Taming a resurgent renormalon

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Bohr-100 meeting, Copenhagen, 2 August 2022

Perturbative expansions in quantum field theory **diverge** for at least **two reasons**.
The **number of diagrams** increases dramatically with the loop number.
Renormalization may make the contribution of some diagrams very **large**.
I shall give an example of the **second** problem, from an ultra-violent **renormalon** of ϕ^3 theory in **6 dimensions**, where we can compute to **very high loop-order**.
Taming this renormalon involves my recent work on **resurgence** with **Michael Borinsky** in `arXiv:2202.01513`. This challenge is **much more demanding** than the corresponding problem for Yukawa theory in 4 dimensions.

1. Two sources of divergence
2. **Dyson-Schwinger** equation and **asymptotic** expansion
3. **Padé-Borel** summation with **alternating** signs
4. **Trans-series** and **resurgent** hyperasymptotic expansions
5. Comments and conclusions

1 Two sources of divergence *d'après Gerard, Erice, 1977*

Too many diagrams: *An example of a divergent perturbation expansion in field theory* by Angas **Hurst** in 1951 considers the **super-renormalizable** case of ϕ^3 theory in $D = 4$ space-time dimensions. For any number of external legs, the **number** of primitive n-loop digrams increases **factorially** with n. Below production threshold, each diagram gives a **positive** integral over Feynman parameters, bounded from below **geometrically**.

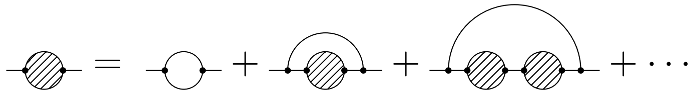
A renormalon: *On high order estimates in QED* by Benny **Lautrup** in 1977 considers the magnetic moment of the electron. At n loops there is a **single** gauge-invariant diagram whose contribution grows **factorially** with n.

Imaginary coupling: *Divergence of perturbation theory in quantum electrodynamics* by Freeman **Dyson** in 1952 remarks that convergence in $\alpha = e^2/(4\pi)$, would make sense of a **non-unitary** theory with **imaginary** coupling giving $\alpha < 0$ and a nightmarish world in which electrons **repel** positrons.

Recent work: *Instantons or renormalons? A comment on ϕ^4 theory in the MS scheme* by Gerald **Dunne** and Max **Meynig** in 2022 uses recent perturbative results to investigate a suggestion by Gerard 't **Hooft** that renormalons do **not** contribute to **anomalous dimensions** in a **mimimal subtraction** scheme.

2 Dyson-Schwinger equation and asymptotic expansion

Consider the perturbation expansion generated by a single divergent diagram, via the **non-linear** Dyson-Schwinger equation



contributing to the **self-energy** term Σ in the inverse propagator $q^2(1-\Sigma)$, for a **massless scalar** particle with a ϕ^3 interaction, in the **critical** space-time dimension $D=6$, for which the coupling constant is **dimensionless**.

The dependence of Σ on the external momentum q comes **solely** from **renormalization**. At n loops, we get a contribution that is a **polynomial** of degree n in the **logarithm** $\log(q^2/\mu^2)$, multiplied by a^n where $a=\lambda^2/(4\pi)^3$, λ is the coupling constant and μ is the **renormalization scale**.

If we use **momentum-space subtraction**, so that Σ vanishes at $q^2=\mu^2$, the dependence on momentum is completely determined by the **anomalous dimension**, with

$$\gamma(a) = -\, q^2 \frac{\mathrm{d}\Sigma}{\mathrm{d}q^2}\bigg|_{q^2=\mu^2} \qquad \text{giving} \qquad \frac{\mathrm{d}\log(1-\Sigma)}{\mathrm{d}\log q^2} = \gamma\left(\frac{a}{(1-\Sigma)^{\mathbf{2}}}\right).$$

How many n-loop Feynman diagrams for this problem?

The number of distinct diagrams at n loops is the number T_n of **rooted trees** with n nodes, which gives the sequence

$$1, 1, 2, 4, 9, 20, 48, 115, 286, 719, 1842, 4766, 12486, 32973, 87811, 235381,$$

up to 16 loops. The **iterated** structure: **tree** = **root** + **branches**, with every branch being itself a tree, gives the asymptotic growth

$$T_n = \frac{b}{n^{3/2}} c^n (1 + O(1/n))$$

$$b = 0.43992401257102530404090339143454476479808540794011 \ldots$$

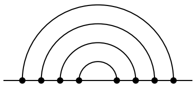
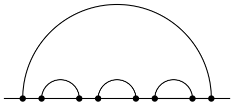
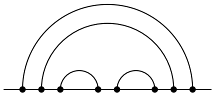
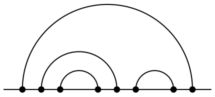
$$c = 2.95576528565199497471481752412319458837549230466359 \ldots$$

At **250 loops** the number of Feynman diagrams is

```
T_250=517763755754613310897899496398412372256908589980657316
271041790137801884375338813698141647334732891545098109934676
```

This is **not** the main source of the problem. If the contribution of each diagram was bounded, there would be a **finite** radius of convergence for the perturbation expansion. The divergence of the series comes from **renormalization**, which makes the n-loop term grow **factorially**. This is called a **renormalon** singularity.

At 4 loops, we have a **rainbow**, a **chain** and two more interesting diagrams:

The sum of rainbows converges. Chains can be summed by Borel transformation.

$$
\begin{aligned}
\gamma_{\text{rainbow}} = \frac{3-\sqrt{5+4\sqrt{1+a}}}{2} &= -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 206\,\frac{a^3}{6^5} + 4711\,\frac{a^4}{6^7} + O(a^5) \\
\gamma_{\text{chain}} = -\int_0^\infty \frac{6\exp(-6z/a)\mathrm{d}z}{(z+1)(z+2)(z+3)} &= -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 170\,\frac{a^3}{6^5} + 3450\,\frac{a^4}{6^7} + O(a^5) \\
\gamma \sim \sum_{n>0} G_n\,\frac{(-a)^n}{6^{2n-1}} &= -\frac{a}{6} + 11\,\frac{a^2}{6^3} - 376\,\frac{a^3}{6^5} + \mathbf{20241}\,\frac{a^4}{6^7} + O(a^5)
\end{aligned}
$$

with **large integers** G_n in the **alternating** asymptotic series for γ. Note that $G_4 = 20241 > 4711 + 3450$, because of two further diagrams, above. In one we have a chain inside a double rainbow. In the other, a double rainbow is chained with the primitive divergence. This interplay is coded by **rooted trees**.

At **500 loops** I determined the **integer** coefficient

```
G500=2062614519660805414511193562652664079058161175768956015206163286705433040976
62369668214104674763068056454522518617422020409397336434904863988900797769773644
47129884863324773181376863120291798830884688213932683869821267125662274428136514
68974978228592824043044373847281757207937081063432528806815509319762088807291996
54549245884853496719417048678199825379018355919198123075612308008976364608893906
00835837012056033720017238115336850340799075684336975651857656078799282745256216
85768456030809283727097722850488278232311177219444745322287340871435443707536590
64304859950724683157717734493071321199539578218428617617722892100276682781401203
04983974209704793621909710059353724523231635766062166284812903992269403282699432
81718327508638643305481989940132234093616573076862094588977827344981584305605437
66475002382217933275761312682929603923397580260987048907414858143897114762331252
08694985337972553885925402003826420205441859988844001867088083850782378303677991
14077650584544145709672328391394562704209221732180879565868213522109303655045186
92714017665002971967455255310508358729281544729403249398746232320441525286283859
23093041626365262630100048817481274793707664791767175677240144896307853488347045
21622394885797995125083750860330519417878429051836575477220881369445751634601965
33191009573619480068718080810533581305996863996579338522874547127421808710757882
86996199556804886954946559116947132125235605586627322129268965041445488085748194
82341875039156647569797757032552836429751077302524927736861138479038542006096835
73747720303607608007740173613335602076396299832459826245418033598839559699294537
37336134624690115674194793212055897162647586497730033948880084738561472545509216
```

with 1675 decimal digits. The number of diagrams has *merely* 231 digits.

This was achieved in work with **Dirk Kreimer** that resulted in a **third-order** differential equation

$$8a^3\gamma\left\{\gamma^2\gamma''' + 4\gamma\gamma'\gamma'' + (\gamma')^3\right\} + 4a^2\gamma\left\{2\gamma(\gamma-3)\gamma'' + (\gamma-6)(\gamma')^2\right\}$$
$$+\,2a\gamma(2\gamma^2+6\gamma+11)\gamma' - \gamma(\gamma+1)(\gamma+2)(\gamma+3) = a$$

with **quartic** non-linearity.

Our interest in this problem came from his discovery of the **Hopf algebra** of the iterated subtraction of subdivergences, whose utility we illustrated in this example, with a single primitive divergence leading to **undecorated** rooted trees.

For the corresponding diagrams in **Yukawa** theory, in its critical dimension $D = 4$, we found a **first-order** equation with merely **quadratic** non-linearity, which we solved using the **complementary error function**, thereby achieving explicit **all-orders** results for both the anomalous dimension and the self energy. The expansion coefficients in this simpler case enumerate **connected cord diagrams**.

We also investigated the $D = 4$ and $D = 6$ examples in the more cumbersome **minimal subtraction** scheme, where one retains finite parts of Σ at $q^2 = \mu^2$. Here one encounters unwieldy products of **zeta values** with weights that increase linearly with the loop-number. Recently **Paul-Hermann Balduf** has shown how to absorb these into a rescaling of μ that can be expanded in the coupling a.

3 Padé-Borel summation with alternating signs

We sought to resum the factorially divergent alternating series by an Ansatz

$$\gamma(a) = -\frac{a}{6\Gamma(\beta)} \int_0^\infty P(ax/3) \exp(-x) x^{\beta-1} \mathrm{d}x, \quad P(z) = \frac{N(z)}{D(z)}.$$

The expansion coefficients of $P(z) = 1 + O(z)$ are obtained from those those of $\gamma(a)/a$ by dividing the latter by factorially increasing factors, producing a function P which we expected to have a **finite** radius of convergence in the **Borel** variable z, with singularities on the **negative** z-axis, as for the sum of chains.

The **Padé** trick is to convert the expansion of P, up to n loops, into a **ratio** N/D of polynomials of degrees close to $n/2$. Then one can **check** how well this method reproduces G_{n+1}. We found that this works rather well with $\beta \approx \mathbf{3}$.

For example, we fitted the first 29 values of G_n with a ratio of polynomials of degree 14 and found a pole, coming from the denominator $D(z)$, at $z = -0.994$. The other 13 poles occurred further to the left, with $\Re z < -1$. Moreover the numerator $N(z)$ gave no zero with $\Re z > 0$. Then this method reproduced the first 15 decimal digits of G_{30}. Gerald Dunne has recently shown that this method works even better with $\beta = \mathbf{\frac{35}{12}}$, for reasons that I shall now explain.

4 Trans-series and resurgent hyperasymptotics

There is an old and rather loose argument, going back to **Freeman Dyson** in 1952, that we should **not expect** realistic field theories to give convergent expansions in the **square** of a coupling constant. If they did, we could get sensible answers for a pathological non-unitary theory with an **imaginary** coupling constant, such as an electrodynamics in which electrons repel positrons.

There is an amusing **converse** of this suggestion. If you find an expansion that is Borel summable, then study it at **imaginary coupling**, where ϕ^3 theory gives the **Yang-Lee edge singularity** in condensed matter physics, using PT symmetry.

So now I recast the Broadhurst-Kreimer problem, in the manner of Borinksy, Dunne and Meynig, by setting $g(x) = \gamma(-3x)/x$, to obtain an ODE that is economically written as

$$(g(x)P-1)(g(x)P-2)(g(x)P-3)g(x) = -3, \quad P = x\left(2x\frac{\mathrm{d}}{\mathrm{d}x}+1\right),$$

and has an **unsummable** formal perturbative solution

$$g_0(x) \sim \sum_{n=0}^{\infty} A_n x^n = \frac{1}{2} + \frac{11}{24}x + \frac{47}{36}x^2 + \frac{2249}{384}x^3 + \frac{356789}{10368}x^4 + \frac{60819625}{248832}x^5 + O(x^6).$$

The expansion coefficients behave as

$$A_n = S_1 \Gamma\left(n + \mathbf{\frac{35}{12}}\right)\left(1 - \mathbf{\frac{97}{48}}\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)\right),$$

at large n, with a **Stokes constant**

$$S_1 = 0.087595552909179124483795447421262990627388017406822\ldots$$

that can be determined, empirically, by considering a solution

$$g(x) = g_0(x) + \sigma_1 x^{-\beta} \exp(-1/x) h_1(x) + O(\sigma_1^2)$$

and retaining terms linear in σ_1 in the ODE. This yields a **linear homogeneous** ODE for $h_1(x)$, which permits a solution that is **finite and regular** at $x = 0$ if and **only if** $\beta = \mathbf{\frac{35}{12}}$. Normalizing σ_1 by setting $h_1(0) = -1$, we obtain the expansion of

$$h_1(x) \sim \sum_{k=0}^{\infty} B_k x^k = -1 + \mathbf{\frac{97}{48}} x + \frac{53917}{13824} x^2 + \frac{3026443}{221184} x^3 + \frac{32035763261}{382205952} x^4 + O(x^5)$$

which gives the **first-instanton** correction to the perturbative solution, suppressed by $\exp(-1/x)$.

By developing the series A_n and B_k, I was able to determine **3000 digits** of S_1 in

$$A_n \sim -S_1 \sum_{k\geq 0} \Gamma\left(n+\frac{35}{12}-k\right) B_k.$$

This is an example of **resurgence**: information about A_n resurges in B_k, and vice versa, because both $A(x) = g_0(x)$ and $B(x) = h_1(x)$ know about the **same** physics.

Hyperasymptotic expansions involve the study of how B_n behaves at large n, which involves another set of numbers C_k, at small k, and so on, and so on.

Large A's need smaller B's, especially to guide them,
and larger B's need smaller C's, and so ad infinitum.

Hyperasymptotic investigation involves terms suppressed by $\exp(-m/x)$, with **action** $m > 1$. For this **third-order** ODE, there are **3 solutions** to the **linearized** problem, namely

$$g(x) = g_0(x) + \sigma_m \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}}\right)^m h_m(x) + O(\sigma_m^2), \quad m \in \{1, 2, 3\},$$

with $h_2/x^5 = C$ and $h_3/x^5 = D$ finite and regular near the origin.

Then we use the linearized ODE to develop the expansions

$$
\begin{aligned}
C(x) = h_2(x)/x^5 &= -1 + \frac{151}{24}x - \frac{63727}{3456}x^2 + \frac{7112963}{82944}x^3 - \frac{7975908763x}{23887872}x^4 + O(x^5),\\
D(x) = h_3(x)/x^5 &= -1 + \frac{227}{48}x + \frac{1399}{4608}x^2 + \frac{814211}{73728}x^3 + \frac{3444654437}{42467328}x^4 + O(x^5).
\end{aligned}
$$

Before presenting the **trans-series**, I remark on some of its general **features**.

1. The terms suppressed by $\exp(-2/x)$ involve σ_2 and σ_1^2. The former are given by C and the latter are determined by an **inhomogeneous** linear ODE, whose solution is **ambiguous**, up to a multiple of the homogeneous solution $h_2 = x^5 C$, since we can **shift** σ_2 by a multiple of σ_1^2.
2. In the terms suppressed by $\exp(-3/x)$ there a **second ambiguity**, since we can shift σ_3 by a multiple of σ_1^3.
3. Ambiguities of inhomogeneous solutions occur at places in expansions where **logarithms** first arise. This happens when the **power** of x in an expansion is a multiple of **5**.
4. The **highest** power of $\log(x)$, in terms with **action** m, is $\lfloor m/2 \rfloor$.

The terms in the **trans-series** with action $m \leq 4$ are of the form

$$g = \sum_{m \geq 0} g_m \left(x^{-\frac{35}{12}} e^{-\frac{1}{x}} \right)^m, \quad L = \frac{21265}{2304} x^5 \log(x),$$

$$g_0 = A, \quad g_1 = \sigma_1 B, \quad g_2 = \sigma_2 x^5 C + \sigma_1^2 (F + CL),$$

$$g_3 = \sigma_3 x^5 D + \sigma_1 \sigma_2 x^5 E + \sigma_1^3 (I + (D + E)L),$$

$$g_4 = \sigma_1 \sigma_3 x^5 G + \sigma_2^2 x^{10} H + \sigma_1^2 \sigma_2 x^5 (J + 2HL) + \sigma_1^4 (K + (G + J)L + HL^2).$$

Denoting the coefficients of x^n in functions by subscripts, we found that the choices

$$\frac{F_5}{2!} = \frac{I_5}{3!} = \frac{32642693907919}{36691771392}$$

greatly simplify of our system of hyperasymptotic expansions. Then

$$B_n \sim -2S_1 \sum_{k \geq 0} F_k \Gamma(n + \tfrac{35}{12} - k)$$

$$+ 4S_1 \sum_{k \geq 0} C_k \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1 \right),$$

$$d_1 = -43.332634728250755924500717390319380703460728022278 \ldots$$

with $\psi(z) = \Gamma'(z)/\Gamma(z) = \log(z) + O(1/z)$, shows the $m = 1$ term, at large n, looking forward to $m = 2$ terms, at small k.

For the asymptotic expansion of the **second-instanton** coefficients, we found

$$C_n \sim -S_1 \sum_{k\geq 0} E_k \Gamma(n + \tfrac{35}{12} - k) + S_3 \sum_{k\geq 0} B_k (-1)^{n-k} \Gamma(n + \tfrac{25}{12} - k).$$

The first sum looks **forwards** to $m = 3$ in the trans-series, where coefficients of

$$E(x) = -4 + \tfrac{371}{12}x - \tfrac{111785}{1152}x^2 + \tfrac{8206067}{18432}x^3 - \tfrac{18251431003}{10616832}x^4 + O(x^5)$$

appear. It does **not contain** the coefficients D_k of the **third instanton**, which **decouples** from the asymptotic expansion for the second instanton.

The second sum has **alternating** signs, looks **backwards** to $m = 1$ and is **suppressed** by a factor of $1/n^{5/6}$. This can be understood using **alien calculus**. Likewise,

$$\begin{aligned} F_n \sim & -3S_1 \sum_{k\geq 0} I_k \Gamma(n + \tfrac{35}{12} - k) \\ & + 2S_1 \sum_{k\geq 0} (3D_k + 2E_k) \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1\right) \\ & - 2S_3 \sum_{k\geq 0} B_k (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{35}{12} - k) + f_1\right) \end{aligned}$$

looks forwards to I_k, D_k and E_k, at $m = 3$, and backwards to B_k at, $m = 1$.

The new constants are

$$\begin{aligned}
S_3 &= 2.17178531405909902116086012279038923024794641930\,27\ldots \\
f_1 &= -40.9036925092285150038144791269013547852636695530\,14\ldots
\end{aligned}$$

Two more were discovered in the backward looking terms of

$$\begin{aligned}
I_n \sim{}& -4S_1\sum_{k\geq 0} K_k\Gamma(n+\tfrac{35}{12}-k) \\
&+2S_1\sum_{k\geq 0}(3G_k+2J_k)\Gamma(n-\tfrac{25}{12}-k)\left(\tfrac{21265}{4608}\psi(n-\tfrac{25}{12}-k)+d_1\right) \\
&-4S_3\sum_{k\geq 0} F_k(-1)^{n-k}\Gamma(n-\tfrac{35}{12}-k)\left(\tfrac{21265}{4608}\psi(n-\tfrac{35}{12}-k)+f_1\right) \\
&-8S_3\sum_{k\geq 0} C_k(-1)^{n-k}\Gamma(n-\tfrac{95}{12}-k)Q(n-\tfrac{95}{12}-k), \\
Q(z) ={}& \left(\tfrac{21265}{4608}\right)^2\left(\psi^2(z)+\psi'(z)\right)+2c_1\left(\tfrac{21265}{4608}\right)\psi(z)+c_2, \\
c_1 ={}& -41.0319567643027105839210681015455094537048978981\,88\ldots \\
c_2/c_1^2 ={}& 1.00020164721319925958228053808383241880115723042\,76\ldots
\end{aligned}$$

We believe that **6 constants suffice** for the complete description of resurgence.

Conjecture: The **trans-series** and its **resurgence** take the forms

$$g(x) = \sum_{m=0}^{\infty} \left(x^{-\frac{35}{12}}\, e^{-\frac{1}{x}} \right)^m \sum_{i=0}^{\lfloor m/2 \rfloor} \sum_{j=0}^{\lfloor (m-2i)/3 \rfloor} \sigma_1^{m-2i-3j} \widehat{\sigma}_2^i \widehat{\sigma}_3^j x^{5(i+j)} \sum_{n \geq 0} a_{i,j}^{(m)}(n) x^n,$$

$$\widehat{\sigma}_2 = \sigma_2 + \tfrac{21265}{2304} \sigma_1^2 \log(x), \quad \widehat{\sigma}_3 = \sigma_3 + \tfrac{21265}{2304} \sigma_1^3 \log(x),$$

$$a_{i,j}^{(m)}(n) \sim -(s+1) S_1 \sum_{k \geq 0} a_{i,j}^{(m+1)}(k) \Gamma(n + \tfrac{35}{12} - k)$$

$$+ S_1 \sum_{k \geq 0} \Big(4(i+1) a_{i+1,j}^{(m+1)}(k) + 6(j+1) a_{i,j+1}^{(m+1)}(k) \Big) \Gamma(n - \tfrac{25}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{25}{12} - k) + d_1 \right)$$

$$+ \tfrac{1}{4} S_3 \sum_{k \geq 0} \Big(4(s+1) a_{i-1,j}^{(m-1)}(k) + 6(j+1) a_{i-2,j+1}^{(m-1)}(k) \Big) (-1)^{n-k} \Gamma(n + \tfrac{25}{12} - k)$$

$$- 2(s - 2i - 1) S_3 \sum_{k \geq 0} a_{i,j}^{(m-1)}(k) (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k) \left(\tfrac{21265}{4608} \psi(n - \tfrac{35}{12} - k) + f_1 \right)$$

$$- S_3 \sum_{k \geq 0} \Big(8(i+1) a_{i+1,j}^{(m-1)}(k) + 6(j+1) a_{i,j+1}^{(m-1)}(k) \Big) (-1)^{n-k} \Gamma(n - \tfrac{95}{12} - k) Q(n - \tfrac{95}{12} - k)$$

$$- (f_1 - c_1) S_3 \sum_{k \geq 0} \Big(2(i+1) a_{i+1,j-1}^{(m-1)}(k) + 6(i+j) a_{i,j}^{(m-1)}(k) \Big) (-1)^{n-k} \Gamma(n - \tfrac{35}{12} - k),$$

with $s = m - 2i - 3j$ and $Q(z) = \left(\tfrac{21265}{4608} \right)^2 \left(\psi^2(z) + \psi'(z) \right) + 2c_1 \left(\tfrac{21265}{4608} \right) \psi(z) + c_2$.

5 Comments and conclusions

1. The conjecture exhibits **17 resurgent terms**, all of which have been intensively tested at **high precision**, for all **actions** $m \leq 8$.
2. The **6 Stokes constants** have been determined to better than **1000 digits**.
3. Excellent **freeware**, from `Pari-GP` in Bordeaux, was vital to this enterprise.
4. First and second **derivatives** of Γ and **suppressions** by $1/n^{5/6}$ make Richardson acceleration infeasible. I used systematic **matrix inversion**.
5. The presence of **logarithms** in trans-series has been ascribed to **resonant actions**. Michael Borinsky and I find this **misleading**. We showed that a closely analogous **second-order** problem is both resonant and **log-free**.
6. We have been guided by helpful advice from **Gerald Dunne** and encouraged by the programme and workshops on *Applicable Resurgent Asymptotics* hosted by the **Isaac Newton Institute** in Cambridge.
7. For physicists who wonder, as I did, why one might consider **imaginary** coupling, I remark that the idea goes back 70 years, to **Freeman Dyson**, who was a notable inquirer into both mathematics and quantum field theory.