Double Copy Variations

Clifford Cheung
Caltech

2108.02276 (CC, Mangan)
2201.05147 (CC, Parra-Martinez, Sivaramakrishnan)
2204.07130 (CC, Mangan, Parra-Martinez, Shah)

“ *the theory* ”

action

↓

amplitude

“ *the observable* ”

action

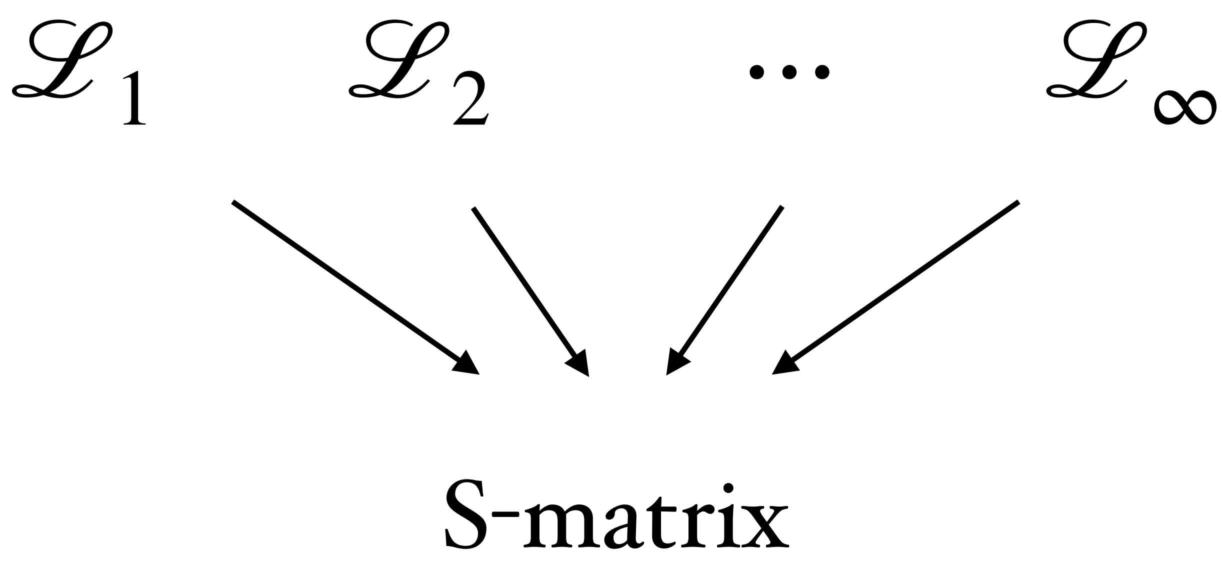
“ S-matrix program ”

amplitude

The fields of QFT are integration variables of the path integral. You can always change variables.

$$Z[J] \sim \int [d\phi]\, e^{iS[\phi]+i\int J\phi}$$

Thus, Lagrangians are infinitely redundant!



Scattering amplitudes are a powerful diagnostic for identifying structure in QFT and gravity.

Scattering amplitudes are a powerful diagnostic for identifying structure in QFT and gravity.

- dual conformal symmetry
- amplitude = *vol* (polytope)
- amplitude = *correlator* (CFT)
- (gauge) ^{2} = gravity
- … and much, much more …

Bern, Carrasco, and Johansson (BCJ) discovered a hidden duality structure in gauge theory + gravity.

$$(\text{gauge})^2 = \text{gravity}$$

Color - Kinematics Duality: scattering exhibits an isomorphism between color and kinematics.

Double Copy: swapping color for kinematics yields the correct amplitudes of new theories.

In three-particle scattering, double copy is trivial.

3pt gluon

$$A(1_a^- 2_b^- 3_c^+) = \frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} f_{abc}$$

$$A(1_a^+ 2_b^+ 3_c^-) = \frac{[12]^3}{[13][32]} f_{abc}$$

3pt graviton

$$M(1^- 2^- 3^+) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2}$$

$$M(1^+ 2^+ 3^-) = \frac{[12]^6}{[13]^2[32]^2}$$

Simply replace f_{abc} with the kinematic structure.

In four-particle scattering, we see a small miracle.

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$c_s = f_{abe} f_{cde}$ $\quad$ $c_t = f_{bce} f_{ade}$ $\quad$ $c_u = f_{cae} f_{bde}$

Here n_s, n_t, n_u are non-unique functions of $p_i p_j$, $p_i e_j$, $e_i e_j$ that satisfy kinematic Jacobi identities.

$$c_s + c_t + c_u = 0$$

(mathematical identity)

$$n_s + n_t + n_u = 0$$

(true on-shell)

Perhaps color and kinematics are interchangeable since they satisfy the same algebraic identities?

Bern, Carrasco, Johansson (0805.3993)

Perhaps color and kinematics are interchangeable since they satisfy the same algebraic identities?

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

4pt gluon
(*polarization* = e_μ)

Bern, Carrasco, Johansson (0805.3993)

Perhaps color and kinematics are interchangeable since they satisfy the same algebraic identities?

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

4pt gluon
(*polarization* = e_μ)

$\downarrow\downarrow\downarrow$ " *double copy* "

$$M_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

4pt graviton
(*polarization* = $e_\mu e_\nu$)

Bern, Carrasco, Johansson (0805.3993)

Perhaps color and kinematics are interchangeable since they satisfy the same algebraic identities?

$$A_4 = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

4pt gluon
(*polarization* = e_μ)

$\downarrow\downarrow\downarrow$ " *double copy* "

$$M_4 = \frac{n_s^2}{s} + \frac{n_t^2}{t} + \frac{n_u^2}{u}$$

4pt graviton
(*polarization* = $e_\mu e_\nu$)

Double copy is proven at tree + recycled to loop integrands via unitarity to SUGRA, and LIGO.

Bern, Carrasco, Johansson (0805.3993)

Double copy is weirdly ubiquitous among “nice” theories with very few coupling constants.

$\mathcal{N}>4$ supergravity	• $\mathcal{N}=4$ SYM theory • SYM theory ($\mathcal{N}=1,2,4$)	[1, 2, 31, 291, 292]	
$\mathcal{N}=4$ supergravity with vector multiplets	• $\mathcal{N}=4$ SYM theory • YM-scalar theory from dim. reduction	[1, 2, 31, 293]	• $\mathcal{N}=2\times\mathcal{N}=2$ construction is also possible
pure $\mathcal{N}<4$ supergravity	• (S)YM theory with matter • (S)YM theory with ghosts	[188]	• ghost fields in fundamental rep
Einstein gravity	• YM theory with matter • YM theory with ghosts	[188]	• ghost/matter fields in fundamental rep
$\mathcal{N}=2$ Maxwell-Einstein supergravities (generic family)	• $\mathcal{N}=2$ SYM theory • YM-scalar theory from dim. reduction	[120]	• truncations to $\mathcal{N}=1,0$ • only adjoint fields
$\mathcal{N}=2$ Maxwell-Einstein supergravities (homogeneous theories)	• $\mathcal{N}=2$ SYM theory with half hypermultiplet • YM-scalar theory from dim. reduction with matter fermions	[121, 294]	• fields in pseudo-real reps • include Magical Supergravities
$\mathcal{N}=2$ supergravities with hypermultiplets	• $\mathcal{N}=2$ SYM theory with half hypermultiplet • YM-scalar theory from dim. red. with extra matter scalars	[121, 240]	• fields in matter representations • construction known in particular cases
$\mathcal{N}=2$ supergravities with vector/ hypermultiplets	• $\mathcal{N}=1$ SYM theory with chiral multiplets • $\mathcal{N}=1$ SYM theory with chiral multiplets	[239, 241, 295]	• construction known in particular cases
$\mathcal{N}=1$ supergravities with vector multiplets	• $\mathcal{N}=1$ SYM theory with chiral multiplets • YM-scalar theory with fermions	[188, 239, 241, 295]	• fields in matter reps • construction known in particular cases
$\mathcal{N}=1$ supergravities with chiral multiplets	• $\mathcal{N}=1$ SYM theory with chiral multiplets • YM-scalar with extra matter scalars	[188, 239, 241, 295]	• fields in matter reps • construction known in particular cases
Einstein gravity with matter	• YM theory with matter • YM theory with matter	[1, 188]	• construction known in particular cases

$R+\phi R^2+R^3$ gravity	• YM theory + $F^3+F^4+\ldots$ • YM theory + $F^3+F^4+\ldots$	[296]	• extension to $\mathcal{N}\leq 4$ replacing one of the factors by undeformed SYM theory
Conformal (super)gravity	• DF^2 theory • (S)YM theory	[152, 153]	• $\mathcal{N}\leq 4$ • involves specific gauge theory with dimension-six operators
3*D* maximal supergravity	• BLG theory • BLG theory	[119, 243, 297]	• 3*D* only
YME supergravities	• SYM theory • YM + ϕ^3 theory	[120, 125, 133, 134, 140, 214, 216, 257, 283, 285, 289]	• trilinear scalar couplings • $\mathcal{N}=0,1,2,4$ possible
Higgsed supergravities	• SYM theory (Coulomb branch) • YM + ϕ^3 theory with extra massive scalars	[122]	• $\mathcal{N}=0,1,2,4$ possible • massive fields in supergravity
$U(1)_R$ gauged supergravities	• SYM theory (Coulomb branch) • YM theory with SUSY broken by fermion masses	[123]	• $0\leq\mathcal{N}\leq 8$ possible • SUSY is spontaneously broken • only theories with Minkowski vacua
gauged supergravities (nonabelian)	• SYM theory (Coulomb branch) • YM + ϕ^3 theory with massive fermions	[284]	• SUSY is spontaneously broken • only theories with Minkowski vacua

DBI theory	• NLSM • (S)YM theory	[125, 126, 285, 298–301]	• $\mathcal{N}\leq 4$ possible • also obtained as $\alpha'\to 0$ limit of abelian Z-theory
Volkov-Akulov theory	• NLSM • SYM theory (external fermions)	[125, 302–308]	• restriction to external fermions from supersymmetric DBI
Special Galileon theory	• NLSM • NLSM	[125, 285, 301, 306, 309]	• theory is also characterized by its soft limits
DBI + (S)YM theory	• NLSM + ϕ^3 • (S)YM theory	[125, 126, 156, 285, 298–300, 306, 310]	• $\mathcal{N}\leq 4$ possible • also obtained as $\alpha'\to 0$ limit of semi-abelianized Z-theory
DBI + NLSM theory	• NLSM • YM + ϕ^3 theory	[125, 126, 156, 285, 298–300]	

Bern, Carrasco, Chiodaroli, Johansson, Roiban (1909.01358)

Double copy transcends gauge theory + gravity!

Double copy transcends gauge theory + gravity!

- gluon $\otimes$ gluon = graviton

Double copy transcends gauge theory + gravity!

- gluon $\otimes$ gluon = graviton
- pion $\otimes$ pion = special Galileon

Double copy transcends gauge theory + gravity!

- gluon $\otimes$ gluon = graviton
- pion $\otimes$ pion = special Galileon
- gluon $\otimes$ pion = Born-Infeld photon

Double copy transcends gauge theory + gravity!

- gluon $\otimes$ gluon = graviton
- pion $\otimes$ pion = special Galileon
- gluon $\otimes$ pion = Born-Infeld photon

Is the double copy just the tetrad formalism? No.

Double copy transcends gauge theory + gravity!

- gluon $\otimes$ gluon = graviton
- pion $\otimes$ pion = special Galileon
- gluon $\otimes$ pion = Born-Infeld photon

Is the double copy just the tetrad formalism? No.
Is it just open/closed string duality? Unclear.
Anyway, a QFT fact deserves a QFT explanation.

The double copy is a proven fact about on-shell, flat-space, tree-level scattering amplitudes.

Why is it true?

The double copy is a proven fact about on-shell, flat-space, tree-level scattering amplitudes.

Why is it true? When is it true?

The double copy is a proven fact about on-shell, flat-space, tree-level scattering amplitudes.

Why is it true? When is it true?

- which theories?
- higher-loops?
- non-perturbatively?
- curved spacetime?
- classical solutions?

The double copy is a proven fact about on-shell, flat-space, tree-level scattering amplitudes.

Why is it true? When is it true?

- which theories?
- higher-loops?
- non-perturbatively?
- curved spacetime?
- classical solutions?

We don't understand double copy. And the stakes are not low: $(\text{lattice QCD})^2 = \text{QG}$?

My claim: there is a simple field theory origin of the double copy for the pion and its cousins:

color algebra $V^a T^a$

$\downarrow\downarrow\downarrow$

diff algebra $V_\mu \partial^\mu$

Proof of principle duality map for EOM, actions, loops, classical solutions, currents, integrability.

Outline

1. Double Copy via Equations of Motion

2. Double Copy in Curved Spacetime

3. Double Copy at all Perturbative Orders

double copy via
equations of motion

The usual textbook Lagrangian for the NLSM is

$$\mathscr{L}^{\text{NLSM}} = \frac{1}{2} j_\mu^a j^{a\mu} + \pi^a J^a$$

← on-shell external source

where the chiral current is

$$j_\mu^a = i\,\text{tr}[g^{-1}\partial_\mu g\, T^a] \quad \text{where} \quad g = e^{i\pi} \quad \text{or} \quad \frac{1 + i\pi/2}{1 - i\pi/2} \quad \text{or} \quad \cdots$$

and the EOM says the chiral current is conserved

$$\partial^\mu j_\mu^a = J^a$$

← conserved modulo external sources

Let us define a first-order formulation of NLSM,

(a) $\partial_{[\mu} j^a_{\nu]} + f^{abc} j^b_\mu j^c_\nu = 0 \xrightarrow{\text{implicit}} j^a_\mu = i\,\mathrm{tr}[g^{-1}\partial_\mu g\, T^a]$

(b) $\partial^\mu j^a_\mu = J^a$

$\partial^\mu (\mathrm{a})_{\mu\nu} + \partial_\nu (\mathrm{b})$ $\quad \Box\, j^a_\mu + f^{abc} j^{b\nu} \partial_\nu j^c_\mu = \partial_\mu J^a$

Feynman propagator

cubic self-interaction

chiral current sourced by derivative of scalar source

Let us define a first-order formulation of NLSM,

(a) $\partial_{[\mu} j^a_{\nu]} + f^{abc} j^b_\mu j^c_\nu = 0 \xrightarrow{\text{implicit}} j^a_\mu = i\,\mathrm{tr}[g^{-1}\partial_\mu g\, T^a]$

(b) $\partial^\mu j^a_\mu = J^a$

$\partial^\mu (\text{a})_{\mu\nu} + \partial_\nu (\text{b})$ $\quad \Box\, j^a_\mu + f^{abc} j^{b\nu} \partial_\nu j^c_\mu = \partial_\mu J^a$

The chiral current is agnostic about field basis redundancy (also see Freedman-Townsend, 1981).

We want to scatter scalars, not chiral currents,

$$j^a_\mu = -\,\partial_\mu \pi^a + \cdots \qquad \longrightarrow \qquad \pi^a = -\,\frac{q^\mu j^a_\mu}{q\partial} + \cdots$$

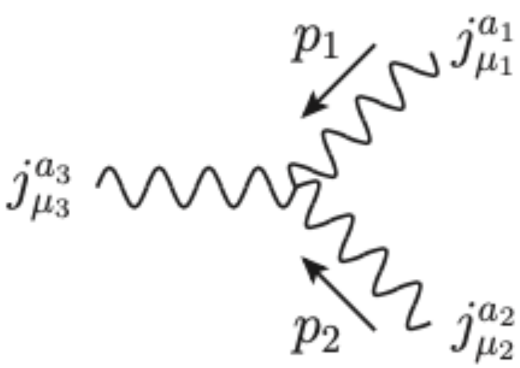
for reference q. All nonlinear field ambiguities vanish on-shell, so π^a = exotically polarized j^a_μ.

$$\langle \pi^a(p)\rangle_J = \tilde{\varepsilon}^\mu(p)\langle j^a_\mu(p)\rangle_J \qquad \text{where} \qquad \tilde{\varepsilon}_\mu(p) = \frac{iq_\mu}{pq}$$

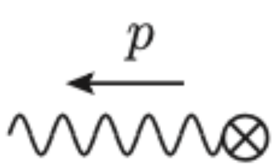
$$\langle \pi^{a_1}(p_1)\pi^{a_2}(p_2)\cdots\pi^{a_n}(p_n)\rangle_{J=0} = \left[\left(\prod_{i=1}^{n-1}\frac{1}{i}\frac{\delta}{\delta J^{a_i}(p_i)}\right)\tilde{\varepsilon}^\mu(p_n)\langle j^{a_n}_\mu(p_n)\rangle_J\right]_{J=0}$$

NLSM Feynman Rules

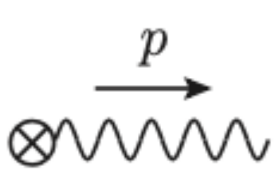
$$= \frac{i\delta^{a_1 a_2}\eta^{\mu_1\mu_2}}{p^2}$$



$$= -if^{a_1a_2a_3}(ip_2^{\mu_1}\eta^{\mu_2\mu_3} - ip_1^{\mu_2}\eta^{\mu_1\mu_3})$$

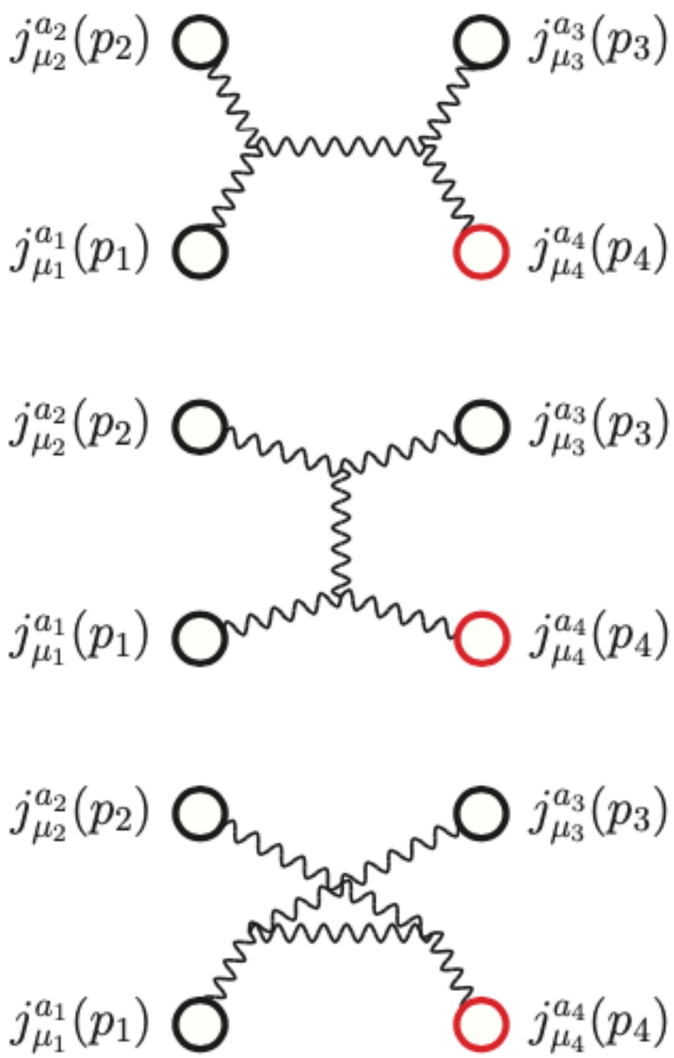


$$= \varepsilon_\mu = ip_\mu$$



$$= \tilde{\varepsilon}_\mu = \frac{iq_\mu}{pq}$$

The kinematic Jacobi identity holds off-shell!



$$n_s = p_1^{\mu_2}(p_1+p_2)^{\mu_3}\eta^{\mu_1\mu_4} + p_2^{\mu_1}p_3^{\mu_2}\eta^{\mu_3\mu_4} - \{1 \leftrightarrow 2\}$$

$$n_t = p_2^{\mu_3}(p_2+p_3)^{\mu_1}\eta^{\mu_2\mu_4} + p_3^{\mu_2}p_1^{\mu_3}\eta^{\mu_1\mu_4} - \{2 \leftrightarrow 3\}$$

$$n_u = p_3^{\mu_1}(p_3+p_1)^{\mu_2}\eta^{\mu_3\mu_4} + p_1^{\mu_3}p_2^{\mu_1}\eta^{\mu_2\mu_4} - \{3 \leftrightarrow 1\}$$

$\longrightarrow$ $n_s + n_t + n_u = 0$ But why ???

BAS $$\Box \phi^{a\bar{a}} + \frac{1}{2} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = J^{a\bar{a}}$$

isomorphic

NLSM $$\Box j^a_\mu + f^{abc} j^{b\nu} \partial_\nu j^c_\mu = \partial_\mu J^a$$

Define " $\otimes$ NLSM " double copy acting on fields,

$$V^a \overset{\text{NLSM}}{\to} V_\mu$$

$$f^{abc} V^b W^c \overset{\text{NLSM}}{\to} V^\nu \partial_\nu W_\mu - W^\nu \partial_\nu V_\mu$$

$$J^a \overset{\text{NLSM}}{\to} \partial_\mu J$$

By inspection, kinematic algebra = diff algebra!

$$[\, V_\mu \partial^\mu, W_\nu \partial^\nu \,] = (V^\nu \partial_\nu W_\mu - W^\nu \partial_\nu V_\mu) \partial^\mu$$

This implements the double copy at the level of fields. So we can apply it to conserved currents.

color current $\mathscr{K}^{\bar{a}}_{\alpha} = f^{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{b}}\overleftrightarrow{\partial}_{\alpha}\phi^{a\bar{c}}$

$\downarrow\downarrow\downarrow$

kinematic current $\mathscr{K}_{\mu\alpha} = j^{a\nu}\partial_{\nu}\overleftrightarrow{\partial}_{\alpha}j^{a}_{\mu}$

Note: the fundamental BCJ relation is literally the conservation equation after color-stripping.

Coleman-Mandula forbids a kinematic symmetry.

$$\mathcal{K}_{\mu\alpha} = -\Box\, T^{\text{NLSM}}_{\mu\alpha} + \text{improvement terms}$$

Indeed, there's no symmetry since charges vanish!

current $\quad \partial^\alpha J_\alpha = 0 \qquad Q = \int d^3x\, J_0(x) \qquad \partial_0 Q = 0$

derivative of current $\quad \partial^\alpha \partial_\mu J_\alpha = 0 \qquad Q_\mu = \int d^3x\, \partial_\mu J_0(x) \qquad \partial_0 Q_\mu = 0$

$$\longrightarrow \quad a^\mu Q_\mu = \lim_{a\to 0} \int d^3x\, \left[J_0(x+a) - J_0(x)\right] = 0$$

BAS $\otimes$ NLSM = NLSM

	fields	EOM
BAS	$\phi^{a\bar{a}}$	$\Box\phi^{a\bar{a}} + \frac{1}{2} f^{abc} f^{\bar{a}\bar{b}\bar{c}} \phi^{b\bar{b}} \phi^{c\bar{c}} = J^{a\bar{a}}$
NLSM	j_μ^a	$\Box j_\mu^a + f^{abc} j^{b\nu} \partial_\nu j_\mu^c = \partial_\mu J^a$

NLSM $\otimes$ NLSM = SG

	fields	EOM
NLSM	j_μ^a	$\partial_{[\mu} j_{\nu]}^a + f^{abc} j_\mu^b j_\nu^c = 0$ $\partial^\mu j_\mu^a = J^a$
SG new form!	$j_{\mu\bar{\mu}}$	$\partial_{[\mu} j_{\nu]\bar{\mu}} + j_\mu{}^{\bar{\nu}} \partial_{\bar{\nu}} j_{\nu\bar{\mu}} - j_\nu{}^{\bar{\nu}} \partial_{\bar{\nu}} j_{\mu\bar{\mu}} = 0$ $\partial^\mu j_{\mu\bar{\nu}} = \partial_{\bar{\nu}} J$

YM ⊗ NLSM = BI

	fields	EOM
YM	A^a_μ	$F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + f^{abc} A^b_\mu A^c_\nu$
	$F^a_{\mu\nu}$	$\partial^\mu F^a_{\mu\nu} + f^{abc} A^{b\mu} F^c_{\mu\nu} = J^a_\nu$
BI	$A_{\mu\bar{\mu}}$	$F_{\mu\nu\bar{\mu}} = \partial_{[\mu} A_{\nu]\bar{\mu}} + A_\mu{}^{\bar{\nu}} \partial_{\bar{\nu}} A_{\nu\bar{\mu}} - A_\nu{}^{\bar{\nu}} \partial_{\bar{\nu}} A_{\mu\bar{\mu}}$
new form!	$F_{\mu\nu\bar{\mu}}$	$\partial^\mu F_{\mu\nu\bar{\mu}} + A^{\mu\bar{\nu}} \partial_{\bar{\nu}} F_{\mu\nu\bar{\mu}} - \partial_{\bar{\nu}} A^\mu{}_{\bar{\mu}} F_{\mu\nu}{}^{\bar{\nu}} = \partial_{\bar{\mu}} J_\nu$

double copy in
curved spacetime

To double copy in curved spacetime we need to generalize flat-space amplitudes and kinematics.

flat space	AdS	symmetric space
on-shell amplitude	boundary correlator	"on-shell correlator"
plane wave	bulk-boundary propagator	linearized solutions
delta function	contact correlator	contact correlator
momentum	conformal generator	"isometric momentum"

Herderschee, Roiban, Teng (2201.05067)

A symmetric manifold has the Killing vectors

$$K_A = K_A^\mu \partial_\mu \qquad \text{where} \qquad [K_A, K_B] = F_{AB}{}^C K_C$$

The spacetime and Killing metrics are related via

$$g_{\mu\nu} = g_{AB} K_\mu^A K_\nu^B$$

⟵ isometric indices are overcomplete

We use this to map spacetime indices, μ, ν, ρ, etc. to isometric indices, A, B, C, etc.

By defining a shorthand for the Lie derivative,

$$\mathbb{D}_A = \mathscr{L}_{K_A} \qquad \text{where} \qquad [\mathbb{D}_A, \mathbb{D}_B] = F_{AB}{}^{C}\mathbb{D}_C$$

we then recast certain derivatives of a scalar

$$\nabla^2\phi = \mathbb{D}^2\phi \qquad \text{and} \qquad \nabla_\mu\phi\,\nabla^\mu\phi = \mathbb{D}_A\phi\mathbb{D}^A\phi$$

Thus, any Lagrangian $\mathscr{L}(\phi, \partial_\mu\phi)$ can be recast in terms of isometric variables.

The external wavefunction of leg i is a solution to the linearized equations of motion labeled by p_i,

$$\mathbb{D}^2\psi(p_i, x) = 0$$

where the analog of momentum conservation is

$$(\mathbb{D} + \mathbb{D}_i)\psi(p_i, x) = 0$$

and "isometric momentum" $\mathbb{D}_i$ is an isometry transformation on p_i and $[\mathbb{D}_{iA}, \mathbb{D}_{jB}] = \delta_{ij} F_{AB}{}^{C} \mathbb{D}_{iC}$.

The analog of the momentum delta function is

$$\Delta_n = \Delta(p_1, \cdots, p_n) = \int_x \prod_{i=1}^{n} \psi(p_i, x)$$

The $\mathbb{D}_i$ are non-commutative on-shell momenta!

$$\left(\sum_{i=1}^{n} \mathbb{D}_i \right) \Delta_n = 0 \qquad \text{and} \qquad \mathbb{D}_i^2 \Delta_n = 0$$

"momentum conservation" "on-shell condition"

Finally, we define an "on-shell correlator",

$$A(p_1, \cdots, p_n) = \left(\prod_{i=1}^{n} \int_{x_i} \psi(p_i, x_i)\, \nabla^2_{x_i} \right) \langle \phi(x_1) \cdots \phi(x_n) \rangle$$

which has a differential representation,

$$A(p_1, \cdots, p_n) = \mathbb{A}(\mathbb{D}_1, \cdots, \mathbb{D}_n) \Delta(p_1, \cdots, p_n)$$

This generalizes the AdS construction of Roiban et al. (2106.10822) to arbitrary symmetric spaces.

Tree-level on-shell correlators are obtained from Feynman rules with non-commutative momenta.

i) ϕ^3 theory :

$$\mathscr{L} = \frac{1}{2}\nabla\phi^2 + \frac{\kappa}{6}\phi^3$$

Tree-level on-shell correlators are obtained from Feynman rules with non-commutative momenta.

i) ϕ^3 theory :

$$\mathscr{L} = \frac{1}{2}\nabla\phi^2 + \frac{\kappa}{6}\phi^3$$

$$\mathbb{A}_3 = \kappa$$

$$\mathbb{A}_4 = \frac{\kappa^2}{\mathbb{D}_{12}^2} + \frac{\kappa^2}{\mathbb{D}_{23}^2} + \frac{\kappa^2}{\mathbb{D}_{31}^2}$$

$$\mathbb{A}_5 = \frac{\kappa^3}{\mathbb{D}_{12}^2\mathbb{D}_{34}^2} + \cdots$$

← propagator denominators all mutually commute!

Tree-level on-shell correlators are obtained from Feynman rules with non-commutative momenta.

ii) $\nabla\phi^4$ theory :

$$\mathscr{L} = \frac{1}{2}\nabla\phi^2 + \frac{\lambda}{8}\nabla\phi^4$$

Tree-level on-shell correlators are obtained from Feynman rules with non-commutative momenta.

ii) $\nabla\phi^4$ theory :

$$\mathscr{L} = \frac{1}{2}\nabla\phi^2 + \frac{\lambda}{8}\nabla\phi^4$$

$$\mathbb{A}_4 = \lambda(\mathbb{D}_{123}\cdot\mathbb{D}_1)(\mathbb{D}_2\cdot\mathbb{D}_3) + \cdots$$

$$\mathbb{A}_6 = \lambda(\mathbb{D}_{12345}\cdot\mathbb{D}_1)(\mathbb{D}_2\cdot\mathbb{D}_{345})\frac{\lambda}{\mathbb{D}^2_{345}}[(\mathbb{D}_{345}\cdot\mathbb{D}_3)(\mathbb{D}_4\cdot\mathbb{D}_5) + \cdots] + \cdots$$

$$+\lambda(\mathbb{D}_{12345}\cdot\mathbb{D}_{123})\frac{\lambda}{\mathbb{D}^2_{123}}[(\mathbb{D}_{123}\cdot\mathbb{D}_1)(\mathbb{D}_2\cdot\mathbb{D}_3) + \cdots](\mathbb{D}_4\cdot\mathbb{D}_5) + \cdots$$

In curved spacetime, the NLSM is obtained via

$$V^a \overset{\text{NLSM}}{\to} V_A$$

$$f^{abc} V^b W^c \overset{\text{NLSM}}{\to} V^B \mathbb{D}_B W_A - W^B \mathbb{D}_B V_A$$

$$J^a \overset{\text{NLSM}}{\to} \mathbb{D}_A J$$

The kinematic algebra defines gauged isometries,

$$[\, V_A \mathbb{D}^A, W_B \mathbb{D}^B \,] \sim (V^B \mathbb{D}_B W_A - W^B \mathbb{D}_B V_A) \mathbb{D}^A$$

double copy at
all perturbative orders

In two-dimensional spacetime, we can say more.

$$\lim_{N\to\infty} U(N) \sim \mathrm{Diff}_{S^1\times S^1}$$

Concretely, for odd N there is a generator basis,

$$[\mathbf{T}_{p_i}, \mathbf{T}_{p_j}] = i f_{p_i p_j}{}^{p_k} \mathbf{T}_{p_k}$$

where each of N^2 generators is labeled by a two-vector, so e.g. for $\mathbf{T}_p$ has $p \in \mathbb{Z}_N \times \mathbb{Z}_N$.

Jens Hoppe (1989)

In this basis the $U(N)$ structure constants are

$$f_{p_i p_j}{}^{p_k} = -\frac{N}{2\pi}\sin\left(\frac{2\pi}{N}\langle ij\rangle\right)\delta_{p_i+p_j,p_k}$$

$$\overset{N\to\infty}{=} -\langle ij\rangle\delta_{p_i+p_j,p_k}$$

where $\langle ij\rangle = \epsilon^{\mu\nu}p_{i\mu}p_{j\nu}$. Note this is identical to

$$\{A,B\} = \frac{\partial A}{\partial t}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial t} = -\epsilon^{\mu\nu}\partial_\mu A\partial_\nu B$$

so $N\to\infty$ the limit yields the Poisson algebra.

In two dimensional spacetime, the double copy is literally taking the $N \to \infty$ limit of $U(N)$.

color algebra $V^a T^a$

$\downarrow\downarrow\downarrow$ $N \to \infty$

Poisson ~ diff algebra $\epsilon^{\mu\nu}\partial_\mu V \partial_\nu$

This can be applied off-shell to Lagrangians, correlators, currents, classical solutions.

Mechanically, " $\otimes$ NLSM " double copy becomes

$$V^a \overset{\text{NLSM}}{\rightarrow} V_\mu$$

$$f^{abc}V^bW^c \overset{\text{NLSM}}{\rightarrow} \epsilon^{\mu\nu}\partial_\mu V \partial_\nu W$$

$$g_{ab}V^aW^b \overset{\text{NLSM}}{\rightarrow} \int VW$$

The last line defines a bilinear form for Poisson and diff algebras, derived explicitly from the $N \to \infty$ limit of $U(N)$.

This gives an all orders in perturbation theory, Lagrangian-level definition of the double copy.

	Lagrangian
BAS	$\frac{1}{2}\partial_\mu\phi_{a\bar{a}}\partial^\mu\phi^{a\bar{a}}+\frac{1}{6}f_{abc}f_{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}}$
ZM - NLSM	$\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi^a+\frac{1}{6}f_{abc}\epsilon^{\mu\nu}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c$
SG	$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{6}\epsilon^{\mu\nu}\epsilon^{\bar{\mu}\bar{\nu}}\phi\partial_\mu\partial_{\bar{\mu}}\phi\partial_\nu\partial_{\bar{\nu}}\phi$

Feynman diagrams yield the off-shell numerators,

$$n_s = \langle 12 \rangle \langle 34 \rangle \qquad n_t = \langle 23 \rangle \langle 14 \rangle \qquad n_u = \langle 31 \rangle \langle 24 \rangle$$

which automatically satisfy the kinematic Jacobi identity, which is literally the Schouten identity!

$$n_s + n_t + n_u = 0$$

The Feynman rules manifest the double copy at all orders in perturbation theory, i.e. all loops.

Notably, ZM - NLSM - PCM have integrable properties and infinite towers of currents.

$$A_\mu = \frac{1}{1-\lambda^2}(\epsilon^{\mu\nu}\partial_\nu\phi^a + \lambda\partial_\mu\phi^a)T^a$$ (Lax connection)

$$W = P\exp\left[-\int^x dx' A(x')\right]$$ (Wilson line)

$$J_\mu = \epsilon^{\mu\nu}\partial_\nu W = \sum_{k=0}^{\infty}\lambda^{-k}J_\mu^{(k)}$$ where $$\partial^\mu J_\mu^{(k)} = 0$$

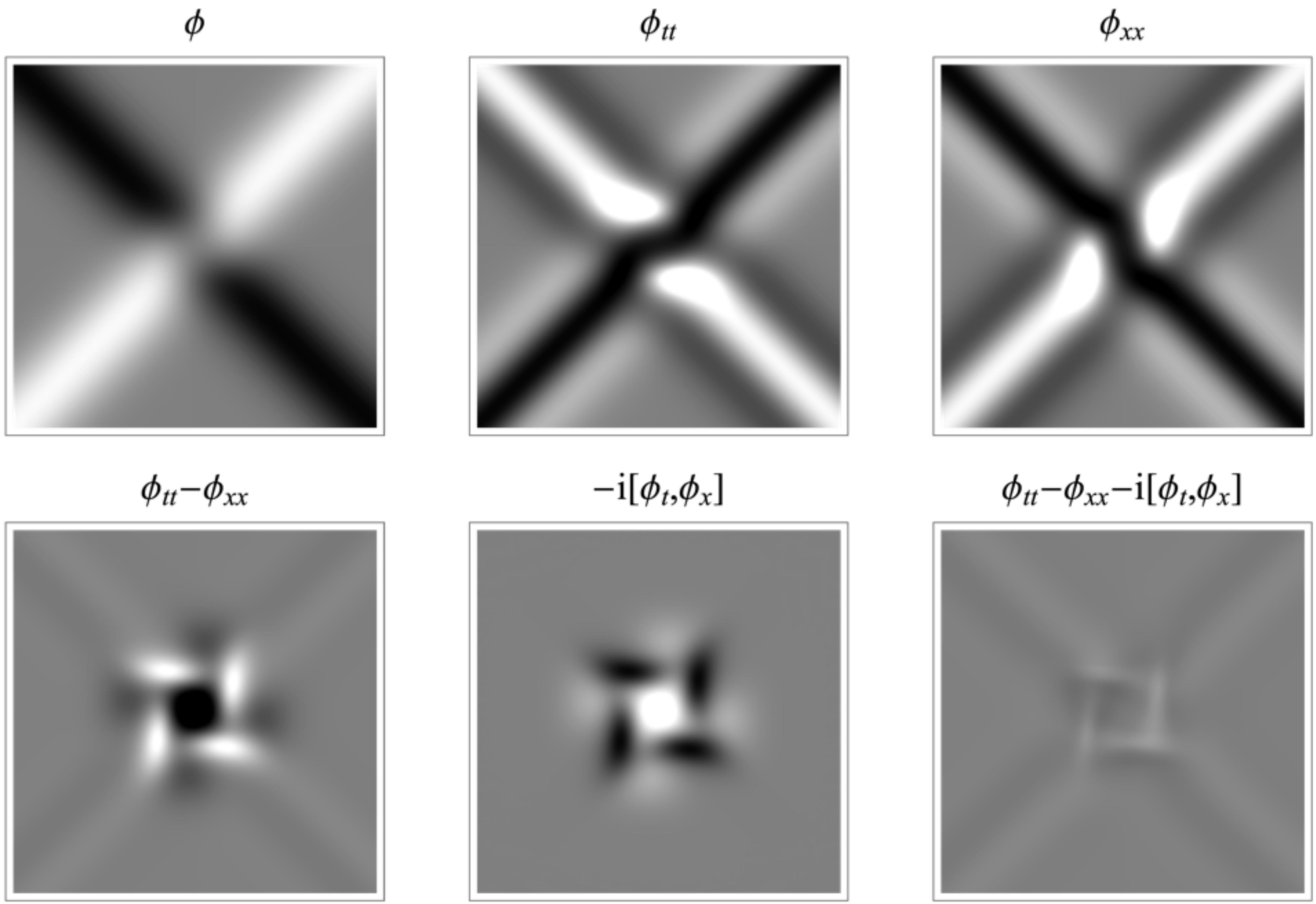
We can explicitly double copy every quantity above simply by sending $\phi^a T^a \to \epsilon^{\mu\nu}\partial_\mu\phi\partial_\nu$.

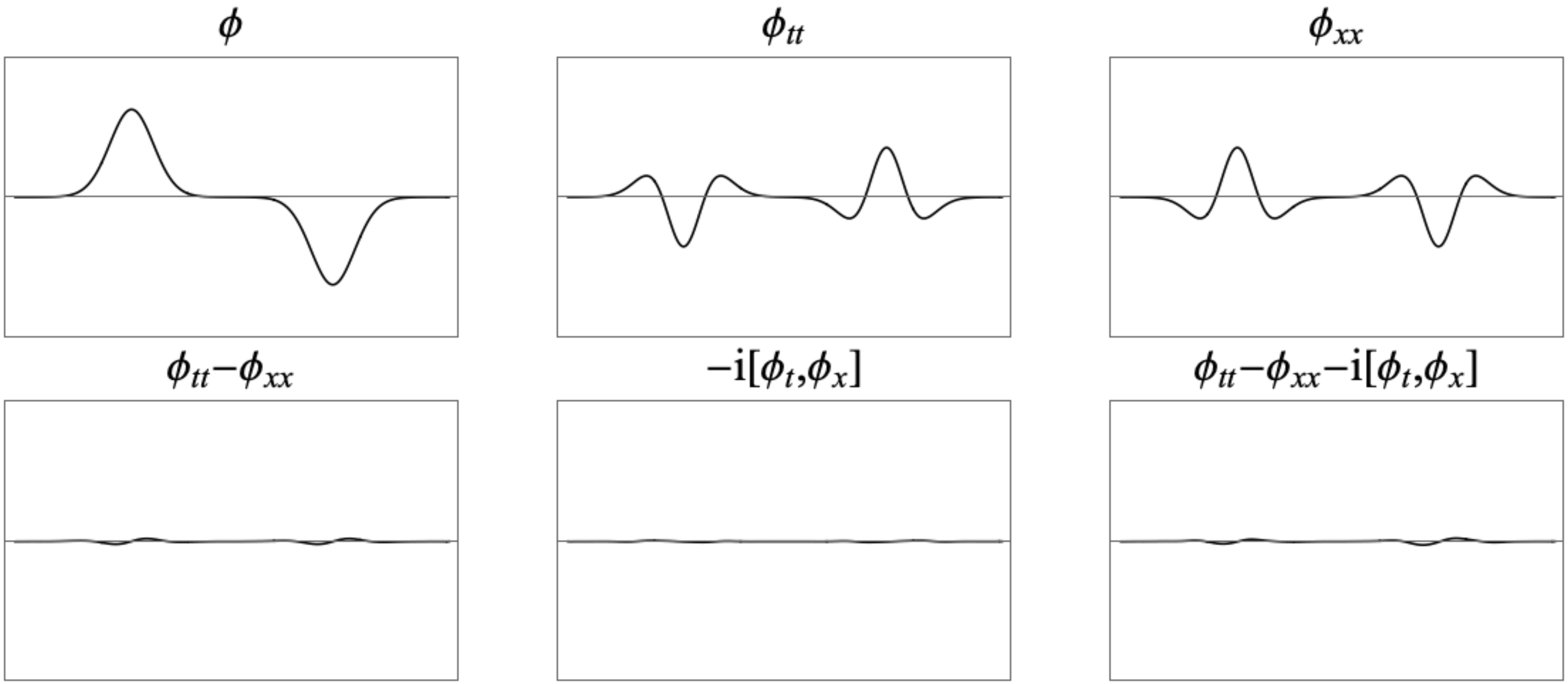
Amusingly, we can double copy arbitrary large-field, non-perturbative classical solutions!

SG solution: $$\phi(x) = \int_p e^{ipx}\tilde{\phi}(p)$$

NLSM solution: $$\phi^a(x)T^a = \int_p e^{ipx}\tilde{\phi}(p)\mathbf{T}_p$$

i) Take any solution $\phi(x)$ of SG, ii) compute its Fourier transform $\tilde{\phi}(p)$, iii) convolve this $\mathbf{T}_p$. The result is automatically a NLSM solution!


ϕ
ϕtt
ϕxx
ϕtt−ϕxx
−i[ϕt,ϕx]
ϕtt−ϕxx−i[ϕt,ϕx]


ϕ
ϕ_{tt}
ϕ_{xx}
$\phi_{tt}-\phi_{xx}$
$-i[\phi_t,\phi_x]$
$\phi_{tt}-\phi_{xx}-i[\phi_t,\phi_x]$

conclusions

- Scattering amplitudes have uncovered hidden structures lurking inside real-world theories like gravitons, gluons, and pions.

- For scalar theories like the NLSM and SG, the origin of the double copy is the replacement of the color algebra to diff algebra. This gives a QFT definition of the double copy.

- We apply the double copy to off-shell equations of motion and Lagrangians, curved spacetime, and non-perturbative, large-field configurations.

thank you!