Leading Singularities of Gravity Amplitudes

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P.B. and Freddy Cachazo - work-in-progress
Leading Singularities of Gravity Amplitudes

Motivations

Leading Singularity Technique

N=8 SUGRA

Conclusion

Outline

1. Motivations

2. Leading Singularity Technique

3. $N = 8$ SUGRA

4. Conclusion
Outline

1. Motivations
2. Leading Singularity Technique
3. $\mathcal{N} = 8$ SUGRA
4. Conclusion
Perturbative structures of field theories seems to be simpler than what appears from Feynman diagrams.

S-matrix analysis

1. Tree level singularities: poles only

\[ \downarrow \]

A class of theory is \textit{fully} determined by the 3-particle amplitude.

2. Loop singularities: poles and branch cuts

\[ \downarrow \]

At one loop, a class of theories is determined by the quadruple cuts.

Attention on $\mathcal{N} = 8$ Supergravity
Investigating loop-orders

- unitarity-based method (Bern, Carrasco, Dixon, Dunbar, Johansson, Kosower, Morgan, Roiban,...)

- leading singularity:
  1. $\mathcal{N} = 4$ SYM (Cachazo & Skinner; Cachazo; Cachazo, Spradlin & Volovich; Spradlin, Volovich & Wen)
  2. $\mathcal{N} = 8$ Supergravity (Cachazo & Skinner; Arkani-Hamed, Cachazo, Kaplan)
$N = 8$ Supergravity at 1-loop

N. Arkani-Hamed, F. Cachazo, J. Kaplan

\[ \Delta_3 = \sum_i \Delta_4^{(i)} \]
\[ \Delta_2 = \sum_i \Delta_3^{(i)} \]

Triple and double cuts in terms of quadruple cuts

Boxes reproduce Feynman diagrams singularities
(No-triangle hypothesis proven by E. Bjerrum-Bohr & P. Vanhove)
$N = 8$ Supergravity at higher loops

- at 1 loop: absence of triangles & bubbles $\equiv$ amplitude fully determined by leading singularities
- what about higher loops?

$\downarrow$

Conjecture (Arkani-Hamed, Cachazo, Kaplan): The full S-matrix is determined by its leading singularities

$\downarrow$

Consequences, if true:
- tree level amplitudes determine the *full* S-matrix
- no UV-divergencies

Worth to explore!
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Leading Singularity Technique

Amplitude as sum of Feynman diagrams and in terms of an integral basis \( \{ l_i \}_{i \in I} \),

\[
I_i(k) = \int \prod_{i=1}^{L} d^d l^{(i)} \mathcal{I}_i(k, l)
\]

\[
M_n = \sum_{f \in F} \int \prod_{i=1}^{L} d^d l^{(i)} F_f(k, l) = \sum_{i \in I} c_i(k) I_i(k)
\]

Singularities: poles and branch cuts.

**Leading Singularities** : Highest codimension singularities

In \( d \)-dimensions the discontinuity across the leading singularity is computed by a \( d \)-dimensional cut

\[
\downarrow
\]

Quadruple-cut computes the leading singularity in 4-dimensions
Leading Singularity Technique

Quadruple cut

\[ \frac{1}{l^2 + i\epsilon} \rightarrow \delta^+(l^2) \]

4 $\delta$-functions $\rightarrow l$ is localized in $l^{(i)} \in \mathbb{C}^4$

↓

the integral is given by the jacobian of the change of variable from $l$ to the argument of $\delta$ evaluated at $l^{(i)}$
Leading Singularity Technique

Contour integral point of view

\[ \sum_{f \in F} \int_{\gamma} \prod_{i=1}^{L} d^4 l^{(i)} \ F_f(k, l) = \sum_{i \in I} c_i(k) \ \int_{\gamma} \prod_{i=1}^{L} d^4 l^{(i)} \ J_i(k, l) \]

\[ \gamma = T^{4L} \subset C^{4L} \]

Massless case: \( \forall \) loop, \( T^4 \) localizes the \( l \)-integral onto

\[ l^{(i)} \in C^4 \ (i = 1, 2) \]

Algebraic linear equations for the coefficients \( c_i(k) \)
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$\mathcal{N} = 8$ Sugra: 2-loop 5-particle MHV Amplitude

Two sectors:

\[
\begin{cases}
\text{planar} \\
\text{non-planar}
\end{cases}
\]

Two loops $\rightarrow$ integration contour $\gamma = T^8$

Three classes of contours $T^8$:

1. on diagrams with 8 internal propagators
2. on diagrams with 7 internal propagators
3. on diagrams with 6 internal propagators
$\mathcal{N} = 8$ Sugra: Planar Sector

Three topologies

- topology 1
- topology 2
- topology 3
$\mathcal{N} = 8$ Sugra: Planar Sector - Topology 1

Expansion on integrals which share the same singularities

$$T^8 = \{(p, q) \in (C^4 \times C^4) \mid |p| = \varepsilon, |(p + k_1)| = \varepsilon, |(p + k_1 + q)| = \varepsilon, |(p - k_2)| = \varepsilon, |q| = \varepsilon, |(q - k_5)| = \varepsilon, |(q + k_1)| = \varepsilon, |(q + k_1 + k_2)| = \varepsilon\}.$$
\[ \mathcal{N} = 8 \text{ Sugra: Planar Sector - Topology 1} \]

\[ \sum_{\text{multiplet } s=1}^{4} \prod_{s=1}^{4} M^{\text{tree}}_{3}(s) \bigg|_{p^{(i)}} = stu M^{\text{tree}}_{4}, \quad I = \left( \prod_{s=1}^{4} \int_{z_i=0} dz_i \right) |J|^{-1} \]

Two equations

\[ B - \frac{s_{25}}{\hbar(2, 5, 4, 3)} C - \frac{s_{25}}{\hbar(2, 5, 3, 4)} D = 0 \]

\[ B - \frac{s_{25}}{\hbar(2, 3, 4, 5)} C - \frac{s_{25}}{\hbar(2, 4, 3, 5)} D = -s_{12}^3 s_{15} \frac{1}{N(5)} \langle 1, 2 \rangle^8 f(1, 3, 4, 2, 5) \]

2 equations in 3 unknowns: not enough!
\( \mathcal{N} = 8 \) Sugra: Planar Sector - Topology 3

Expansions on integrals which share the same singularities

\[
T^8 = \left\{ (p, q) \in (C^4 \times C^4) \mid |p^2| = \varepsilon, |(p + k_1)^2| = \varepsilon, |(p + k_1 + q)^2| = \varepsilon, |(p - k_2)^2| = \varepsilon, 
|q^2| = \varepsilon, |(q - k_5)^2| = \varepsilon, |(q - k_5 - k_4)^2| = \varepsilon, |(q + k_1 + k_2)^2| = \varepsilon \right\},
\]

Two solutions: the lhs vanishes on one solution and is zero on the other one.

Contradiction?
\[ \mathcal{N} = 8 \text{ Sugra: Planar Sector - Topology 3} \]

**NO!**: The integral expansion is simply not complete!

\[ \downarrow \]

Introduce integrals with share the same singularity BUT do not contribute to other topologies

\[ = C_4 + C' + C'' + \ldots \]
\( \mathcal{N} = 8 \) Sugra: Planar Sector - Topology 3

We need minimal set which makes the system consistent

\[
\begin{align*}
  1^- & \quad 2^- & \quad 3^+ & \quad 4^+ & \quad 5^+ \\
  4^- & \quad 3^- & \quad 1^+ & \quad 2^+ & \quad 3^+ \\
\end{align*}
\]

\[= C_4 + C'\]

Analogous expansion related to \( D \) needed (3 ↔ 4)

\[\downarrow\]

Eqs from top 1 + Eqs from top 3 = 6 eqs in 5 unknowns

most likely it does not have solution
Surprisingly, it has a unique solution!

\[
B = -s_{12}^2 s_{15} s_{25} (s_{15} + s_{25}) \frac{\langle 1, 2 \rangle^8}{N(5)} \frac{\mathcal{h}(1, 3, 4, 2) \mathcal{h}(1, 4, 3, 2)}{\mathcal{h}(1, 2, 3, 5) - \mathcal{h}(1, 5, 3, 2)}
\]

\[
C = s_{12}^2 s_{23} s_{34} s_{45} s_{51} \frac{\langle 1, 2 \rangle^8}{N(5)} \frac{\mathcal{h}(1, 3, 4, 2) \mathcal{h}(1, 4, 5, 2)}{\mathcal{h}(1, 2, 3, 5) - \mathcal{h}(1, 5, 3, 2)}
\]

\[
C' = s_{12} s_{34} s_{45} \frac{\langle 1, 2 \rangle^8}{N(5)} \frac{\mathcal{h}(1, 3, 4, 2) \mathcal{h}(1, 4, 5, 2) \mathcal{h}(1, 5, 3, 2)}{\mathcal{h}(1, 2, 3, 5) - \mathcal{h}(1, 5, 3, 2)}
\]

\[
D = -s_{12}^2 s_{24} s_{43} s_{35} s_{51} \frac{\langle 1, 2 \rangle^8}{N(5)} \frac{\mathcal{h}(1, 4, 3, 2) \mathcal{h}(1, 3, 5, 2)}{\mathcal{h}(1, 2, 4, 5) - \mathcal{h}(1, 5, 4, 2)}
\]

\[
D' = -s_{12} s_{34} s_{35} \frac{\langle 1, 2 \rangle^8}{N(5)} \frac{\mathcal{h}(1, 4, 3, 2) \mathcal{h}(1, 3, 5, 2) \mathcal{h}(1, 5, 4, 2)}{\mathcal{h}(1, 2, 4, 5) - \mathcal{h}(1, 5, 4, 2)}
\]
Expansion on integrals which share the same singularities

\[ = N + C^4 + P + Q^4 \]

Topology 2 has a non-planarity feature built-in!
\( \mathcal{N} = 8 \) Sugra: Planar Sector - Topology 2

2 equations in 2 unknowns! (C and P already determined!)

\[
N = s_{12} s_{23} s_{34} s_{45} s_{51} \frac{\langle 1, 2 \rangle^8}{N(5)} h(1, 4, 5, 2) \frac{s_{14} h(1, 3, 4, 2) - s_{25} h(2, 4, 5, 3)}{f(3, 1, 4, 2, 5) - f(3, 2, 5, 1, 4)}
\]

\[
Q = s_{12} s_{23} s_{34} s_{45} s_{51} s_{13} s_{35} \frac{\langle 1, 2 \rangle^8}{N(5)} h(1, 4, 5, 2) \frac{h(1, 3, 4, 2) + h(2, 4, 5, 3)}{f(3, 2, 5, 1, 4) - f(3, 1, 4, 2, 5)}
\]

Appearance of the non-planar integral

↓

the two sectors cannot be disentangled
Higher number of topologies
First three topologies: 6-internal propagators
Question: how can we integrate on a $T^8$?
Answer: integrating out 1 loop variable → two extra propagators from the jacobian.

However

the $T^8$ does not have solution: a simultaneous factorization in these two channels cannot occur

These three topologies are “not relevant” (?)
$\mathcal{N} = 8$ Sugra: Non-Planar Sector

Example: first non-planar topology

\[ \begin{array}{c}
      5^+ \\
    4^+ \\
    3^+ \\
    2^- \\
    1^- \\
\end{array} = c_1 \begin{array}{c}
    5^+ \\
    4^+ \\
    3^+ \\
    2^- \\
    1^- \\
\end{array} + \ldots \]

The amplitude cannot show a simultaneous factorization in the channels $(q + k_1)^2$ and $(q + k_2)^2$

\[ \downarrow \]

\[ c_1 = 0 \]

*The coefficients of the integral expansion are determined by the study of the topologies from the fourth on*
How to check that the final answer is correct?

Three limits to check:
- collinear limit
- multi-particle limit
- soft limit

Problem with the soft-limit: explicit expression not known for all the integrals → IR analysis hard...
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Conclusion

- Perturbative structure of field theories is more intriguing than what appears from Feynman diagrams.
- Study of poles at tree level: recursion relations.
- Branch-cuts at loop level: complicated structure.

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special singularities: leading singularities

↑

computation of residues!

- \( \mathcal{N} = 8 \) supergravity at 1-loop as well.
- Other loops? 2-loop 5-particle amplitude as a first check.
- Coeffs of the integral expansion never computed before.
- More insights towards a proof of the leading singularity conjecture?