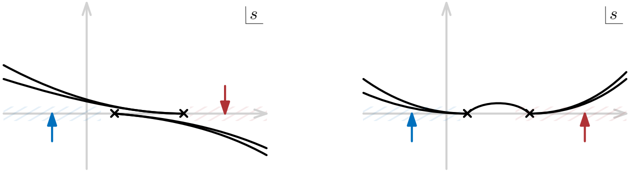
What is the $i\varepsilon$ for the S-matrix?

Hofie Sigridar Hannesdottir

Institute for Advanced Study



2204.02988 with Sebastian Mizera

Congratulations NBI on >100 years!

Morgunblaðið

33. argangur. 275. tbl. — Laugardagur 4. ágúst 1951. Prentsmiðja Morgunblaðsins.

Prófessor Niels Bohr og frú hans að Bessastöðum

Í gær var prófessor Niels Bohr og frú hans, ásamt sendiherra Dana, frú Bodil Begtrup, boðið til hádegisverðar að Bessastöðum. Var þessi mynd tekin fyrir sunnan forsetabústaðinn að afloknum hádegisverðinum. Á myndinni eru herra Sveinn Björnsson, frú Bohr, forsetafrúin, Georgía Björnsson, prófessor Niels Bohr og frú Bodil Begtrup. Sjá grein á bls. 6. — (Ljósm. Mbl. Ól. K. M.)

Bretar viðurkenna þjóðnýtingu í grundvallaratriðum

Bresk samninganefnd er væntanleg til Teheran í kvöld

Formaður nefndarinnar vongóður.

Einkaskeyti til Mbl. frá Reuter—NTB

LUNDÚNUM, 3. ágúst. — Í dag lagði af stað bresk sendinefnd til Teheran undir forystu Stokes, innsiglisvarðar konungs. Á nefndin að reyna að komast að samkomulagi við Persa um lausn olíudeilunnar. Í fylgd með nefndarmönnum er Shephard, sendiherra Breta í Persíu, sem hefir dvalist í Lundúnum um nokkurra daga skeið.

STOKES VONGÓÐUR

Stokes sagði við brottförina, að hann væri bjartsýnn og lausn fengist í öllum atriðum, ef báðir aðilar sýndu góðvild og skilning. Bjóst hann við að koma til Teheran á laugardagskvöld og fara til Abadan á mánudag, til að setja sig sem best inn í deiluna. Stokes sagðist hafa fengið kveðju frá Persastjórn og taldi að það væri góðs viti.

VIÐURKENNA ÞJÓÐNÝTINGU Í GRUNDVALLARATRIÐUM

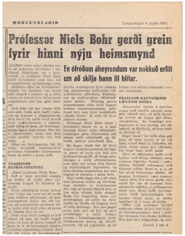
Orðsendingar þær, sem ríkisstjórnum Bretlands og Persíu hefir farið á milli að undanförnu, hafa nú verið birtar. Í orðsend-

Flugvjelin Aries 3

FAIRBANKS, 3. ágúst. — Flugvjelin Aries 3, sem flaug til Alaska um norðurskautið frá Íslandi 24 f. m., lagði í dag af stað frá Fairbanks í Alaska áleiðis til Bretlands. Flýgur hún enn um norðurskautið og Ísland eða svipaða leið og hún fór norður um. — Reuter-NTB.

Býflugurnar unnu á nautinu

Stokkhólmi, 3. ágúst. — Nýlega unnu býflugur á bola í Svíþjóð. Rjeðust að nautinu, þar



MORGUNBLAÐIÐ Laugardagur 4. ágúst 1951.

Prófessor Niels Bohr gerði grein fyrir hinni nýju heimsmynd

En ófróðum áheyrendum var nokkuð erfitt um að skilja hann til hlítar.

Picture: Hólmfríður Dagný Friðjónsdóttir

Aug. 3rd 2022

Motivation

What are the imprints of causality on the S-matrix?

Different notions of causality for the S-matrix, with rich history
(*microcausality*, *macrocausality*, *Bogoliubov causality*, *no Shapiro time advances*)

Bogoliubov, Schutzer, Tiomno, van Kampen, Gell-Mann, Goldberger, Thirring, Wanders, Iagolnitzer, Eden, Landshoff, Peres, Branson, Omnes, Chandler, Pham, Stapp, Rohrlich, Stoddart, 't Hooft, Veltman, Adams, Arkani-Hamed, Dubovsky, Grinstein, O'Connell, Wise, Giddings, Porto, Camanho, Edelstein, Maldacena, Zhiboedov, Tomboulis, Minwalla

Here: *Implement* the causal $i\varepsilon$ prescription in perturbation theory and study its *implications*.

Motivation

Causality generally thought as encoded in the **complex analytic structure** of the S-matrix.

Complexification of S-matrix standard at this point, multiple practical reasons *(dispersion relations, on-shell recursion relations, crossing symmetry).*

Can we complexify the S-matrix, while being consistent with causality?

$$\mathbf{S}(s, t_*) \overset{?}{=} \lim_{\varepsilon \to 0^+} \mathbf{S}_{\mathbb{C}}(s + i\varepsilon, t_*)$$

Notation

Transfer matrix and matrix elements:

$$S = \mathbb{1} + iT \qquad \langle \text{out} | \, T \, | \text{in} \rangle = \boldsymbol{\delta}_{\text{in, out}} \, \mathbf{T}_{\text{in} \to \text{out}}$$

For 2→2 scattering:

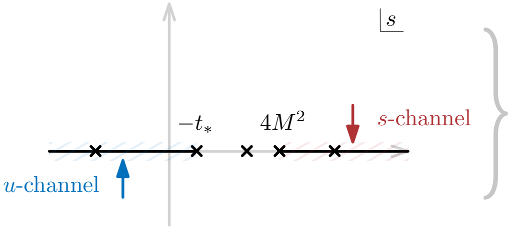
$$s = (p_1 + p_2)^2 \qquad t = (p_2 + p_3)^2 \qquad u = (p_1 + p_3)^2$$

Momentum conservation, solve for u:

$$s + t + u = \sum_{i=1}^{4} M_i^2$$

Introduction

For 2→2 scattering of lightest particle at low momentum transfer $|t_*|$:



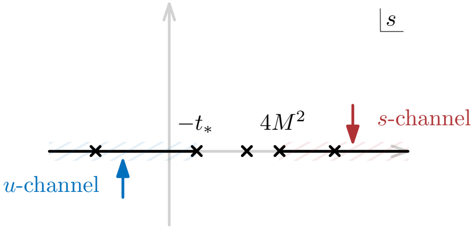
$$\mathbf{T}(s,t_*) = \lim_{\varepsilon\to 0^+} \mathbf{T}_{\mathbb{C}}(s+i\varepsilon, t_*)$$

Real on s-axis, so by the Schwarz reflection principle

$$\operatorname{Im}\mathbf{T}(s,t_*) = \operatorname{Disc}_s \mathbf{T}_{\mathbb{C}}(s,t_*)$$

Introduction

For 2→2 scattering of lightest particle at low momentum transfer $|t_*|$:



$$\mathbf{T}(s, t_*) = \lim_{\varepsilon \to 0^+} \mathbf{T}_{\mathbb{C}}(s + i\varepsilon, t_*)$$

How does this picture extend to
- massless particles?
- UV/IR divergences?
- unstable particles?

Motivation

When is the imaginary part (*unitarity, cutting rules*)

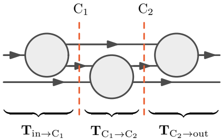
$$\operatorname{Im}\mathbf{T}(s,t_*) = \tfrac{1}{2i}\Big(\mathbf{T}(s,t_*) - \overline{\mathbf{T}(s,t_*)}\Big)$$

equal to the discontinuity (*dispersion relations*)?

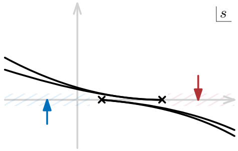
$$\operatorname{Disc}_s \mathbf{T}_{\mathbb{C}}(s,t_*) = \lim_{\varepsilon\to 0^+} \tfrac{1}{2i}\Big(\mathbf{T}_{\mathbb{C}}(s+i\varepsilon,t_*) - \mathbf{T}_{\mathbb{C}}(s-i\varepsilon,t_*)\Big)$$

Outline

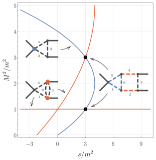
1. Unitarity:
   - *Normal and anomalous thresholds*



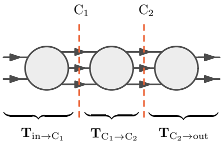
2. Causality:
   - *Feynman iε, kinematic iε, branch-cut deformations*

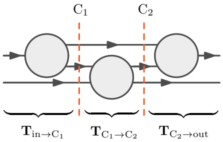


3. Locality (time permitting):
   - *Fluctuations around branch points*



1. Unitarity


C_1
C_2
$\mathbf{T}_{\text{in}\to C_1}$
$\mathbf{T}_{C_1\to C_2}$
$\mathbf{T}_{C_2\to \text{out}}$


C_1
C_2
$\mathbf{T}_{\text{in}\to C_1}$
$\mathbf{T}_{C_1\to C_2}$
$\mathbf{T}_{C_2\to \text{out}}$

Holomorphic Cutting Rules

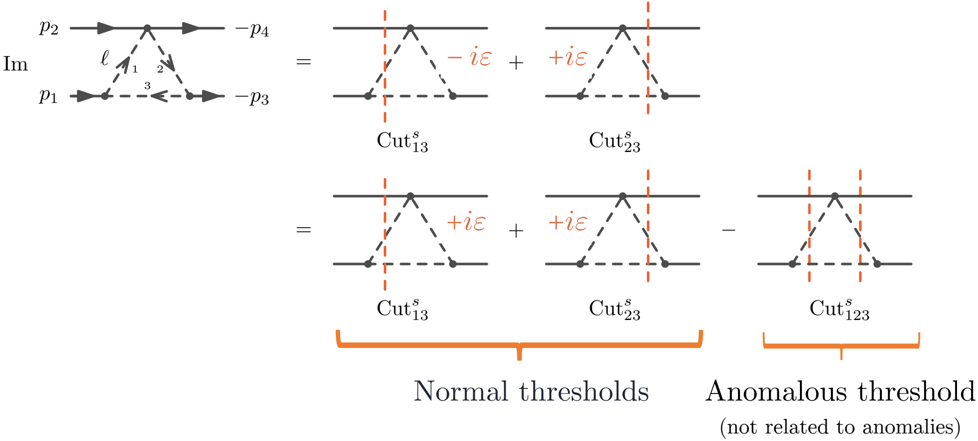
Use $SS^\dagger = \mathbb{1}$, and $S = \mathbb{1} + iT$,

$$\frac{1}{2i}(T - T^\dagger) = \frac{1}{2}TT^\dagger_, \qquad \operatorname{Im} \mathbf{T}_{\text{in}\to\text{in}} = \frac{1}{2} \sum\!\!\!\!\!\!\int_I \boldsymbol{\delta}_{\text{in},\, I} |\mathbf{T}_{\text{in}\to I}|^2.$$

Expand in $T^\dagger = T(\mathbb{1} - iT^\dagger)$:

$$\boxed{\frac{1}{2i}(T - T^\dagger) = -\frac{1}{2} \sum_{c=1}^{\infty} (-iT)^{c+1}.}$$

Normal & Anomalous Thresholds

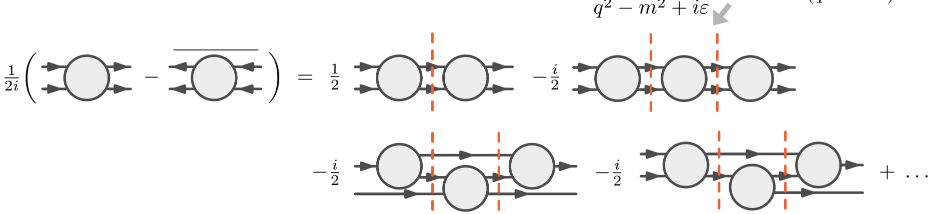


Holomorphic Cutting Rules

In general, expansion implies the *holomorphic cutting rules*

$$\frac{1}{2i}\left(\mathbf{T}_{\mathrm{in}\to\mathrm{out}} - \overline{\mathbf{T}_{\mathrm{out}\to\mathrm{in}}}\right) = \frac{i}{2} \sum_{\substack{\text{holomorphic}\\ \text{cuts C}}} (-i)^c \,\mathrm{Cut}_{\mathrm{C}}\, \mathbf{T}_{\mathrm{in}\to\mathrm{out}},$$

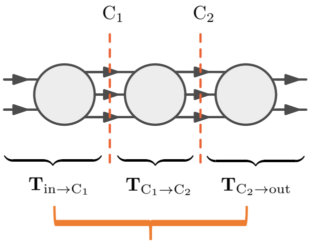
$$\frac{-1}{q^2 - m^2 + i\varepsilon} \to i\,2\pi\delta^+(q^2 - m^2)$$



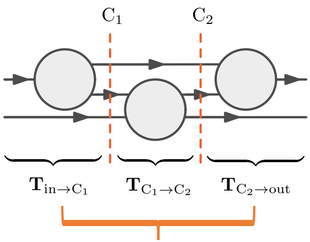
See also Cutkosky, Coster, Stapp, Bourjaily, HSH, McLeod, Schwartz, Vergu, Matak, Blazek

Normal & Anomalous Thresholds

Thresholds: Classical configurations when phase space opens up, potential branch points of amplitudes



Normal thresholds

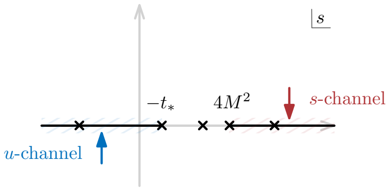


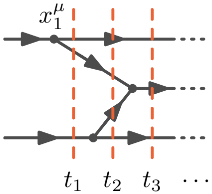
Anomalous thresholds

(not related to anomalies)

Why not in previous 2→2 example?

Thresholds occur when process is allowed classically



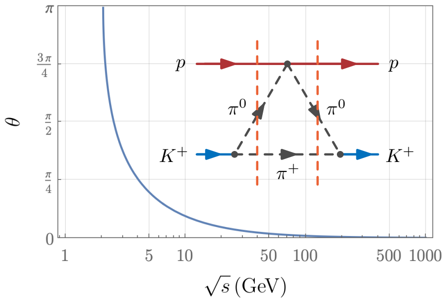


Lightest particle cannot decay

→ the two incoming (outgoing) particles meet at a vertex

→ only **normal thresholds** in physical regions

Anomalous Threshold in Standard Model

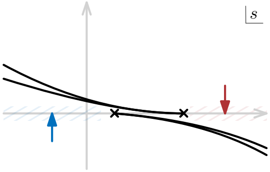


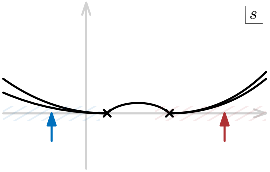
Location of peak

(finite width will result in Breit-Wigner like softening)

$$\cos\theta = 1 - \frac{2s\left(m_{K^+}^2 - (m_{\pi^0}+m_{\pi^+})^2\right)\left(m_{K^+}^2 - (m_{\pi^0}-m_{\pi^+})^2\right)}{m_{\pi^+}^2\left(s-(m_{K^+}+m_p)^2\right)\left(s-(m_{K^+}-m_p)^2\right)}.$$

2. Causality





Algebraic conditions for Causality

Goal: Find $i\varepsilon$ prescription consistent with causality

Here: Investigate in perturbation theory

Result: **algebraic** conditions for branch cuts, branch points and causality in terms of worldline action $\mathcal{V}$:

$$\begin{aligned} \mathcal{V} = 0 &\quad \text{for any } \alpha\text{'s} &\Leftrightarrow&\quad \text{branch cut} \\ \partial_{\alpha_e}\mathcal{V} = 0 &\quad \text{for any } \alpha\text{'s} &\Leftrightarrow&\quad \text{branch point} \\ \operatorname{Im}\mathcal{V} > 0 &\quad \text{for all } \alpha\text{'s} &\Leftrightarrow&\quad \text{causal branch} \end{aligned}$$

SCHWINGER-PARAMETRIZATION OF BUBBLE

$$\mathcal{I}_{\text{bub}}(s) = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^{1,\mathrm{D}-1}} \frac{\mathrm{d}^{\mathrm{D}}\ell}{i\pi^{\mathrm{D}/2}} \frac{1}{[\ell^2 - m_1^2 + i\varepsilon][(p-\ell)^2 - m_2^2 + i\varepsilon]}$$

Introduce Schwinger parameters α_e for every internal line:

$$\frac{-1}{q_e^2 - m_e^2 + i\varepsilon} = \frac{i}{\hbar} \int_0^\infty \mathrm{d}\alpha_e \exp\left[\frac{i}{\hbar}(q_e^2 - m_e^2 + i\varepsilon)\alpha_e\right]$$

Schwinger-parametrization of bubble

Performing momentum integrals results in

$$\mathcal{I}_{\text{bub}} = (-i\hbar)^{\text{D}/2-2} \lim_{\varepsilon\to 0^+} \int_0^\infty \frac{\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2}{(\alpha_1+\alpha_2)^{\text{D}/2}} \exp\left[\frac{i}{\hbar}\Big(\mathcal{V} + i\varepsilon(\alpha_1+\alpha_2)\Big)\right]$$

where $\mathcal{V}$ is the worldline action:

$$\mathcal{V} = s\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2} - m_1^2\alpha_1 - m_2^2\alpha_2$$

Schwinger-parametrization

For any Feynman integral:

$$\mathcal{I} = (-i\hbar)^{-d} \lim_{\varepsilon\to 0^+} \int_0^\infty \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathcal{U}^{\mathrm{D}/2}} \mathcal{N} \exp\left[\frac{i}{\hbar}\left(\mathcal{V} + i\varepsilon \textstyle\sum_{e=1}^{\mathrm{E}} \alpha_e\right)\right]$$

with

$$\mathcal{V} = \frac{\mathcal{F}}{\mathcal{U}} \qquad \mathcal{U} = \sum_{\substack{\text{spanning}\\ \text{trees}\, T}} \prod_{e\notin T} \alpha_e \qquad \mathcal{F} = \sum_{\substack{\text{spanning}\\ \text{two-trees}\\ T_L \sqcup T_R}} p_L^2 \prod_{e\notin T_L\sqcup T_R} \alpha_e - \mathcal{U} \sum_{e=1}^{\mathrm{E}} m_e^2 \alpha_e$$

Takeaway points:

Algebraic formula for any Feynman integral

Integration over exponential of worldline action

Singularities and branch cuts

$$\mathcal{I} = \Gamma(d) \lim_{\varepsilon \to 0^+} \int \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathrm{GL}(1)} \frac{\widetilde{\mathcal{N}}}{\mathcal{U}^{\mathrm{D}/2}(-\mathcal{V} - i\varepsilon)^d},$$

$\mathcal{V}$: ratio of polynomials in α's, homogenous with degree 1

Using this Schwinger-parametrization, we find

$$\mathcal{V} = 0 \quad \text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch cut}$$

$$\partial_{\alpha_e}\mathcal{V} = 0 \quad \text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch point}$$

Causality requires

$$\operatorname{Im}\mathcal{V} > 0 \quad \text{for all } \alpha\text{'s} \quad \Leftrightarrow \quad \text{causal branch}$$

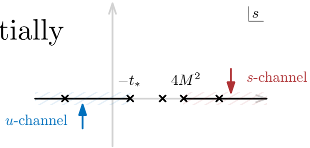
Physical sheet

$$\begin{aligned} \mathcal{V} = 0 \quad &\text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch cut} \\ \partial_{\alpha_e}\mathcal{V} = 0 \quad &\text{for any } \alpha\text{'s} \quad \Leftrightarrow \quad \text{branch point} \end{aligned}$$

Feynman integral lays out *branch points and branch cuts*

Physical sheet: Values of external variables accessible from physical region using analytic continuation, without crossing branch cuts

By definition: Singularities with all $\alpha_e \geqslant 0$ potentially on the physical sheet, others are not

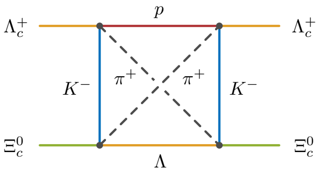


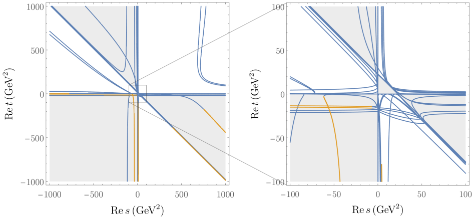
Landau equations

We found condition for leading-singularity branch points:

$$\partial_{\alpha_e} \mathcal{V} = 0$$

Referred to as **Landau equations**, give conditions for singularities

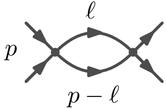


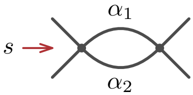


Bjorken, Landau, Nakanishi, Brown,
Mühlbauer, Klausen, Mizera, Telen.

Landau equations in momentum space

Either $\alpha_i = 0$ or $\ell_i^2 = m_i^2$, & $\sum \pm\alpha_i \ell_i = 0$ around every loop.





For bubble integral,

$$\ell^2 = m_1^2 \qquad (p-\ell)^2 = m_2^2 \qquad \alpha_1 \ell^\mu + \alpha_2 (\ell - p)^\mu = 0$$

Solutions are codimension ≥ 1 constraints on *external* kinematics:

$$s = (m_1 + m_2)^2 \qquad s = (m_1 - m_2)^2$$

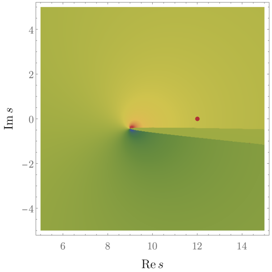
Back to causality

We are now equipped with the condition:

$$\operatorname{Im}\mathcal{V} > 0 \quad \text{for all } \alpha\text{'s} \qquad \Leftrightarrow \qquad \text{causal branch}$$

Can we exploit it to improve on $i\varepsilon$ prescription?

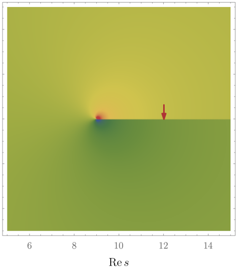
Different $i\varepsilon$ prescriptions, overview



Feynman $i\varepsilon$

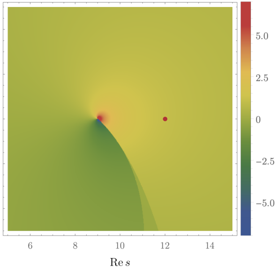
-displaces branch points

-unphysical mass scale ε



Kinematic $i\varepsilon$

-does not work for unstable particles & higher multiplicity

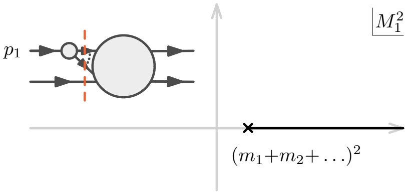


Branch-cut deformations

-reveal physical sheet without modifying branch points

Problem with kinematic $i\mathcal{E}$

Branch cut when $M_1 > m_1+m_2+\ldots$



On shell, capture $M_1^2 \pm i\varepsilon$ as $s \mp i\varepsilon + t + u = \sum_{i=1}^{4} M_i^2$

$\rightarrow$ ***Singularity along entire s-axis***

Branch-cut Deformations

Perform **phase rotations** of Schwinger parameters

$$\hat{\alpha}_e = \alpha_e \exp(i\varepsilon\,\partial_{\alpha_e}\mathcal{V}) = \alpha_e\left[1 + i\varepsilon\,\partial_{\alpha_e}\mathcal{V} + \mathcal{O}(\varepsilon^2)\right]$$

$$\hat{\mathcal{V}} = \mathcal{V} + i\varepsilon\sum_{e=1}^{\mathrm{E}}\alpha_e\underbrace{(\partial_{\alpha_e}\mathcal{V})^2}_{} + \mathcal{O}(\varepsilon^2)$$

$= 0$ at branch points

> 0 away from branch points

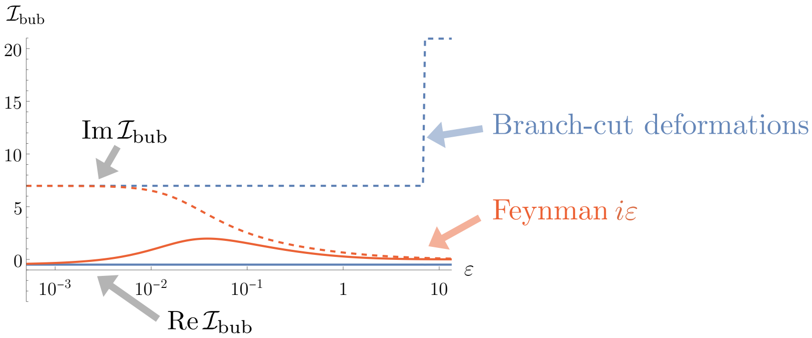
Im $\hat{\mathcal{V}} \geq 0$: **Reveal physical sheet** without modifying branch points

Branch-cut Deformations

Advantages over Feynman $i\varepsilon$:

(i) Rotates *branch cuts*

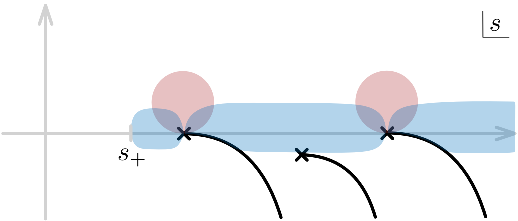
(ii) ε is *small*, not infinitesimal



Analyticity from Branch-cut Deformations

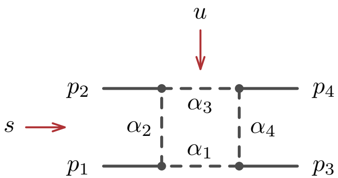
For 2→2 scattering of *stable* particles in perturbation theory, *regardless* of existence of Euclidean region:

Analyticity in a strip around s-channel physical-region



Example I: Necessity of Deformations

Box diagram, external masses $\boldsymbol{M{=}0}$, internal masses $\boldsymbol{m}$

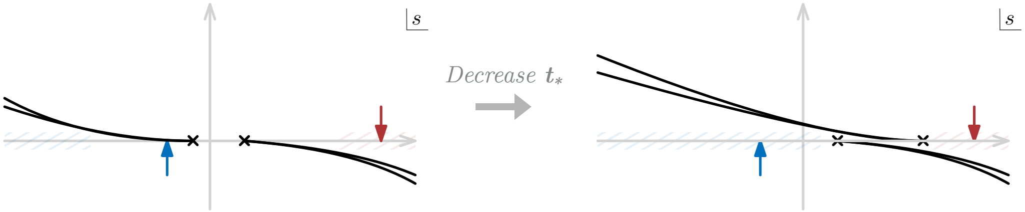


Action:

$$\mathcal{V}_{\text{box}} = \frac{s\,\alpha_1\alpha_3 + u\,\alpha_2\alpha_4}{\alpha_1{+}\alpha_2{+}\alpha_3{+}\alpha_4} - m^2(\alpha_1{+}\alpha_2{+}\alpha_3{+}\alpha_4)$$

Branch points: $s = 4m^2,\ u = 4m^2,\ su + 4m^2 t = 0$

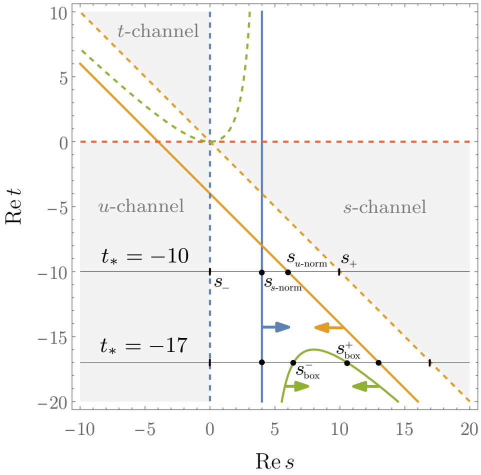
When do branch-cuts overlap?



Use $s > 4m^2$, $u > 4m^2$, $s + t_* + u = 0$:

$$\boldsymbol{t_* < -8m^2}$$

Landau Curves



Analytic Expression

The box example is simple enough for an analytic expression:

$$\mathcal{I}_{\text{box}}(s,t) = \lim_{\varepsilon\to 0^+} \left[\mathcal{I}^{\mathbb{C},s}_{\text{box}}(s+i\varepsilon,t) + \mathcal{I}^{\mathbb{C},u}_{\text{box}}(s-i\varepsilon,t)\right]$$

with

$$\begin{aligned}\mathcal{I}^{\mathbb{C},s}_{\text{box}} = -\frac{xy}{8m^4\beta_{xy}}\Bigg\{ &\log\left(\frac{\beta_x-1}{\sqrt{x}}\right)\left[\log\left(\frac{\beta_{xy}-1}{\beta_{xy}-\beta_x}\right) - \log\left(\frac{\beta_{xy}+1}{\beta_{xy}+\beta_x}\right)\right] \\ &+ \log\left(\frac{\beta_x+1}{\sqrt{x}}\right)\left[\log^-\left(\frac{\beta_{xy}-1}{\beta_{xy}+\beta_x}, \operatorname{Im} s\right) - \log^+\left(\frac{\beta_{xy}+1}{\beta_{xy}-\beta_x}, -\operatorname{Im} s\right)\right] \\ &+ \operatorname{Li}_2\left(\frac{-\beta_x+1}{\beta_{xy}-\beta_x}\right) + \operatorname{Li}_2^+\left(\frac{\beta_x+1}{\beta_{xy}+\beta_x}, \operatorname{Im} s\right) \\ &- \operatorname{Li}_2\left(\frac{\beta_x-1}{\beta_{xy}+\beta_x}\right) - \operatorname{Li}_2^-\left(\frac{-\beta_x-1}{\beta_{xy}-\beta_x}, -\operatorname{Im} s\right)\Bigg\}\end{aligned}$$

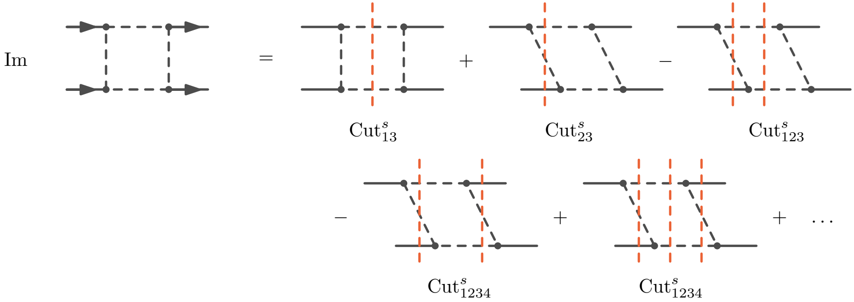
$$x = -\frac{4m^2}{s}, \qquad y = -\frac{4m^2}{u},$$

$$\beta_x = \sqrt{1+x} \qquad \beta_y = \sqrt{1+y}$$

$$\beta_{xy} = -i\sqrt{-1-x-y}\,.$$

Unitarity Cuts

s-channel cuts:



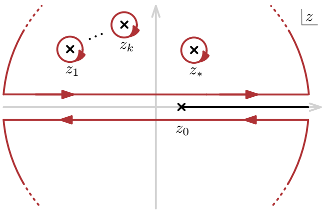
$$\mathrm{Cut}^s_{13}\,\mathcal{I}_{\mathrm{box}} = \frac{\pi x y}{16\beta_{xy}}\left\{\log\left[-(\beta_{xy}-\beta_x)^2\right] - \log\left[-(\beta_{xy}+\beta_x)^2\right]\right\}\Theta(s-4m^2)$$

Use Cuts for Dispersion Relations?

Imaginary part $\leftrightarrow$ *Unitarity cuts*

Discontinuity $\leftrightarrow$ *Dispersion relations*

Since Im $\neq$ Disc, can we still use dispersion relations?



Relating Im to Disc

Using the Schwinger-parametrized form, find

$$\operatorname{Im}\mathcal{I} = \mathcal{I}_D^+ + \mathcal{I}_D^-, \qquad \operatorname{Disc}_s \mathcal{I} = \mathcal{I}_D^+ - \mathcal{I}_D^-.$$

with

$$\mathcal{I}_D^\pm(s,t) = \pi \int \frac{\mathrm{d}^{\mathrm{E}}\alpha}{\mathrm{GL}(1)} \frac{\widetilde{\mathcal{N}}}{\mathcal{U}^{\mathrm{D}/2}} \delta^{(d-1)}(\mathcal{V})\, \Theta(\pm\partial_s \mathcal{V}),$$

Learn:

1. *Im and Disc split into two components with* $\Theta(\pm\partial_s\mathcal{V})$
2. *Im* $=$ *Disc when* $\mathcal{I}_D^- = 0$

Relating Im to Disc

Here, we use

$$\mathcal{V}_{\text{box}} = \frac{s\,\alpha_1\alpha_3 + u\,\alpha_2\alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - m^2(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$$

to get

$$\partial_s \mathcal{V}_{\text{box}} = \frac{\alpha_1\alpha_3}{\alpha_1+\alpha_2+\alpha_3+\alpha_4} - \frac{\alpha_2\alpha_4}{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathcal{V}_s \qquad\qquad \mathcal{V}_u$$

$\mathcal{I}_D^+$ corresponds to $\mathcal{V}_s > \mathcal{V}_u$

$\mathcal{I}_D^-$ corresponds to $\mathcal{V}_s < \mathcal{V}_u$

Relating Im to Disc

Three cases:

a. $t_* > -8m^2$: No overlap of branch cuts

b. $-16m^2 < t_* < -8m^2$: Amplitude splits into components with branch cuts for

$$\textbf{either } s > 4m^2 \quad \textbf{or} \quad u > 4m^2$$

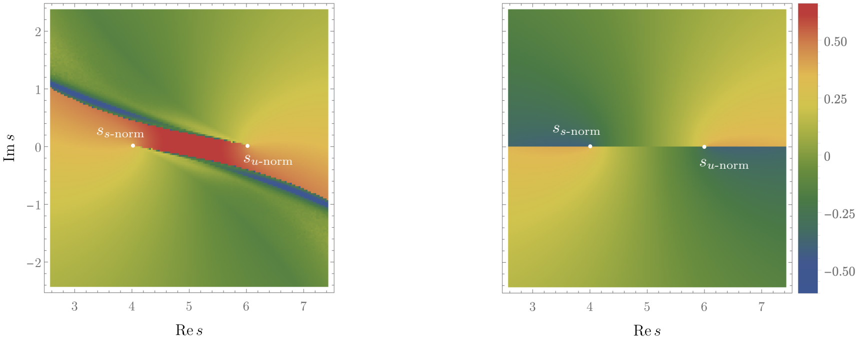
$$\downarrow \qquad\qquad \downarrow$$

$$\mathcal{I}_D^+ \qquad\qquad \mathcal{I}_D^-$$

$$\mathrm{Im}\,\mathcal{I} = \mathcal{I}_D^+ + \mathcal{I}_D^-, \qquad \mathrm{Disc}_s\,\mathcal{I} = \mathcal{I}_D^+ - \mathcal{I}_D^-.$$

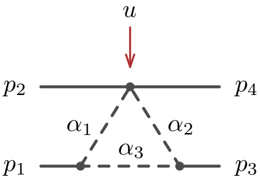
c. $t_* < -16m^2$: Box branch cut spoils the split of the amplitude

ANALYTICALLY CONTINUE CUTS IN s AND u



Example II: Branch-cut along s-axis

Triangle diagram, external masses $M > 2m$



Action:

$$\mathcal{V}_{\text{tri}} = \frac{u\alpha_1\alpha_2 + M^2\alpha_3(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2+\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3)$$

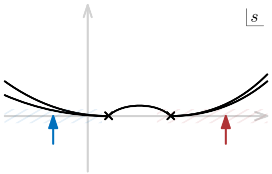
Branch points: $u = 4m^2,\ s_{\text{tri}} = \dfrac{M^4}{m^2} - t$

Approach from LHP

$$\mathcal{V}_{\text{tri}} = \frac{u\alpha_1\alpha_2 + M^2\alpha_3(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2+\alpha_3} - m^2(\alpha_1+\alpha_2+\alpha_3)$$

$$\text{Im}\mathcal{V}_{\text{tri}} = -\text{Im}s\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2+\alpha_3} > 0$$

$\rightarrow$ **Approach physical region from LHP**



Analytic Expressions in LHP and UHP

$$\mathcal{I}_{\text{tri}}^{\text{LHP}}(s,t) = \frac{z}{4M^2\beta_z}\sum_{\zeta\in\{-1,1\}}\Bigg\{\zeta\,\mathrm{Li}_2\left(\frac{1+\frac{z}{2}-\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)+\zeta\,\mathrm{Li}_2\left(1-\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)$$
$$+2\,\mathrm{Li}_2\left(\frac{1+\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\,\mathrm{Li}_2\left(\frac{1-\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\pi i\log\left(\frac{1+\beta_z}{1+\beta_z\beta_{yz}}\right)$$
$$+\zeta\log\left(\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\left[\pi i+\log\left(-1+\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\right]\Bigg\}$$

$$\boxed{\mathcal{I}_{\text{tri}}(s,t) = \lim_{\varepsilon\to 0^+}\mathcal{I}_{\text{tri}}^{\text{LHP}}(s-i\varepsilon,t)}$$

$$\mathcal{I}_{\text{tri}}^{\text{UHP}}(s,t) = \frac{z}{4M^2\beta_z}\sum_{\zeta\in\{-1,1\}}\Bigg\{\zeta\,\mathrm{Li}_2\left(\frac{1+\frac{z}{2}-\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)+\zeta\,\mathrm{Li}_2\left(1-\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)$$
$$+2\,\mathrm{Li}_2\left(\frac{1+\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)-2\,\mathrm{Li}_2\left(\frac{1-\beta_z}{1+\zeta\beta_z\beta_{yz}}\right)+2\pi i\log\left(\frac{1+\beta_z}{1+\beta_z\beta_{yz}}\right)$$
$$+\zeta\log\left(\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\left[-\pi i+\log\left(-1+\frac{1+\frac{z}{2}+\zeta\beta_z}{1+\frac{z}{2}+\zeta\beta_y\beta_z}\right)\right]\Bigg\}$$

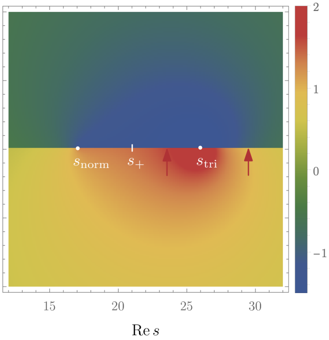
$$y = -\frac{4m^2}{u}\,, \qquad z = -\frac{4M^2}{u}\,,$$
$$\beta_y = \sqrt{1+y}\,, \qquad \beta_z = \sqrt{1+z}\,,$$
$$\beta_{yz} = -i\sqrt{-1+\frac{4y}{z}}\,.$$

Analytic Expression

As expected, ***separate*** analytic expressions in LHP and UHP:

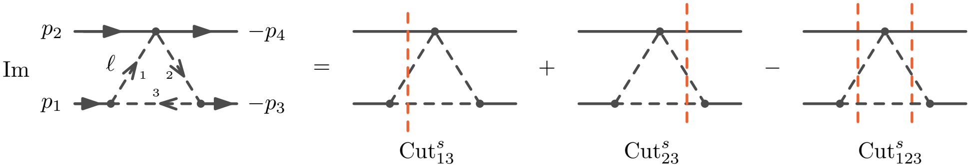
$\mathcal{I}_{\text{tri}}^{\text{UHP}}(s,t)$ →

$\mathcal{I}_{\text{tri}}^{\text{LHP}}(s,t)$ →



Unitarity Cuts

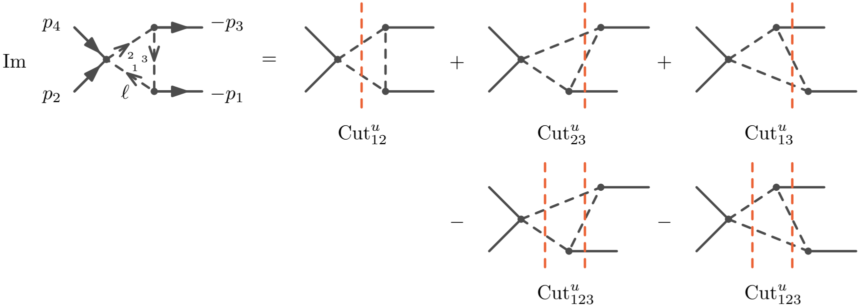
s-channel cuts:



$$\mathrm{Im}\,\mathcal{I}_{\mathrm{tri}} = -\frac{\pi z}{2M^2\beta_z}\begin{cases}\log\left(-\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right) & \text{if } 4M^2-t < s < s_{\mathrm{tri}},\\ \log\left(\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right) & \text{if } s_{\mathrm{tri}} < s,\end{cases}$$

Unitarity Cuts

u-channel cuts:



$$\mathrm{Im}\,\mathcal{I}_{\mathrm{tri}} = -\frac{\pi z}{4M^2\beta_z}\left[2\log\left(-\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right)+\log\left(-\frac{1+\frac{z}{2}+\beta_y\beta_z}{1+\frac{z}{2}-\beta_y\beta_z}\right)-i\pi\right] \quad \text{if } s < s_{\mathrm{norm}}$$

Relating Im to Disc

Recall,

$$\mathrm{Im}\,\mathcal{I} = \mathcal{I}_D^+ + \mathcal{I}_D^-, \qquad \mathrm{Disc}_s\,\mathcal{I} = \mathcal{I}_D^+ - \mathcal{I}_D^-.$$

with splits by

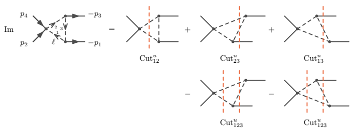
$$\Theta(\pm\partial_s\mathcal{V})$$

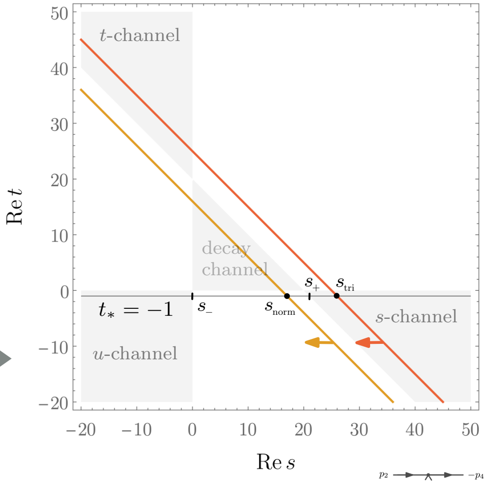
Here:

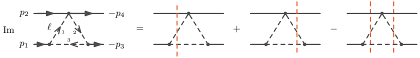
$$\boxed{\mathrm{Im}\,\mathcal{I}_{\mathrm{tri}} = -\mathrm{Disc}_s\,\mathcal{I}_{\mathrm{tri}}}$$

$$\mathrm{Disc}_s\,\mathcal{I}_{\mathrm{tri}} = \frac{\pi z}{4M^2\beta_z}\begin{cases} 2\log\left(-\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right) + \log\left(-\frac{1+\frac{z}{2}+\beta_y\beta_z}{1+\frac{z}{2}-\beta_y\beta_z}\right) - \pi i & \text{if } s < s_{\mathrm{norm}}, \\ 2\log\left(-\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right) & \text{if } s_{\mathrm{norm}} < s < s_{\mathrm{tri}}, \\ 2\log\left(\frac{1-\beta_z\beta_{yz}}{1+\beta_z\beta_{yz}}\right) & \text{if } s_{\mathrm{tri}} < s \end{cases}$$

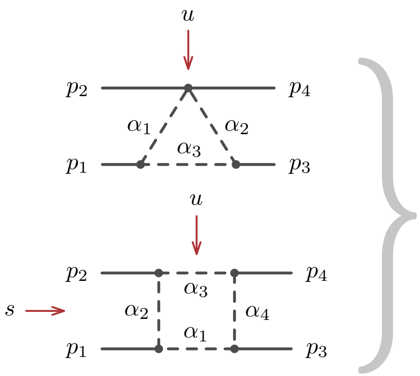
Landau Curves

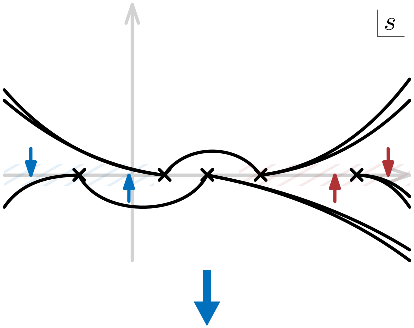






Example III: Summing over Diagrams

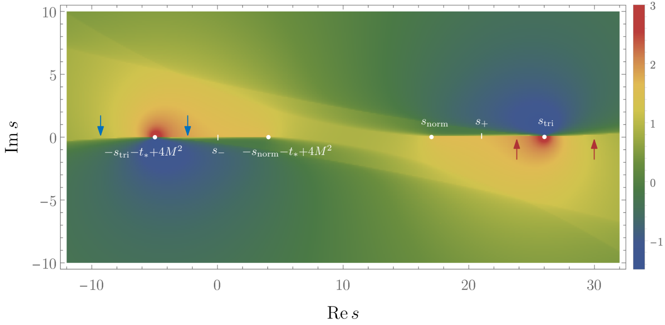




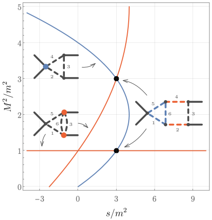
Branch cuts approached from different directions

Example III: Summing over Diagrams

Numerical result:



3. Locality



Fluctuations around Branch Points

In addition to finding saddle points of the action → *Landau branch points*

Fluctuations around saddle points → *expansion around branch points*
(assumes isolated branch points, generic masses)

$$\mathcal{I} \approx \mathcal{I}_0 \lim_{\varepsilon \to 0^+} \begin{cases} [-\Delta\, \partial_\Delta \mathcal{V}^* - i\varepsilon]^\rho & \text{for } \rho < 0, \\ \log[-\Delta\, \mathrm{sgn}(\partial_\Delta \mathcal{V}^*) - i\varepsilon] & \text{for } \rho = 0. \end{cases}$$

Branch point at $\Delta = 0$

$$\rho = \frac{\mathrm{LD} - \mathrm{E} - 1}{2}$$

- L : number of loops
- D : dimensions
- E : number of edges in Feynman diagram

Landau, Pham

Nature of singularities

Isolated codimension-2 branch points (e.g. $s = 3m^2 = M^2$) would imply non-analyticity

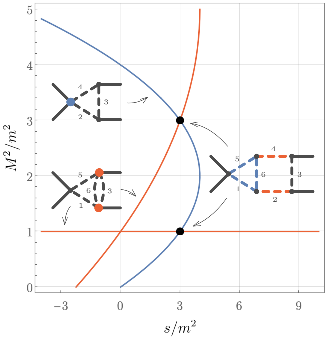
Assuming analyticity, find

$$\mathrm{E} - \mathrm{LD} \leq 1$$

Combining with previous result, singularities are of the form

$$\frac{1}{\Delta} \qquad \frac{1}{\sqrt{\Delta}} \qquad \log(\Delta)$$

Codimension-2 branch points

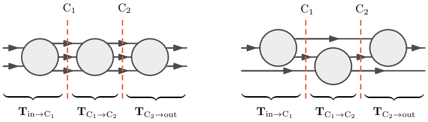


Are codimension-2 branch points always intersections of codimension-1 ones?

Conclusions

Unitarity:

- *Implies anomalous thresholds*

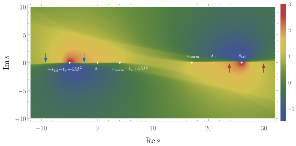


Causality:

- *How to approach the physical regions*
- *Branch-cut deformations preserve analytic structure*

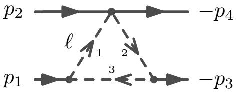
Locality:

- *Kinematic singularities at most poles*



Many open questions

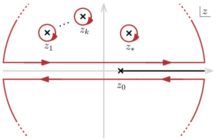
- Unstable particles? Higher-point processes?



- Quantify & measure anomalous thresholds?
  - *What is the error when approaching from UHP?* $(\Gamma/\mathrm{m})^{\#}$?

- Generalized dispersion relations?

- Double dispersion relations?



Thanks!