p_5)(p_3 + p_4)][(p_1 - p_6)(p_3 - p_4)] \\
&\quad + [(p_1 + p_4)(p_3 - p_4)][(p_1 - p_6)(p_2 - p_5)] \\
&\quad + [(p_1 - p_6)(p_2 + p_3)][(p_3 - p_4)(p_2 - p_5)] \}. \tag{13}
\end{aligned}$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams  $D$  are calculated by using eqs. (11)–(13). The result is substituted to eq. (8) to obtain the vectors  $\mathcal{D}_0$  and  $\mathcal{D}_2$ . After generating the vectors  $\mathcal{D}_{0_\alpha}$ ,  $\mathcal{D}_{0_\beta}$ ,  $\mathcal{D}_{0_\gamma}$ ,  $\mathcal{D}_{2_\alpha}$ ,  $\mathcal{D}_{2_\beta}$ , and  $\mathcal{D}_{2_\gamma}$  by the appropriate permutations of momenta, eq. (6) is used to obtain the functions  $A_0$  and  $A_2$ . Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

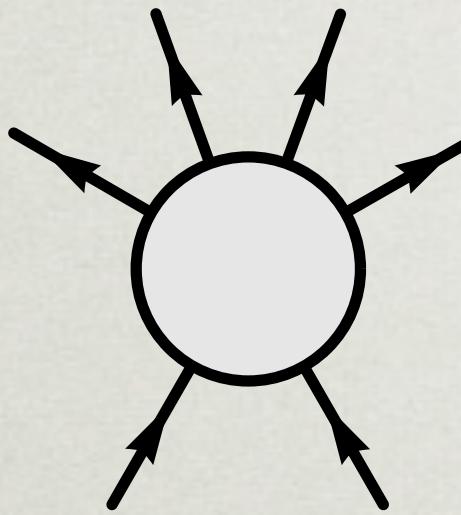
Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

# *Needs (Once) Beyond Our Reach*

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- ◆ Background amplitudes crucial for e.g. colliders



Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

We thank Keith Ellis, Chris Quigg and especially, Estia Eichten for many useful discussions and encouragement during the course of this work. We acknowledge the hospitality of Aspen Center for Physics, where this work was being completed

$$\begin{aligned}
D_2^G(15) &= \frac{-4}{s_{14} s_{36}} [p_1 - p_4] (p_3 - p_6), \\
D_2^G(16) &= \frac{-4}{s_{12} s_{36} t_{124}} [s_{35} - s_{56} + s_{36}] E(p_2, p_2), \\
D_2^G(17) &= \frac{4}{s_{36} s_{45} t_{143}} [s_{23} - s_{26} - s_{36}] E(p_5, p_5), \\
D_2^G(18) &= \frac{-4}{s_{12} s_{36} s_{45}} [2(p_1 + p_2)(p_3 - p_6) - s_{36}] E(p_2, p_5), \\
D_2^G(19) &= \frac{-2}{s_{12} s_{36}} E(p_2, p_3 - p_6), \\
D_2^G(20) &= \frac{2}{s_{36} s_{45}} E(p_3 - p_6, p_5), \\
D_2^G(21) &= \frac{-4}{s_{25} s_{34} t_{134}} [s_{26} - s_{56} + s_{25}] E(p_3, p_3), \\
D_2^G(22) &= \frac{4}{s_{16} s_{23} t_{146}} [s_{23} - s_{35} - s_{25}] E(p_6, p_6), \\
D_2^G(23) &= \frac{4}{s_{16} s_{23} t_{146}} [2(p_1 + p_6)(p_2 - p_3) + s_{25}] E(p_6, p_3),
\end{aligned}$$

$$\begin{aligned}
D_0^S(1) &= \frac{1}{2s_{14}s_{25}s_{36}} \{ [s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5) \\
&\quad - [s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5) \}. \\
(12) \\
\text{The diagrams } D_0^S \text{ are listed below:} \\
D_0^S(1) &= \frac{1}{s_{25}s_{36}\ell_{125}} [s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}], \\
D_0^S(2) &= \frac{1}{s_{14}s_{36}\ell_{124}} [s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}], \\
D_0^S(3) &= \frac{1}{s_{14}s_{36}\ell_{145}} [s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}], \\
D_0^S(4) &= \frac{1}{s_{15}s_{56}\ell_{125}} [s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}], \\
D_0^S(5) &= \frac{1}{s_{15}s_{34}\ell_{156}} [s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}], \\
D_0^S(6) &= \frac{1}{s_{15}s_{34}\ell_{156}} [s_{46} - s_{34} - s_{36}][s_{13} - s_{25} - s_{15}],
\end{aligned}$$

$$S_5 = \frac{1}{s_{14}s_{25}s_{36}} \left\{ [(p_2 + p_5)(p_3 - p_6)][(p_1 - p_4)(p_2 - p_5)] \right. \\ \left. + [(p_2 - p_5)(p_3 - p_6)][(p_1 - p_4)(p_3 + p_6)] \right. \\ \left. + [(p_1 + p_4)(p_2 - p_3)][(p_1 - p_4)(p_3 - p_6)] \right\}, \\ S_5 = \frac{2}{s_{16}s_{34}s_{25}} \left\{ [(p_2 - p_5)(p_3 + p_4)][(p_1 - p_6)(p_3 - p_4)] \right. \\ \left. + [(p_1 + p_6)(p_3 - p_4)][(p_1 - p_6)(p_2 - p_5)] \right. \\ \left. + [(p_1 - p_6)(p_2 + p_3)][(p_3 - p_4)(p_2 - p_5)] \right\}. \quad (13)$$

preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams  $D$  are calculated from eqs. (11)–(13). The result is substituted to eq. (8) to obtain the vectors  $\mathcal{D}_0$ ,  $\mathcal{D}_1$ ,  $\mathcal{D}_2$ ,  $\mathcal{D}_{12}$ ,  $\mathcal{D}_{21}$  and  $\mathcal{D}_3$ , by the appropriate relations of momenta, eq. (6) is used to obtain the functions  $A_0$  and  $A_1$ . Finally, the cross section is calculated by using eq. (5). The FORTRAN 5 program in such a scheme generates ten Monte Carlo points in less than a second on a CDC CYBER 175/975.

In view of the complexity of the final result, it is very important to have some reliable procedures available for numerical calculations. Usually in QCD, the multi-



# Discovery of Shocking Simplicity

- ♦ Within six months, Parke-Taylor stumbled on a simple guess  
—unquestionably a *theorist's delight*:

$$\text{Diagram: } n \text{-point vertex with gluons } 1, 2, \dots, n \text{ entering from the left.} \\ = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle}$$

Amplitude for  $n$ -Gluon Scattering [PRL 56 (1986)]

Stephen J. Parke and T. R. Taylor  
Fermi National Accelerator Laboratory, Batavia, Illinois 60510  
(Received 17 March 1986)

A nontrivial squared helicity amplitude is given for the scattering of an arbitrary number of gluons to lowest order in the coupling constant and to leading order in the number of colors.

$$p_a^\mu \equiv \sigma_{\alpha\dot{\alpha}}^\mu \lambda_a^\alpha \tilde{\lambda}_a^{\dot{\alpha}}$$

$$\langle a b \rangle \equiv \det(\lambda_a, \lambda_b)$$

$$[a b] \equiv \det(\tilde{\lambda}_a, \tilde{\lambda}_b)$$

[van der Waerden (1929)]

# Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Bern, Dixon, Dunbar, Kosower (1994)]

$$\text{Diagram} = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times \\ \left\{ 1 + \sum_{a < b} \text{Diagram} + \dots \right\}$$

The diagram consists of a central black circle with four outgoing arrows labeled 1, 2, 3, and 4. Below it is a series of dots indicating continuation. To the right is an equals sign followed by a fraction. The numerator is  $\langle 1 2 \rangle^4$ . The denominator is the product of scattering amplitudes  $\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle$ . To the right of the fraction is a multiplication sign ( $\times$ ). Below the fraction is a curly brace grouping the term 1 and the sum. The sum is preceded by a plus sign and followed by three dots. The term inside the brace is a diagram showing a horizontal line with two vertices. The top vertex is a white circle connected to a line labeled  $n_a$ . The bottom vertex is a black circle connected to a line labeled  $b$ . A wavy line connects the two vertices.



# Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

[Arkani-Hamed, JB, Cachazo, Trnka (2010)]

$$\text{Diagram} = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times$$
$$\left\{ 1 + \sum_{a < b} \text{Diagram} + \sum_{a < b < c < d} \text{Diagram} + \dots \right\}$$

The diagram consists of a central black circle with four outgoing arrows labeled 1, 2, 3, and 4. Below it, a series of terms are shown. The first term is 1. The second term is a sum over pairs (a, b) where a < b, showing a diagram with two black circles connected by a horizontal line, with a wavy line connecting them to two white circles labeled a and b. The third term is a sum over quadruples (a, b, c, d) where a < b < c < d, showing a diagram with four black circles at the corners of a square, each connected to a white circle labeled a, b, c, or d, with wavy lines connecting the corners.

# Perturbations of Parke/Taylor's Guess



- ◆ What about beyond the leading order of approximation?

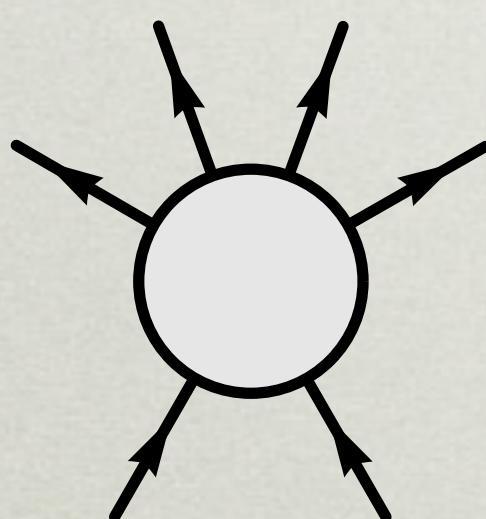
[Arkani-Hamed, JB, Cachazo, Trnka (2011)]

$$\begin{aligned}
 & \text{Diagram of a circular vertex with four external legs labeled } 1, 2, 3, 4 \text{ (with arrows indicating flow).} \\
 & = \frac{\langle 1 2 \rangle^4}{\langle 1 2 \rangle \langle 2 3 \rangle \langle 3 4 \rangle \langle 4 5 \rangle \cdots \langle n 1 \rangle} \times \\
 & \left\{ 1 + \sum_{a < b} \text{Diagram showing two nodes } a \text{ and } b \text{ connected by a wavy line, with other nodes } c, d \text{ and lines.} \right. \\
 & + \sum_{a < b < c < d} \text{Diagram showing a more complex network of nodes } a, b, c, d \text{ and wavy lines.} \\
 & + \sum_{\substack{a < b \leq c < \\ d \leq e < f}} \text{Diagram showing a network of nodes } a, b, c, d, e, f \text{ and wavy lines.} \\
 & + \sum_{\substack{a \leq b < c < \\ d \leq e < f}} \text{Diagram showing a network of nodes } a, b, c, d, e, f \text{ and wavy lines.} \\
 & \left. + \dots \right\}
 \end{aligned}$$



# What Form do Observables Take?

- ♦ But what about after **regularization** and **loop integration**?  
What is the **mathematical form** of the predictions made by QFT?



## The Two-Loop Hexagon Wilson Loop in $\mathcal{N} = 4$ SYM

**Vittorio Del Duca**

*PH Department, TH Unit, CERN CH-1211, Geneva 23, Switzerland  
INFN, Laboratori Nazionali Frascati, 00044 Frascati (Roma), Italy  
E-mail: vittorio.del.duca@cern.ch*

**Claude Duhr**

*Institute for Particle Physics Phenomenology, University of Durham  
Durham, DH1 3LE, U.K.  
E-mail: claude.duhr@durham.ac.uk*

**Vladimir A. Smirnov**

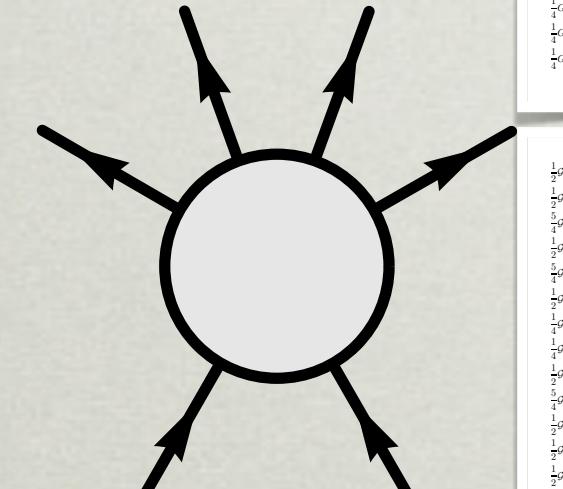
*Nuclear Physics Institute of Moscow State University  
Moscow 119992, Russia  
E-mail: smirnov@theory.sinp.msu.ru*

[Del Duca, Duhr, Smirnov (2010)]



# What Form do Observables Take?

♦ But what about after **regularization** and **loop integration**?  
What is the **mathematical form** of the predictions made by QFT?



**Diagram 1**

Figure 1 shows the mathematical form of observables after regularization and loop integration. The figure consists of two columns of five panels each, with a central title "Diagram 1".

**Left Column:** Panels 1 through 5 show the mathematical expression for the observable  $G$  (Generalized Green's Function). The expressions involve various terms involving  $G$ ,  $\delta$ , and  $\epsilon$  functions, along with complex fractions and powers of  $u_1$  and  $u_2$ .

**Right Column:** Panels 1 through 5 show the mathematical expression for the observable  $H$  (Hadamard function). Similar to the left column, it involves  $H$ ,  $\delta$ , and  $\epsilon$  functions, along with complex fractions and powers of  $u_1$  and  $u_2$ .

**Central Title:** "Diagram 1"

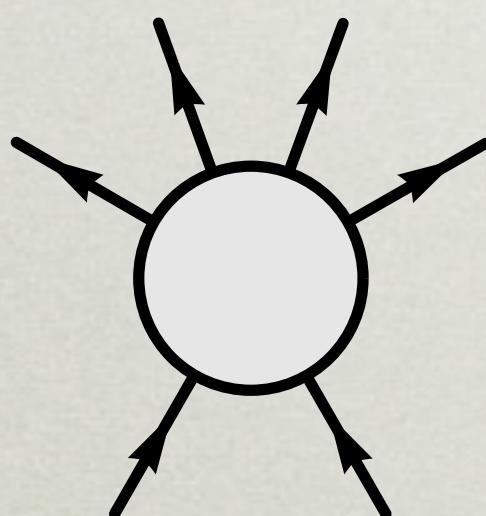
**References:**

- [1] C. Anastasiou, Z. Bern, J. Dittmaier and D.A. Kosower, "Plane amplitudes in maximally supersymmetric Yang-Mills theory," *Phys. Rev. Lett.* **91** (2003) 251602 [[arXiv/hep-th/0306049](#)].
- [2] Z. Bern and V. van Nieuwenhuizen, "Two-loop four-gluon amplitudes in  $N=4$  Yang-Mills," *Phys. Lett. B* **397** (1997) 423 [[hep-th/9709124](#)].
- [3] Z. Bern, M. Czakon, D.A. Kosower, R. Roiban and V.A. Smirnov, "Two-loop iteration of five-point  $N=4$  supersymmetric Yang-Mills amplitudes," *Phys. Rev. Lett.* **99** (2007) 181601 [[arXiv/hep-th/0701085](#)].
- [4] F. Cachazo, M. Spradlin and A. Volovich, "Ramanujan structure within the four-particle tree-level scattering amplitude," *Phys. Rev. Lett.* **91** (2003) 081601 [[hep-th/0305132](#)].
- [5] Z. Bern, J. Dittmaier and V. Smirnov, "The one-loop pentagon to higher orders in epsilon," *JHEP* **1009** (2009) 042 [[arXiv/0905.0997](#) [hep-ph]].
- [6] V. Del Duca, E. de la Ossa and E. P.优 Gómez, "The five-gluon amplitude in the loop expansion," *JHEP* **0909** (2009) 044 [[arXiv/0907.0794](#) [hep-ph]].
- [7] L.F. Alday, J.M.孟, J. M. Perado and J. M. Staudacher, "Scattering into the fifth dimension of  $N=4$  superspace Yang-Mills," *JHEP* **1010** (2010) 027 [[arXiv/0804.1699](#) [hep-th]].
- [8] Z. Bern, J. M.孟, J. M. Perado and J. M. Staudacher, "Five-gluon loop three-loop four-gluon amplitude in  $N=4$  SYM: representation and Regge limits," *arXiv:1001.1358* [hep-th].
- [9] Z. Bern, J. M.孟 and V.A. Smirnov, "Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loop and beyond," *Phys. Rev. D* **72** (2005) 085001 [[hep-th/0412108](#)].
- [10] Z. Bern, J. M.孟, D.A. Kosower, R. Roiban, M. Spradlin, C. Verguts and A. Volovich, "The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory," *Phys. Rev. D* **73** (2006) 024007 [[hep-th/0510169](#)].
- [11] L.F. Alday, J. M.孟 and V.A. Smirnov, "Four-gluon scattering amplitudes via AdS/CFT," *JHEP* **0711** (2007) 055 [[arXiv/0710.1699](#) [hep-th]].
- [12] J.M.孟, D. Ceballos, J. Henn, G. P. Korchemsky and E. Sokatchev, "The hexagon Wilson loop amplitude and its relation to gluon amplitudes," *Phys. Lett. B* **645** (2007) 173 [[arXiv/0707.4138](#) [hep-th]].
- [13] J. M.孟, L. N. Lipatov and A. Sabio Vera, "MHV-Pomeron: Regge gluon amplitudes at high energy," *JHEP* **0909** (2009) 020 [[arXiv/0907.2809](#) [hep-ph]].
- [14] J. M.孟, L. N. Lipatov and A. Sabio Vera, "Supersymmetric Yang-Mills scattering amplitudes at high energies: Regge-like contributions," *arXiv:0907.2810 [hep-ph].*
- [15] R.N. Schabinger, "The Imaginary Part of the  $N=8$  Super-Yang-Mills Two-Loop Six-Point MHV Amplitude in Multi-Regge Kinematics," *JHEP* **0911** (2009) 108 [[arXiv/0910.3933](#) [hep-ph]].



# What Form do Observables Take?

- ♦ But what about after **regularization** and **loop integration**?  
What is the *mathematical form* of the predictions made by QFT?



## Classical Polylogarithms for Amplitudes and Wilson Loops

A. B. Goncharov,<sup>1</sup> M. Spradlin,<sup>2</sup> C. Vergu,<sup>2</sup> and A. Volovich<sup>2</sup>

<sup>1</sup>*Department of Mathematics, Brown University, Box 1917, Providence, Rhode Island 02912, USA*

<sup>2</sup>*Department of Physics, Brown University, Box 1843, Providence, Rhode Island 02912, USA*

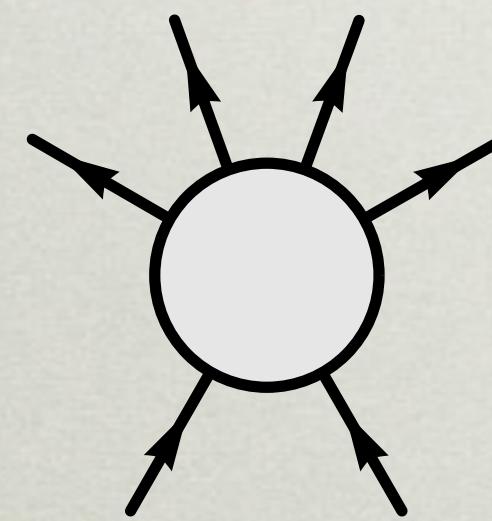
We present a compact analytic formula for the two-loop six-particle maximally helicity violating remainder function (equivalently, the two-loop lightlike hexagon Wilson loop) in  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory in terms of the classical polylogarithm functions  $\text{Li}_k$  with cross-ratios of momentum twistor invariants as their arguments. In deriving our formula we rely on results from the theory of motives.

$$\begin{aligned} R(u_1, u_2, u_3) = & \sum_{i=1}^3 \left( L_4(x_i^+, x_i^-) - \frac{1}{2} \text{Li}_4(1 - 1/u_i) \right) \\ & - \frac{1}{8} \left( \sum_{i=1}^3 \text{Li}_2(1 - 1/u_i) \right)^2 + \frac{J^4}{24} + \frac{1}{2} \zeta_2 (J^2 + \zeta_2) \end{aligned}$$



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What is the ***mathematical form*** of the predictions made by QFT?

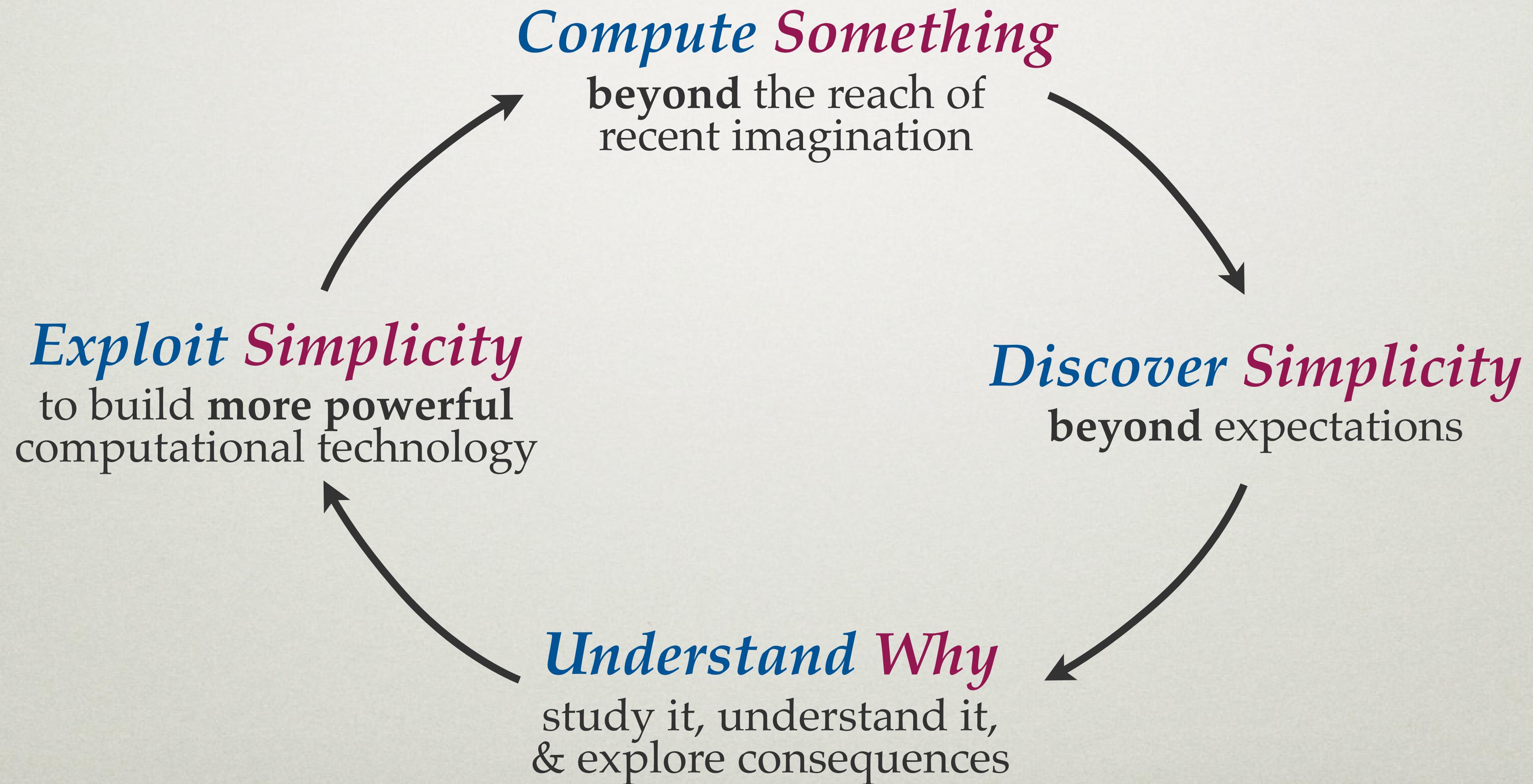


**State of the art:**  
6-point (N)MHV @ (6) 7 loops(!!!)  
7-point (N)MHV @ 4 loops (symbol-level)

[Dixon, *et al* (2019);...]



# *Amplitudes: a Virtuous Cycle*





# What Form do Observables Take?

- ♦ In a general (say, 4d) QFT, it would have *recently been* expected by “experts” that observables took the following general form:

$$\mathcal{A} = \mathcal{A}^{\text{tree}} + \hbar \mathcal{A}^{(L=1)} + \hbar^2 \mathcal{A}^{(L=2)} + \dots + \hbar^L \mathcal{A}^{(L)} + \dots$$

(general dimension  $d$ :  $2 \mapsto \lfloor d/2 \rfloor$ )

$$\begin{aligned} & \text{rational} + \begin{pmatrix} \text{weight-2} \\ \text{polylogs} \end{pmatrix} + \begin{pmatrix} \text{weight-4} \\ \text{polylogs} \end{pmatrix} + \dots + \begin{pmatrix} \text{weight-}2L \\ \text{polylogs} \end{pmatrix} + \dots \\ & + \begin{pmatrix} \text{weight-1} \\ \text{polylogs} \end{pmatrix} + \vdots + \dots + \vdots \\ & + \text{rational} + \begin{pmatrix} \text{weight-1} \\ \text{polylogs} \end{pmatrix} + \dots + \vdots \\ & \qquad \qquad \qquad + \text{rational} \end{aligned}$$

**coefficients:**

*leading singularities*



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$$+ \begin{pmatrix} \text{weight-1} \\ \text{polylogs} \end{pmatrix} + \vdots + \dots + \vdots \quad \text{planar } \mathcal{N}=4?$$
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coefficients:

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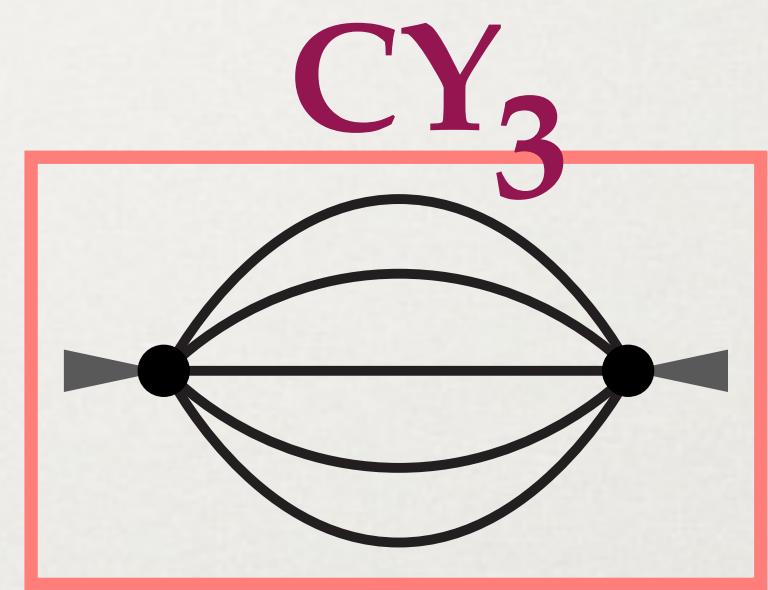
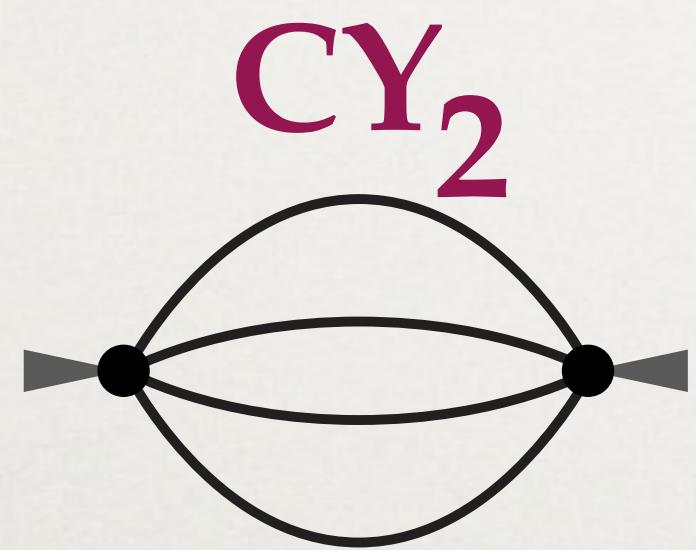
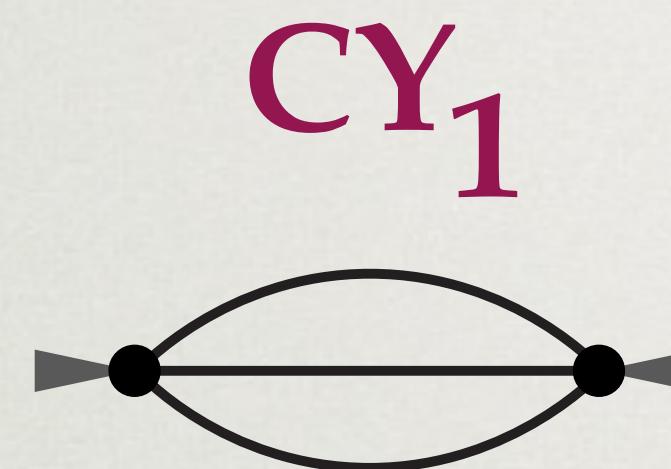
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- ♦ Unfortunately, many pesky counterexamples were to be found:

[Källén, Sabry; Bloch, Kerr, Vanhove; Broadhurst;...]

[Doran, Harder, Thompson (2019)]

sunrises:



...

contributes to electron ( $g-2$ )

} 2d, massive



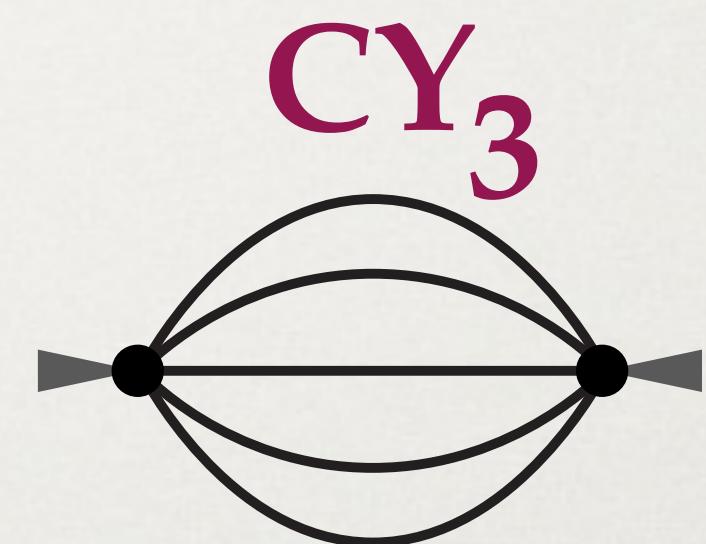
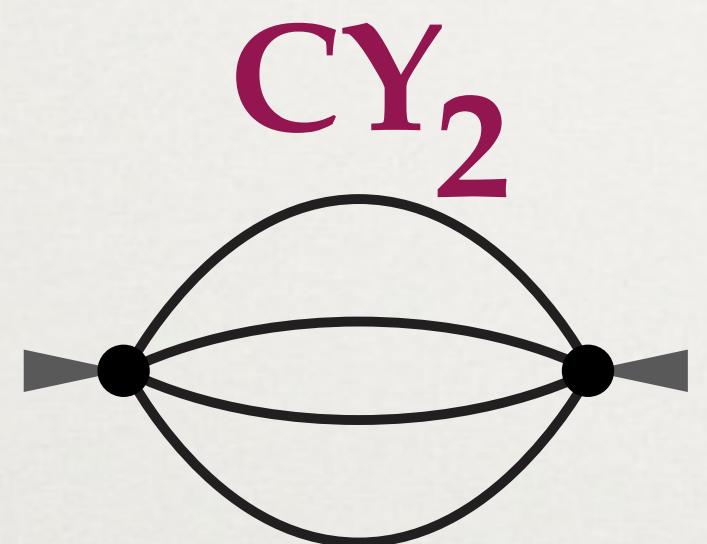
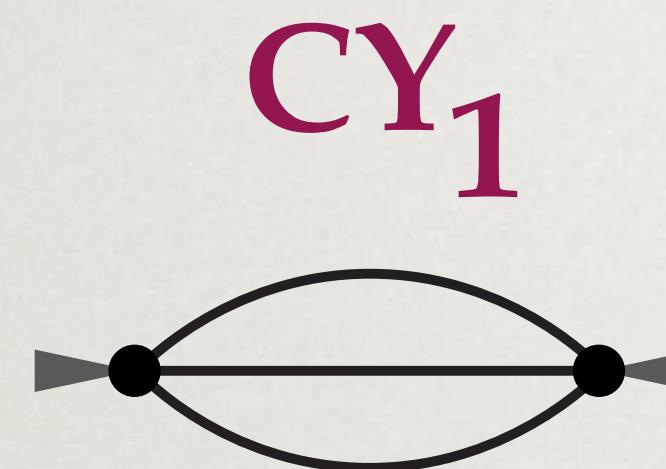
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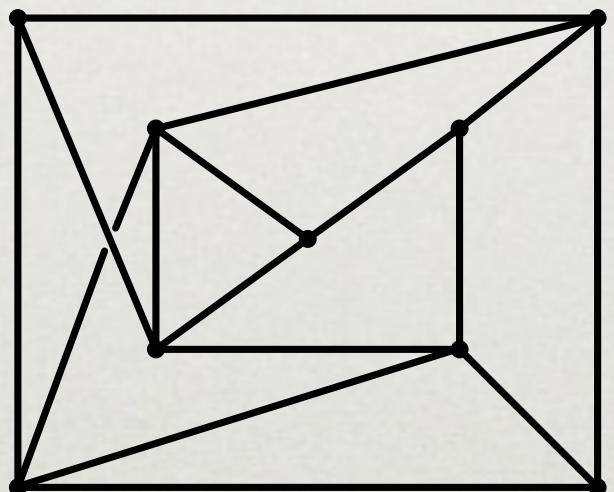
[Doran, Harder, Thompson (2019)]

sunrises:



...

2d, massive



an 8-loop vacuum graph  
evaluating to a **K3 period**

4d, massless

[Brown, Schnetz (2011)]



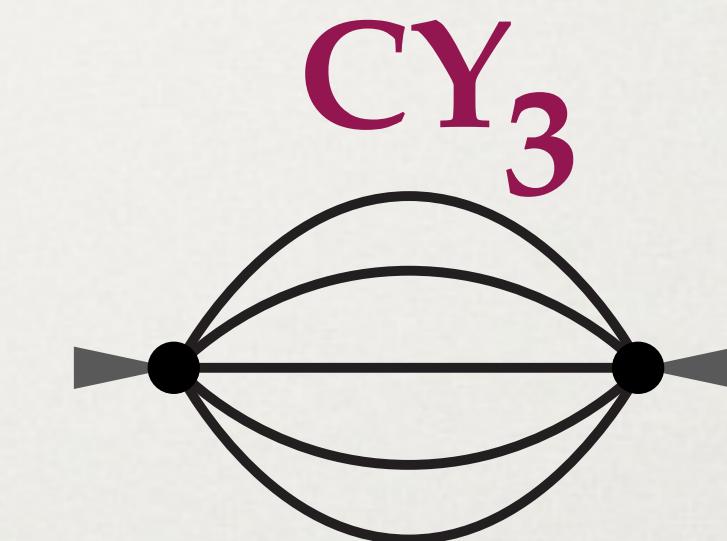
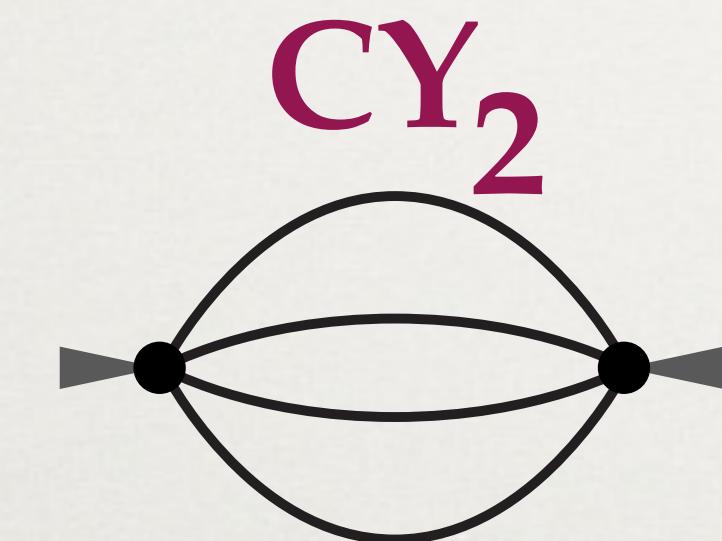
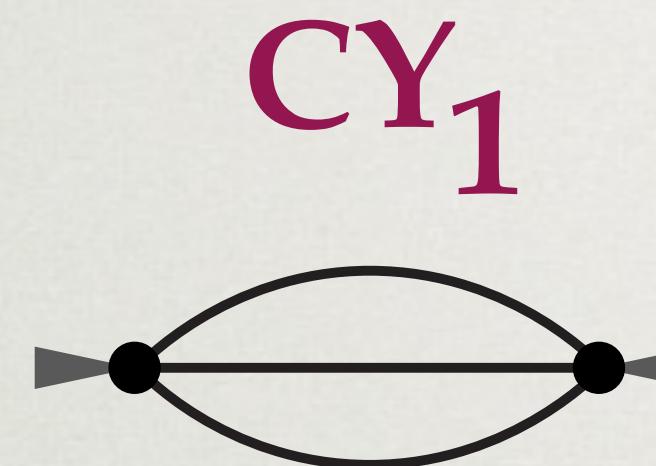
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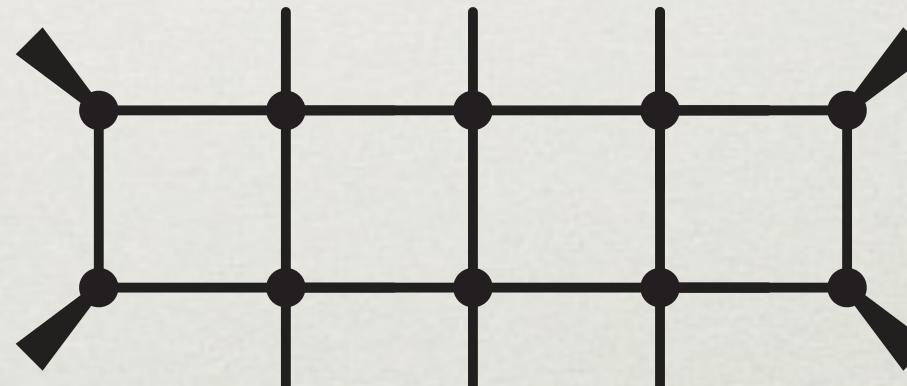
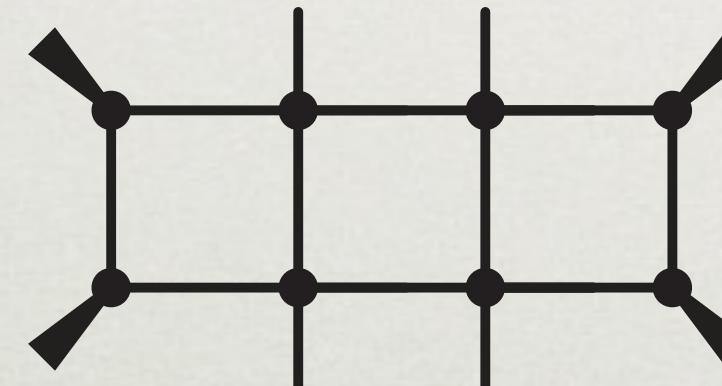
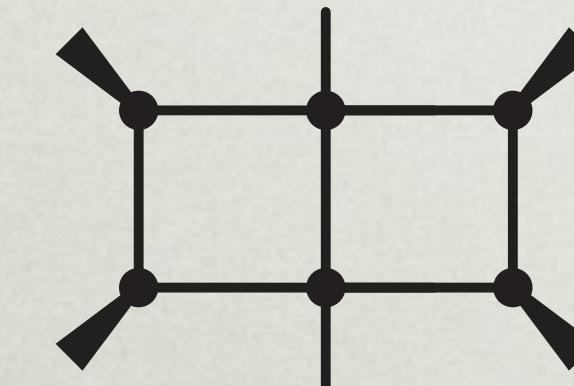
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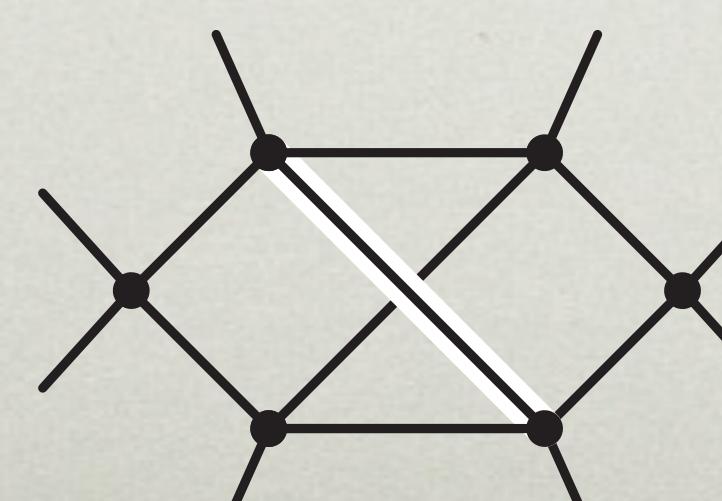
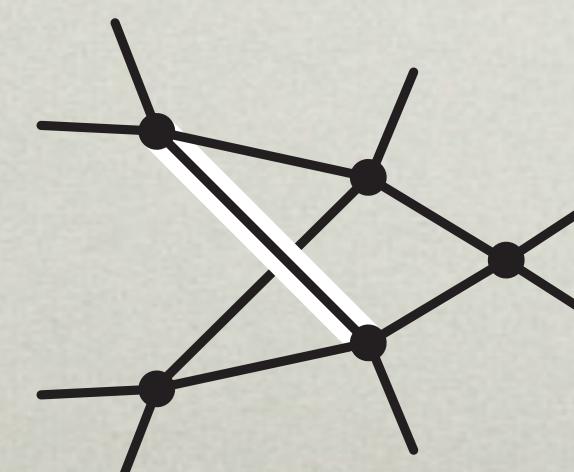
traintracks:



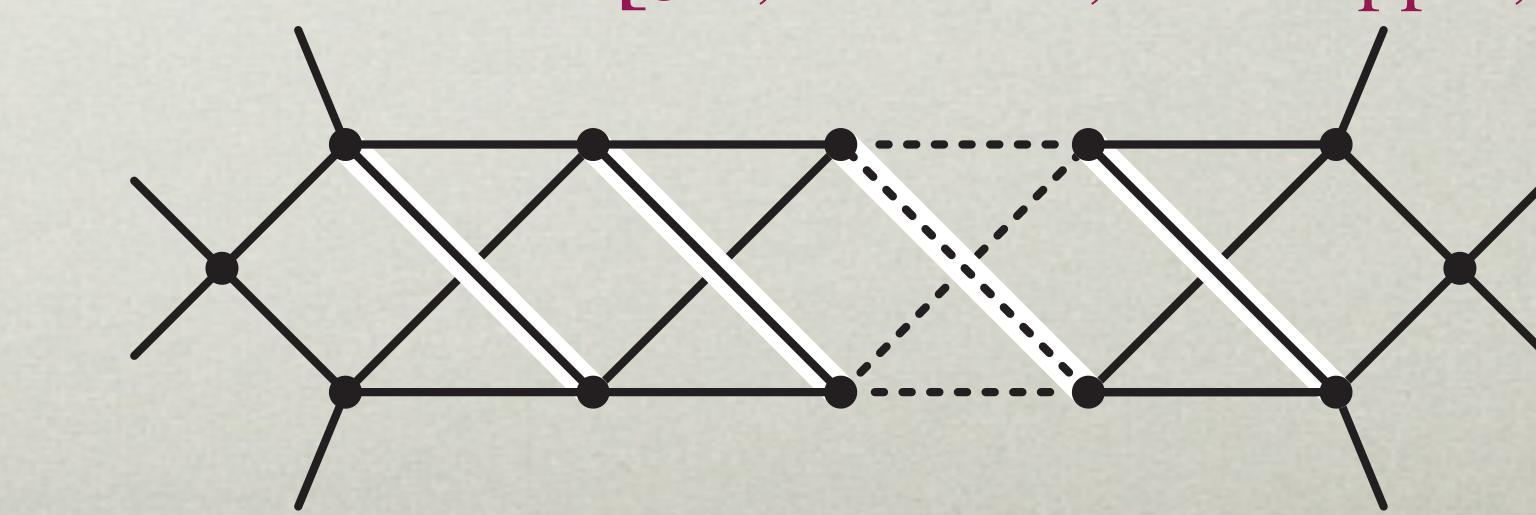
...

4d, massless

tardigrades:



...



**CY<sub>2(L-1)</sub>**

[JB, McLeod, von Hippel, Wilhelm (2018)]

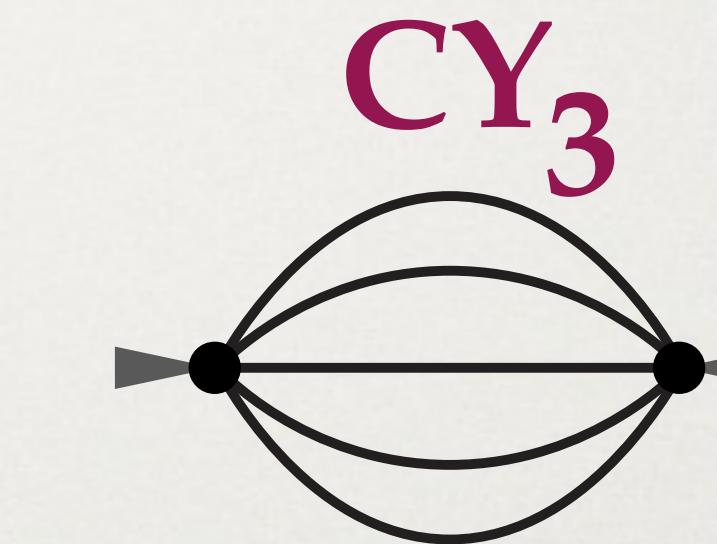
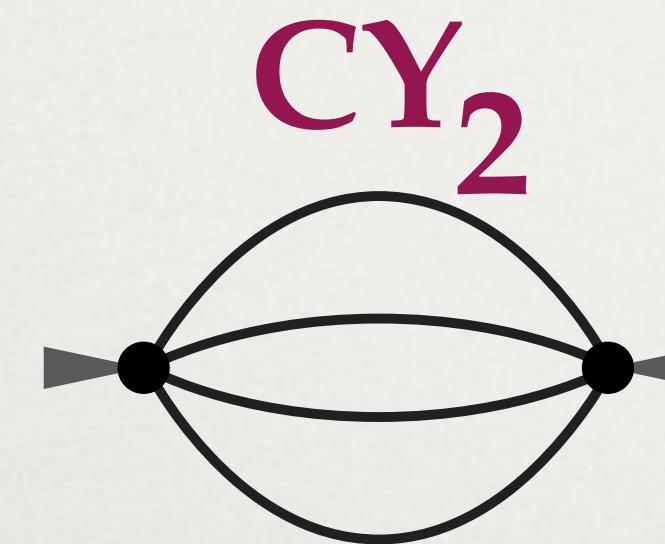
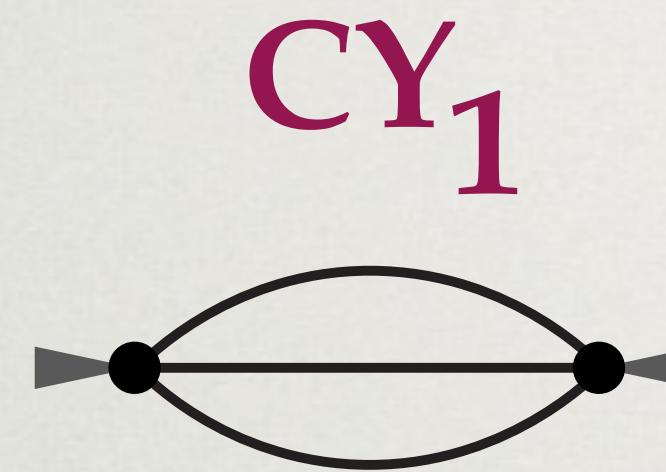


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...

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2d, massive

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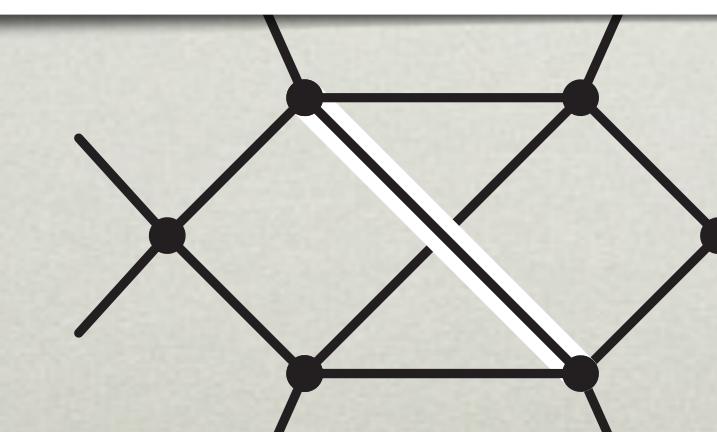
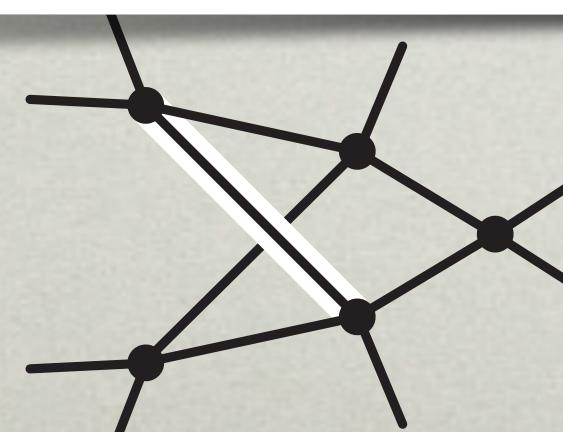
Outside a *small* list of **extremely** limited/simple cases  
(which are widely expected\* to be exceptions):

*almost all* observables in *almost all* QFTs  
are expected to be non-polylogarithmic

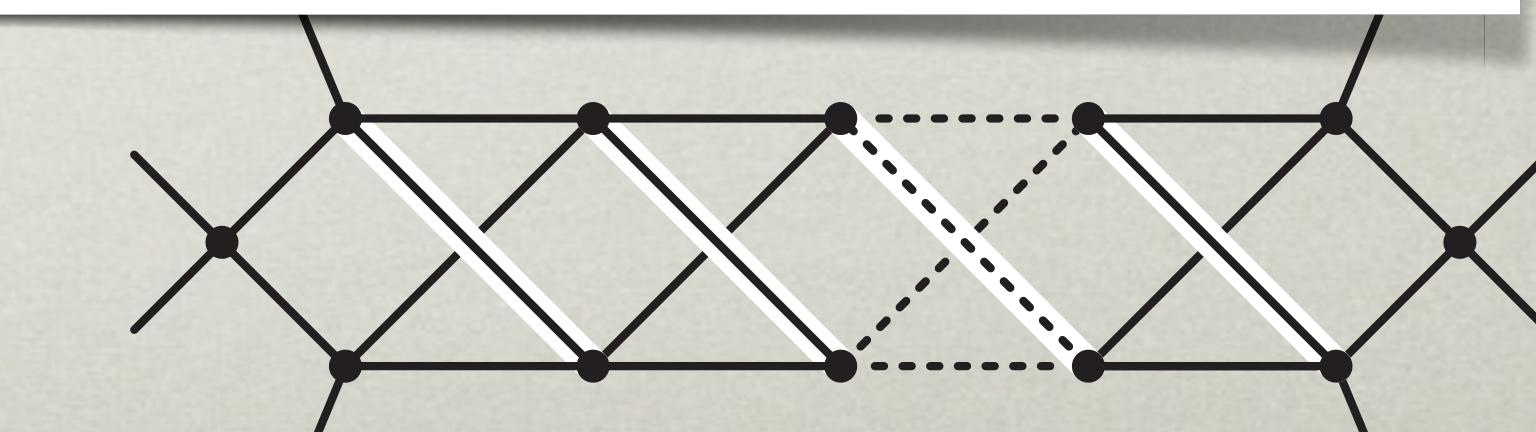
massless

[Wilhelm (2018)]

tardigrades:



...



$CY_{2(L-1)}$

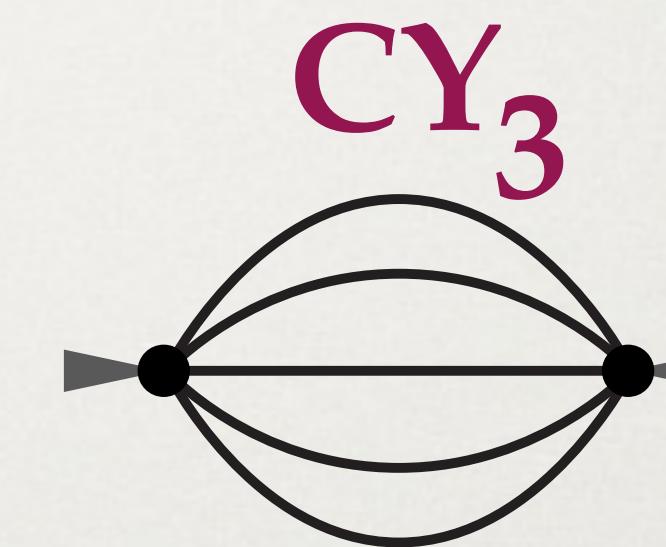
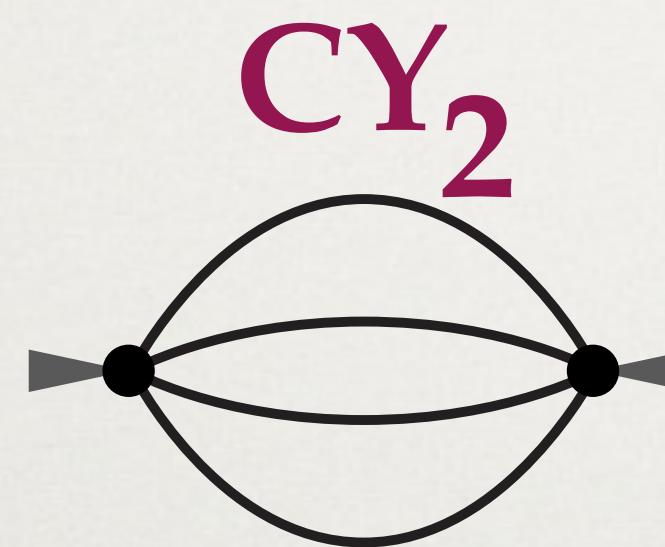
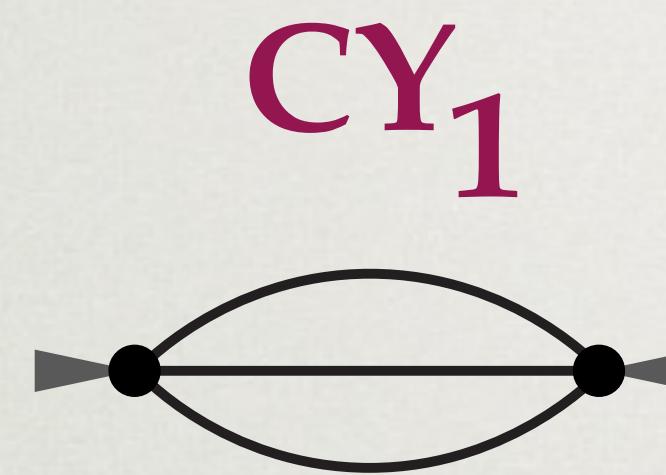


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sunrises:



...

[Doran, Harder, Thompson (2019)]

2d, massive

- ♦ In every instance known, the Calabi-Yau itself is **very simple**:

► degree- $(d+1)$  hypersurface in  $\mathbb{P}^d$  (or multiple-cover thereof)

tra

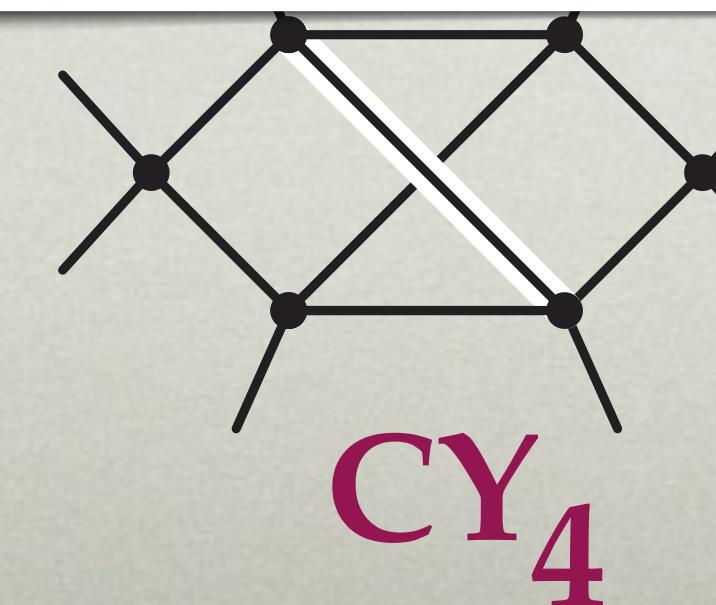
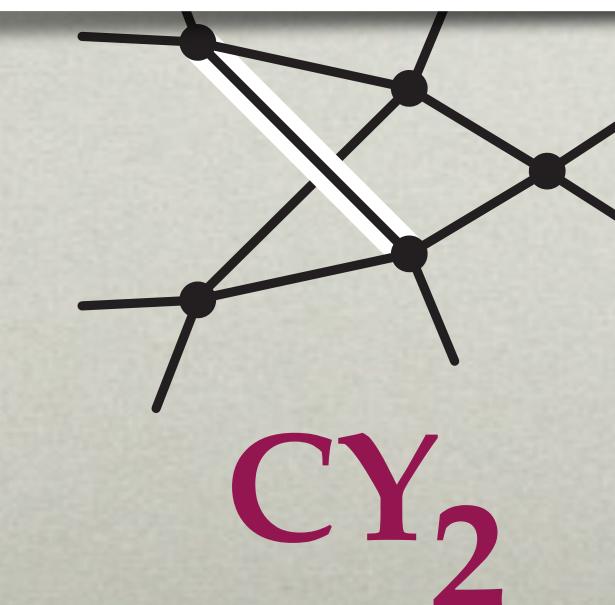
SS

- ♦ kinematic data (momenta / masses) control the moduli

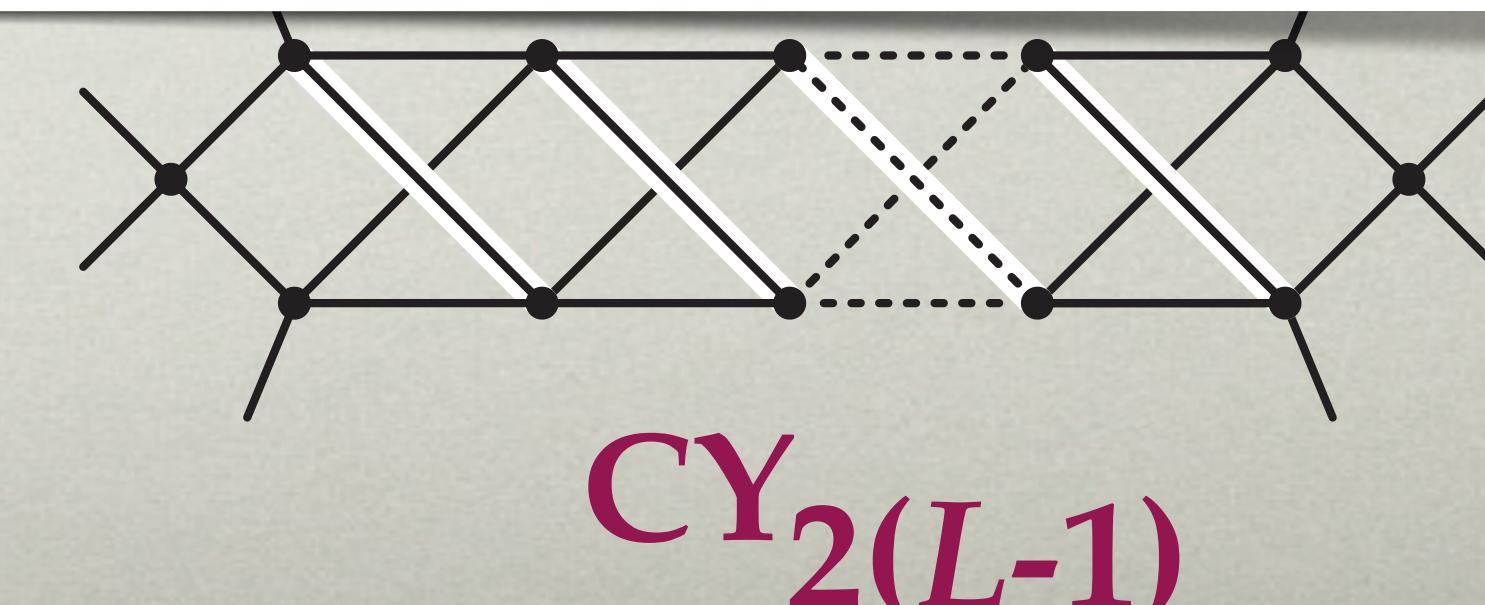
► often **extremely singular**

(2018)]

tardigrades:



...

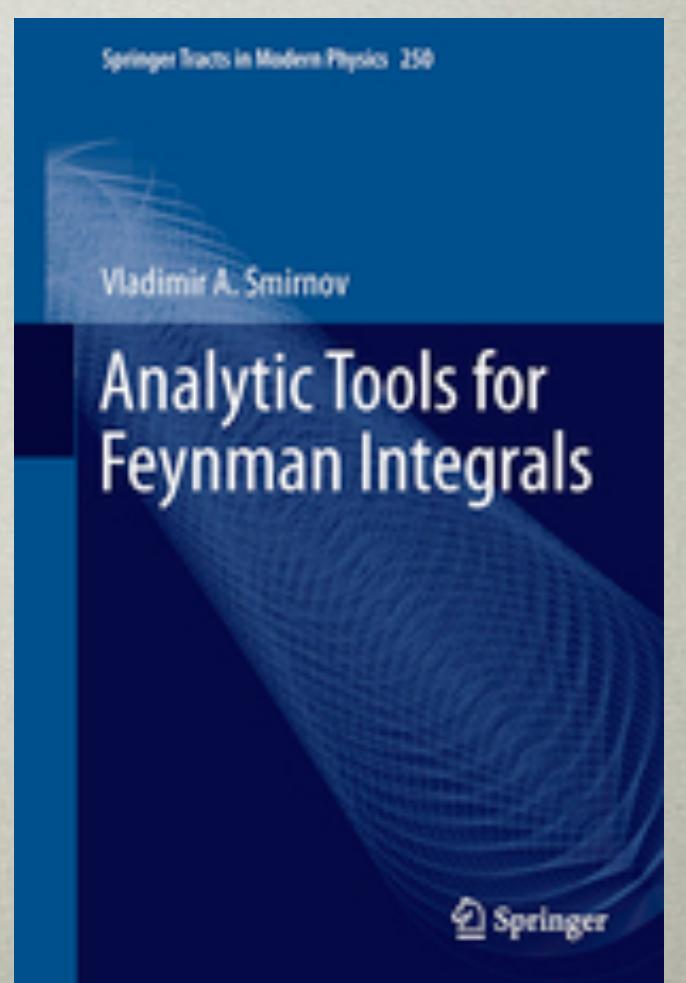




# Why is Perturbation Theory so Hard?

- ♦ Feynman diagrams (esp. with *scalar* numerators) are *horrible*
  - ▶ difficult to integrate, explosive in number, non-physical,...
- ♦ Regularization obscures symmetries (+is technically difficult)
- ♦ Most familiar **mater integrand bases** are *unnecessarily bad*:
  - ▶ don't satisfy nice / canonical differential equations
  - ▶ contain *multiple elliptic*(+worse(!)) geometries,
  - ▶ ...

$$I_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9} = e^{2\gamma_E \epsilon} \iint \frac{d^D k_1 d^D k_2}{(i\pi^{d/2})^2} \frac{P_8^{-a_8} P_9^{-a_9}}{P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7}}$$





# How can We Make it Easier?

- ♦ Use unitarity to choose the *nicest/easiest* integrals to integrate  
*(of course,* integration “ease” changes with time and new methods)
  - ▶ search for as many *pure* integrals as you can
    - those which satisfy nice (canonical) differential equations

**Definition:** a function  $f(s)$  is called *pure* if:

- ▶ there exists a grading of functions by “transcendental” *weight*
- ▶ any **derivative** of  $f(s)$  is *strictly lower in weight*



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e.g.  $g(s) \log(f(s))$  would be *impure*

$$\frac{\partial}{\partial s} [g(s) \log(f(s))] = g'(s) \log(f(s)) + g(s)f'(s)/f(s)$$



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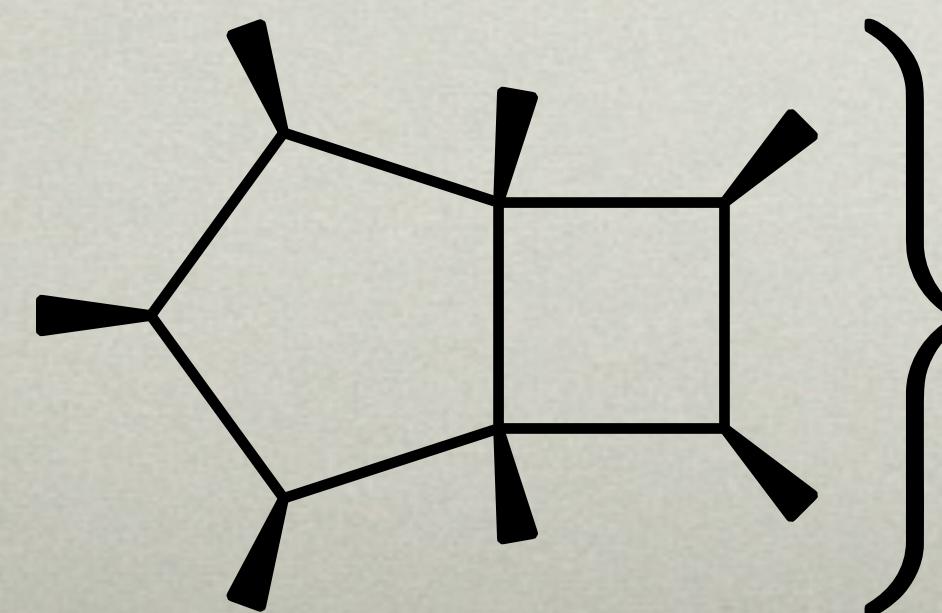
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$$\text{Diagram: A six-pointed star-like graph with points labeled 1 through 6. Point 1 is at the bottom left, 2 is top-left, 3 is top, 4 is top-right, 5 is middle-right, and 6 is bottom-right. Edges connect (1,2), (2,3), (3,4), (4,5), (5,6), (6,1), (1,4), (2,5), and (3,6).}$$
$$\Rightarrow \frac{s_{13}}{s_{12}s_{23}s_{56}} \left( \begin{array}{l} \text{tr}_+ [p_3, p_{12}, p_6, p_1] (\text{Li}_4(\dots)'s + \dots) \\ + \text{tr}_+ [p_{12}, p_6, p_1, p_3] (\text{Li}_4(\dots)'s + \dots) \end{array} \right)$$



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(of course, integration “ease” changes with time and new methods)
  - ▶ search for as many *pure* integrals as you can
    - those which satisfy nice (canonical) differential equations
  
- ♦ Avoid regularization whenever possible:
  - ▶ can all(?) *finite* quantities be computed *without regularization*?
    - without expanding them in terms of divergent integrals?  
(Answer: sometimes)



with numerator ‘1’, this integrates to a sum of  
(an *impure* combination of) **polylogarithms** &  
**elliptic-polylogarithms** involving **4 elliptic curves**

[JB, Langer, Patatoukos (2021); ...]

# *Unitarity-Based Strategies: a modern perspective*



# Generalized Unitarity: a modern take

- ♦ The basic idea behind **unitarity**-based methods is that any *Feynman integrand* is a *rational differential form on loop momenta*
  - as such, it can be expanded into a **basis**  $\mathcal{B}$  of such forms:

$$\mathcal{A} = \sum_{\mathbf{b}^i \in \mathcal{B}} a_i \mathbf{b}^i$$

- ♦ For any fixed QFT (spacetime dimension, particle content), the space of **all amplitude integrands** is **finite-dimensional**
  - *all-multiplicity amplitudes* can be expressed in a ***finite basis!***
- ♦ **Key observation:** viewed as a potential element of *some* basis, *every* Feynman integrand can be interesting!
  - Why not try to find the *best/easiest* integrands—and use these?



# Stratifying Quantum Field Theories

- ♦ QFTs can be partially *ordered* by the scope of the basis required to represent their amplitudes

$$\mathcal{A} = \sum_{\mathbf{b}^i \in \mathfrak{B}} a_i \mathbf{b}^i$$

[Standard Model]  $\succ$  [(Standard Model \ Higgs)]  $\succ$  [QCD]  $\succ$  [Yang-Mills]

[Yang-Mills]  $\succ$  [ $\mathcal{N}=2$  super-Yang-Mills]  $\succ$  [ $\mathcal{N}=4$  super-Yang-Mills]

[ $\mathcal{N}=4$  Yang-Mills]  $\succ$  [planar  $\mathcal{N}=4$  super-Yang-Mills]  $\succ \dots \succ$  [fishnet theory]

This reflects UV behavior (“*power-counting*”) of theories;  
it suggests a possible *stratification of integrand bases*

$$\mathfrak{B}^{\text{SM}} \supset \mathfrak{B}^{\mathcal{N}=2} \supset \mathfrak{B}^{\mathcal{N}=4}$$



# Stratifying Integrand Bases

- ♦ Suppose that a basis could be carved up into subspaces (by any arbitrary means):

$$\mathcal{A} = \sum_{\mathbf{b}^i \in \mathfrak{B}} a_i \mathbf{b}^i$$
$$\mathfrak{B} =: \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \quad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p := \mathcal{A} \cap \mathfrak{B}_p = \sum_{\mathbf{b}_p^i \in \mathfrak{B}_p} a_i \mathbf{b}_p^i$$

- ♦ ¿Is it possible to stratify integrand bases by *physical structure*?

$\{\text{finite}\} \oplus \{\text{divergent}\}$

[JB, Langer, Zhang, (2021)]

[JB, Herrmann, Langer, Patatoukos, *et al* (2021)]



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$$\{\text{finite}\} \oplus \{(\text{UV-divergent})\} \oplus \{(\text{IR-divergent})\}$$

[JB, Langer, Zhang, (2021)]

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- ♦ ¿Is it possible to stratify integrand bases by *physical structure*?

$$\left\{ \text{finite} \right\} \oplus \left\{ \left( \mathcal{O}(1/\epsilon^{2L})\text{-divergent} \right) \oplus \left( \mathcal{O}(1/\epsilon^{2L-1})\text{-divergent} \right) \oplus \cdots \oplus \left( \mathcal{O}(1/\epsilon)\text{-divergent} \right) \right\}$$
$$\oplus \left\{ \left( \log(m)^{2L}\text{-divergent} \right) \oplus \left( \log(m)^{2L-1}\text{-divergent} \right) \oplus \cdots \oplus \left( \log(m)\text{-divergent} \right) \right\}$$

[JB, Langer, Zhang, (2021)]

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- ♦ Suppose that a basis could be carved up into subspaces (by any arbitrary means):

$$\mathcal{A} = \sum_{\mathbf{b}^i \in \mathfrak{B}} a_i \mathbf{b}^i$$
$$\mathfrak{B} =: \bigoplus_{p=0}^{\infty} \mathfrak{B}_p \quad \mathcal{A} = \sum_p \mathcal{A}_p \quad \text{with} \quad \mathcal{A}_p := \mathcal{A} \cap \mathfrak{B}_p = \sum_{\mathbf{b}_p^i \in \mathfrak{B}_p} a_i \mathbf{b}_p^i$$

- ♦ ¿Can we further stratify each part by *transcendental structure*?

$$\left\{ \begin{array}{c} \{\text{finite}\} \\ \overbrace{\{\text{max-weight}\} \oplus \{\text{next-to-max-weight}\} \oplus \cdots \oplus \{\text{rational}\}} \\ \{\text{polylogs}\} \oplus \{\text{elliptic-polylogs}\} \oplus \{\text{K3-polylogs}\} \oplus \cdots \end{array} \right.$$

[JB, Langer, Zhang, (2021)]



# Prescriptive Integrand Bases

- ♦ How *generalized unitarity* has been used to match amplitudes:

$$\mathcal{A} = \sum c_i \mathcal{I}_i^0$$

with coefficients  $c_i$  determined by cuts: a spanning set of cycles  $\{\Omega_j\}$

$$\oint_{\Omega_j} \mathcal{I}_i^0 =: \mathbf{M}_{i,j}$$

$$a_j := \oint_{\Omega_j} \mathcal{A} = \sum_i c_i \mathbf{M}_{i,j} \Rightarrow c_j = \sum_i a_i (\mathbf{M}^{-1})_{i,j}$$

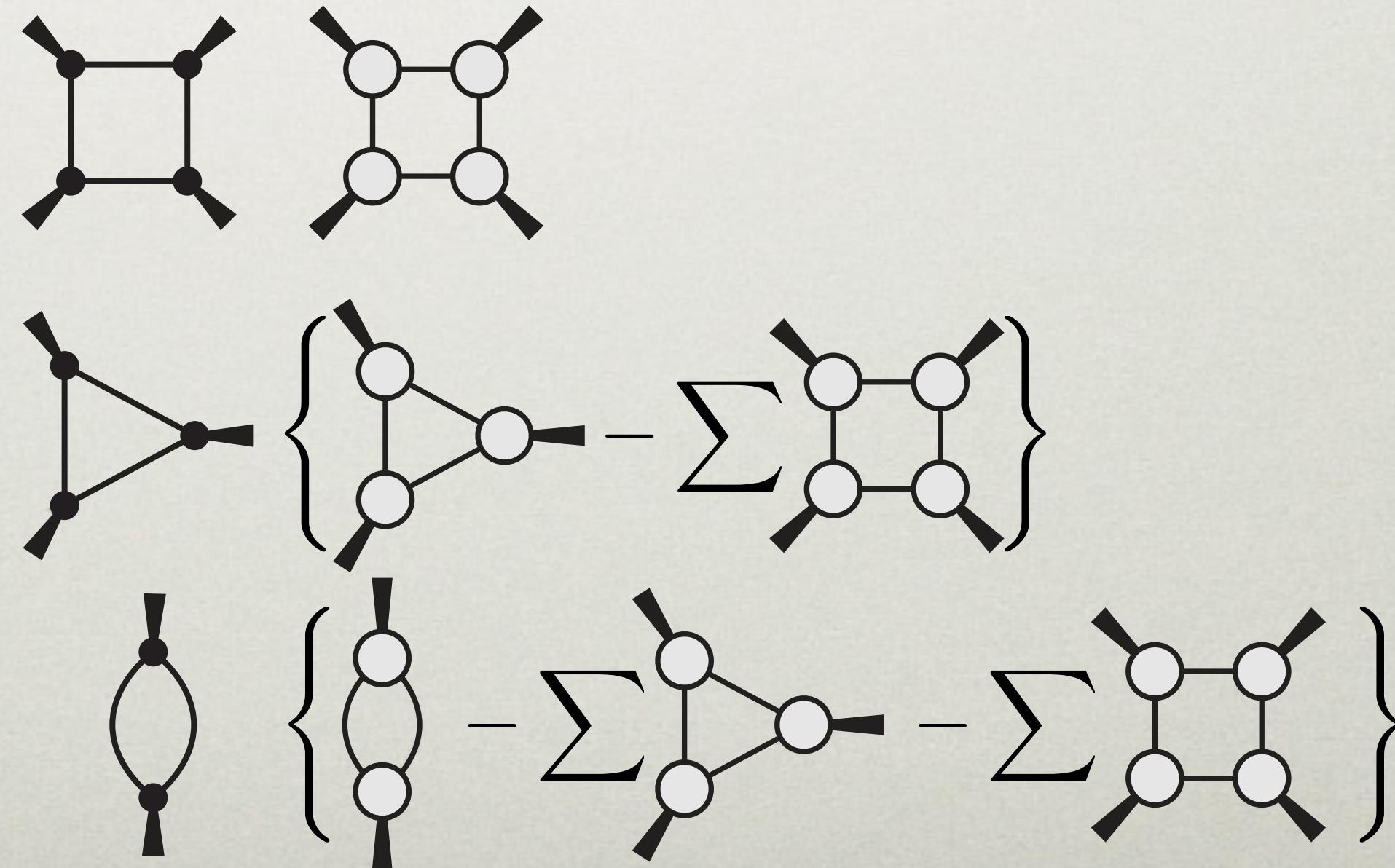


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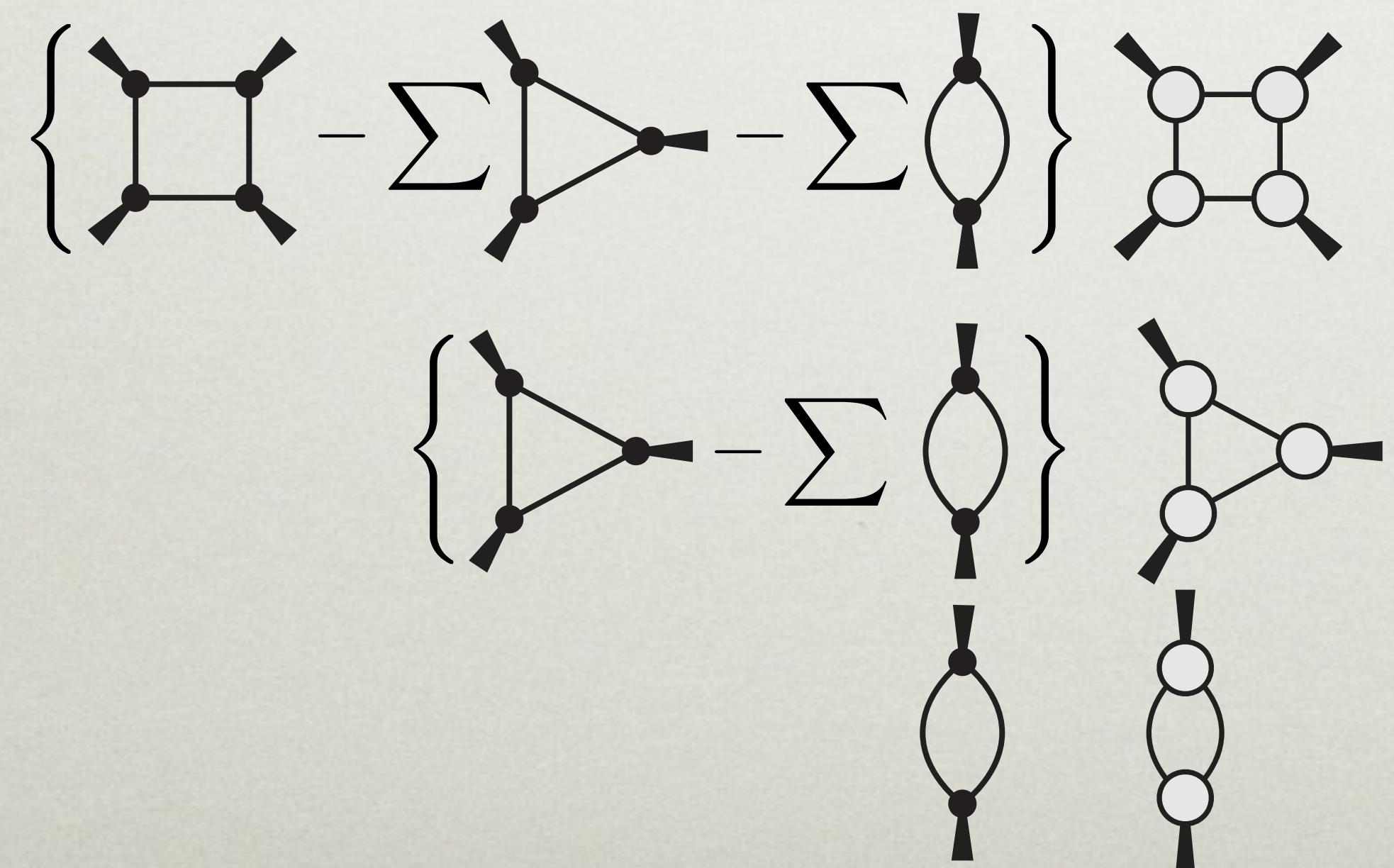


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$$\mathcal{I}_j := \sum_i \mathcal{I}_i^0 (\mathbf{M}^{-1})_{i,j} \quad \oint_{\Omega_j} \mathcal{I}_i = \delta_{i,j}$$

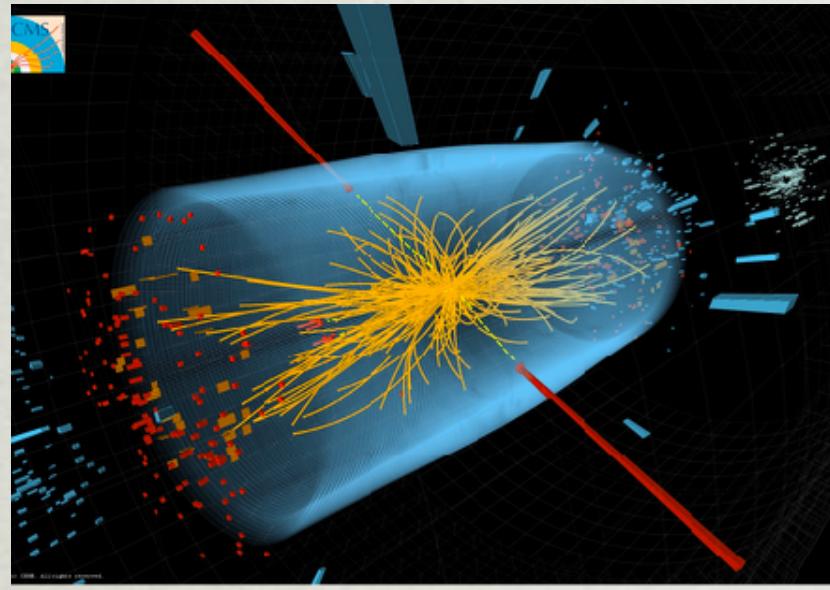
- ♦ A basis is called *prescriptive* if it is the cohomological dual of a spanning set of cycles  $\{\Omega_j\}$



# Strategies for Building Bases

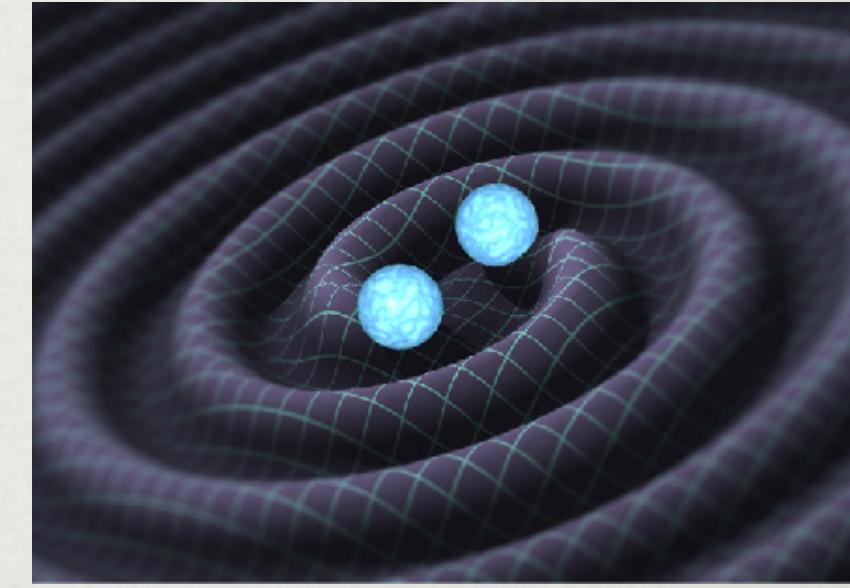
- ♦ Given *some* integrand basis (or strata thereof), one should *diagonalize* the space of integrands according to a **homological/cohomological pairing**:
  - choose a *spanning-set* of **compact, max-dimensional** contours  $\Omega_j$
  - normalize and diagonalize the basis by the requirement
$$\int_{\Omega_j} \mathfrak{b}_i = \delta_{ij}$$
$$\Omega_j : 4L\text{-dimensional compact contours} \quad \left\{ \begin{array}{l} \text{"residues"} \\ \text{elliptic periods} \\ \text{K3 periods, etc.} \end{array} \right.$$
- ♦ This *trivializes* the representation of amplitudes:
  - the **coefficient** of any amplitude in this basis will simply be the *on-shell function* evaluated on the contour (a **leading singularity**)
- ♦ Choosing a **maximal** set of IR/UV-*divergence-probing* contours ensures(?) that the basis is split into finite/divergent subspaces

# *Amplitudes: a Virtuous Cycle*



*Compute Something*

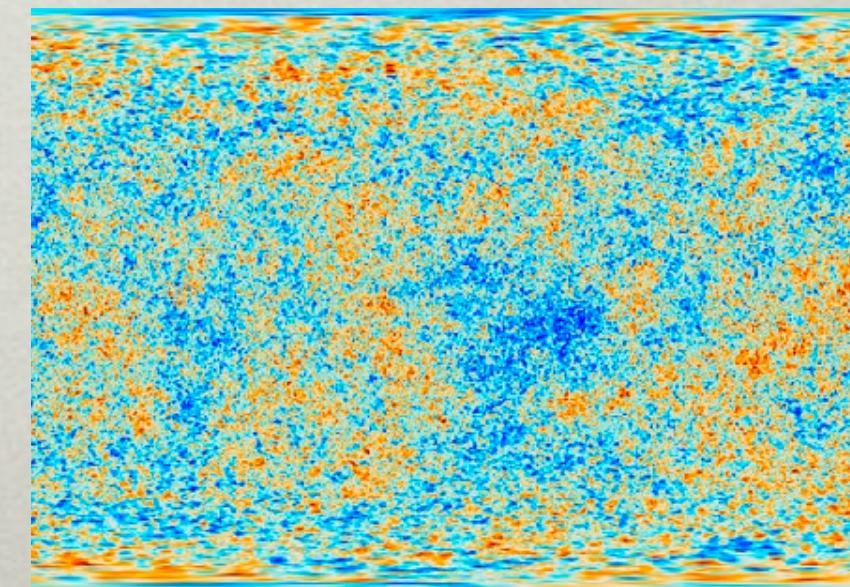
beyond the reach of  
recent imagination



*Exploit Simplicity*  
to build more powerful  
computational technology



*Discover Simplicity*  
beyond expectations

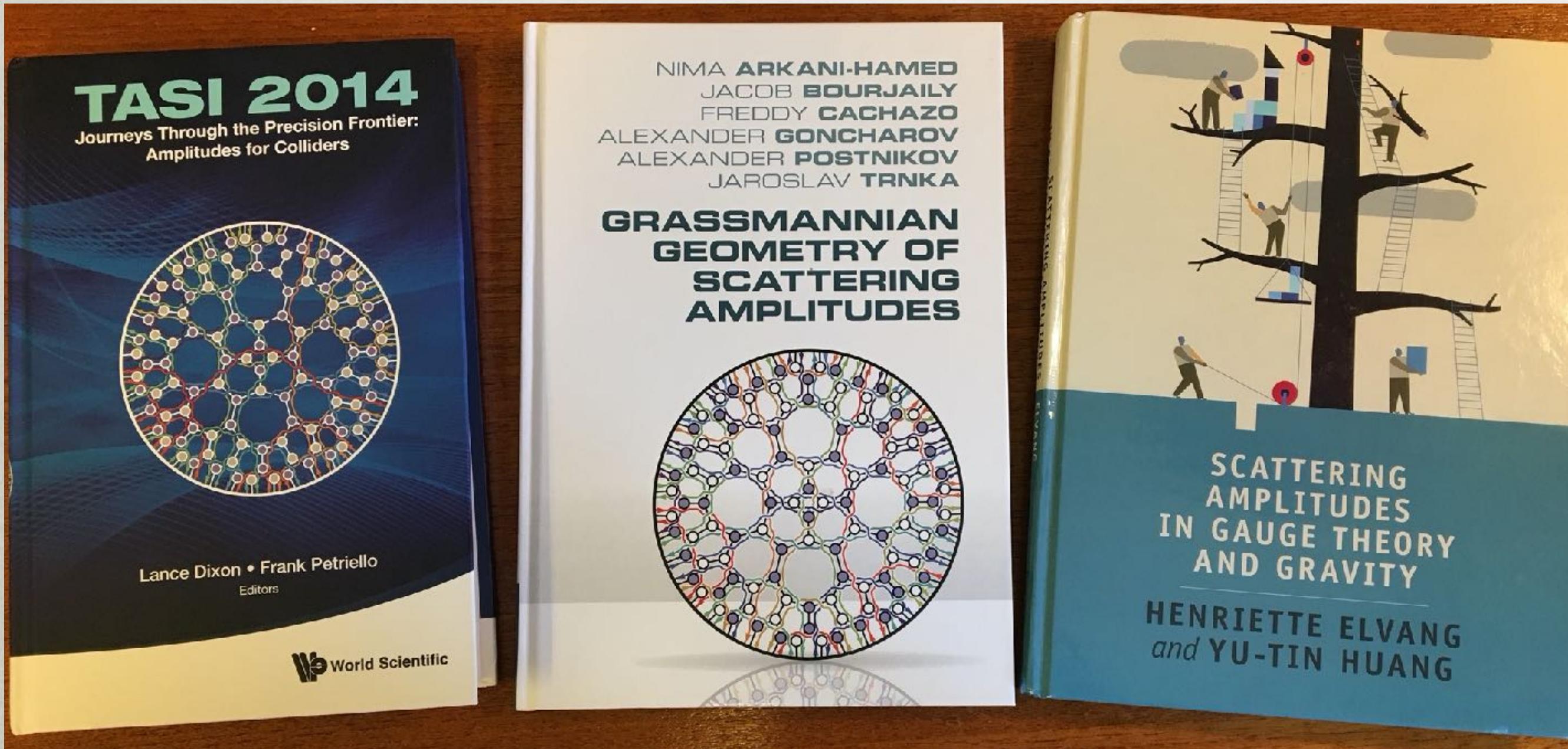


*Understand Why*  
study it, understand it,  
& explore consequences



# Today's Revolution in QFT

- ♦ Defining an *ongoing revolution* in science:
  - when *all the textbooks* of a field become *obsolete*
- ♦ Experts who *need* QFT no longer use **textbook tools**



- ♦ The foundations for *the future's textbooks* are still being discovered every day (it's exciting!)

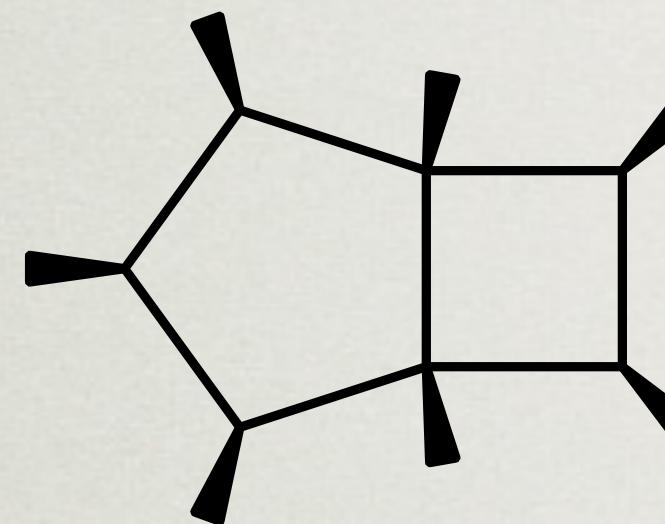
*Thank you!*

*Stratifying Rigidity*



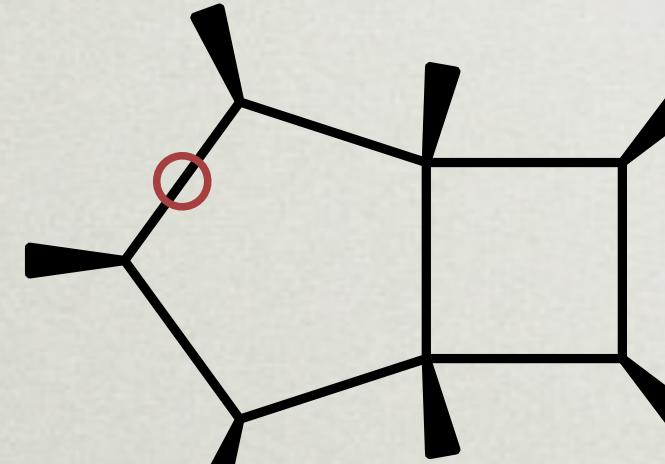
# Diagonalization of Rigidity

- ◆ Consider the following sets of pentabox integrands



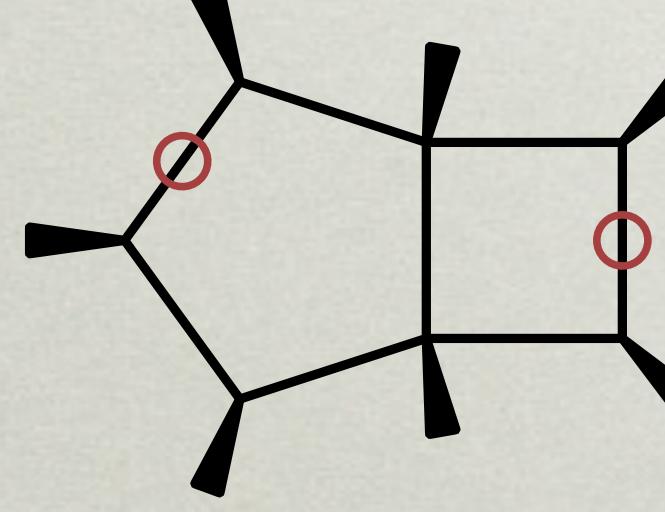
$$\Leftrightarrow \mathcal{I}_0 :=$$

$$\frac{(\ell_1|X)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



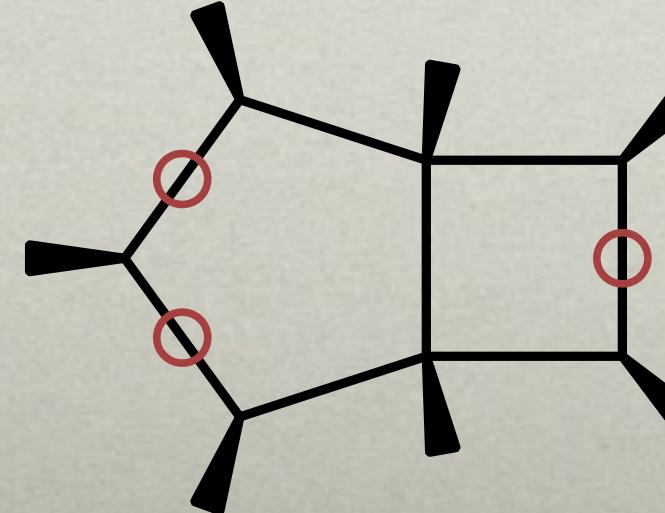
$$\Leftrightarrow \mathcal{I}_1(N) :=$$

$$\frac{(\ell_1|N)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_2(N, M) :=$$

$$\frac{(\ell_1|N)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$



$$\Leftrightarrow \mathcal{I}_3(\vec{N}, M) :=$$

$$\frac{(\ell_1|N_1)(\ell_1|N_2)(\ell_2|M)}{(\ell_1|a)(\ell_1|b)(\ell_1|c)(\ell_1|d)(\ell_1|X)(\ell_2|X)(\ell_1|\ell_2)(\ell_2|e)(\ell_2|f)(\ell_2|g)}$$

[JB, Kalyanapuram (2022)]

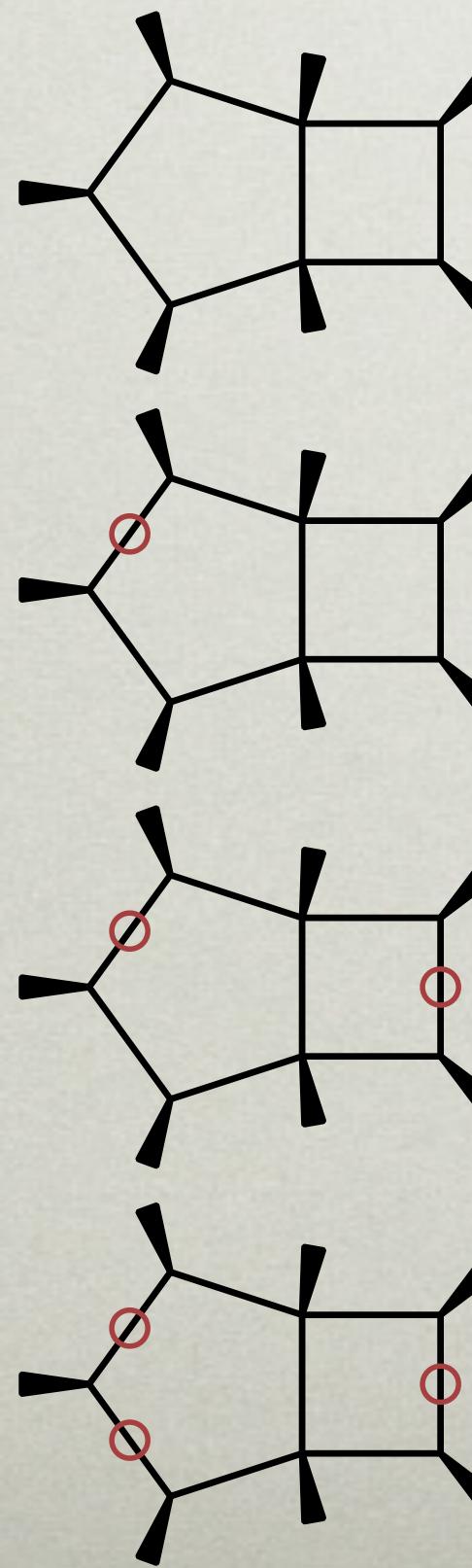


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[JB, Kalyanapuram (2022)]

pure



# ints	# polylogs	# elliptics	# impure/mixed
1	0	0	1
6	0	4 (4)	2
36	10	12 (12)	14
120	76	44 (22)	0