D-dimensional Cuts and Extracting Rational Terms

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Introduction

- Accurate predictions for complicated processes essential for the LHC
  $\Rightarrow$ NLO computations for $t\bar{t}$+jets,VV+jets etc.
- Traditional methods to calculate scattering amplitudes very inefficient

n	4	5	6	7	8	9	10
$0 \to n(g)$	4	25	120	2485	34300	559405	10525900

On-Shell Methods

- Physical d.o.f. $\Rightarrow$ efficient evaluation of multi-particle processes.
- Insight into hidden structures:
  - Multi-Loop $\mathcal{N} = 4$ SYM
  - Finiteness of $\mathcal{N} = 8$ Super-Gravity
- Insight from Twistor space: Complex analysis and factorisation

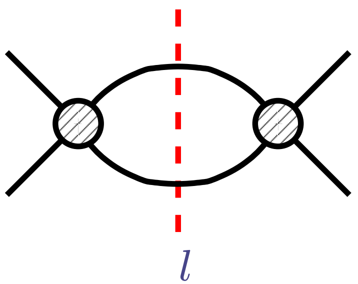
Tree-level: MHV Rules (CSW), On-shell recursion (BCFW)

Recycling Trees Into Loops

- Traditional unitarity methods construct amplitudes from double cuts

[Bern,Dixon,Dunbar,Kosower (1994)]

$$\frac{i}{l^2 - m^2 + iO^+} \to \delta^{(+)}(l^2 - m^2)$$

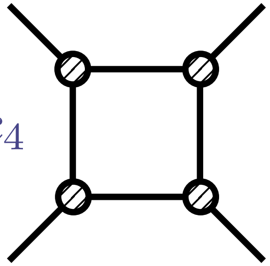
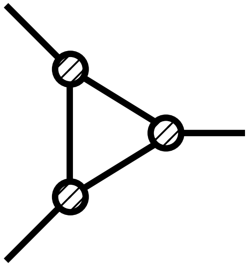
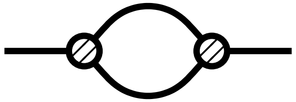


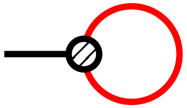
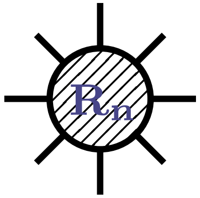
- Fit loop amplitudes to a basis of known integral functions

[Ellis,Zanderighi]

[Denner,Nierste,Scharf]

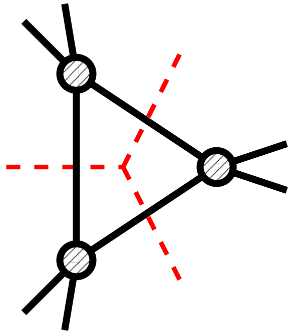
[qcdloop.fnal.gov]

$A_n^{(1)} = \sum_{K_4} c_4$  $+ \sum_{K_3} c_3$  $+ \sum_{K_2} c_2$ 

$+ \sum_{K_1} c_1$  $+$ 

Recycling Trees Into Loops

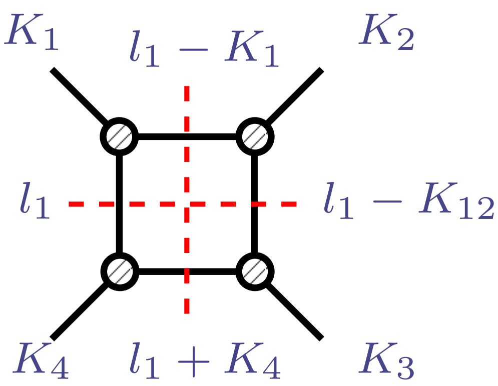
- Improve fitting using multiple cuts



$e^+e^- \to 4$ partons [Bern,Dixon,Kosower (1997)]
Quadruple cuts [Britto,Cachazo,Feng (2004)]

- Completely determines amplitudes in super-symmetric theories
- Rational terms missed by 4-d cuts:
  - Recursion relations [Bern,Dixon,Kosower] [Berger,Bern,Dixon,Forde,Kosower]
  - D-dimensional cuts [Britto,Feng,Mastrolia] [Anastasiou,Britto,Feng,Kunszt] [Ellis,Giele,Kunszt,Melnikov]

Generalised Unitarity With Complex Momenta



Parametrise loop momentum to solve on-shell constraints [Britto,Cachazo,Feng]

$$l_1 = aK_4^\flat + bK_1^\flat + c|K_4^\flat\rangle[K_1^\flat| + d|K_1^\flat\rangle[K_4^\flat|$$

$$K_4^\flat = K_4 - \frac{S_4}{\gamma}K_4 \qquad K_1^\flat = K_1 - \frac{S_1}{\gamma}K_2$$

$$S_i = K_i^2 \qquad \gamma = K_1 \cdot K_4 \pm \sqrt{(K_1 \cdot K_4)^2 - S_1 S_4}$$

- Solving constraints fixes all four coefficients $\rightarrow$ 2 solutions

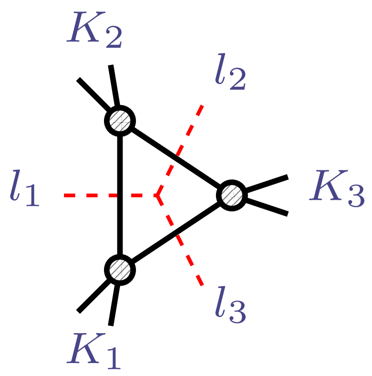
$$\{l_1^2 = 0, (l_1 - K_2)^2 = 0, (l_1 - K_{23})^2 = 0, (l_1 + K_1)^2 = 0\}$$

- Box coefficient given as sum over both solutions

$$C_4 = \frac{i}{2}\sum_{c=c_\pm} A_1 A_2 A_3 A_4(l_1(c))$$

Direct Extraction Of Triangle Coefficients

- Examine analytic behaviour of triple cut [Forde]



$$l_1 = aK_3^\flat + bK_1^\flat + t|K_3^\flat\rangle[K_1^\flat| + \frac{d}{t}|K_3^\flat\rangle[K_1^\flat|$$

$$\Rightarrow \int d^4 l \Pi \delta(l_i^2) A_1 A_2 A_3 \rightarrow \int dt J_t A_1 A_2 A_3$$

- Consider t as a complex variable then one can see that

$$\int dt J_t A_1 A_2 A_3 = \int dt J_t \underbrace{\mathrm{Inf}_t[A_1 A_2 A_3]} + \sum_i \frac{\mathrm{Res}_{t=t_i}(A_1 A_2 A_3)}{(t-t_i)}$$

Laurent series around $t = \infty$
$\mathrm{Inf}_t[X(t)] = x_0 + x_1 t + x_2 t^2 + x_3 t^3$

- Integrals over non-zero powers of t vanish

$$C_3 = -\sum_\gamma \mathrm{Inf}_t[A_1 A_2 A_3]|_{t^0}$$

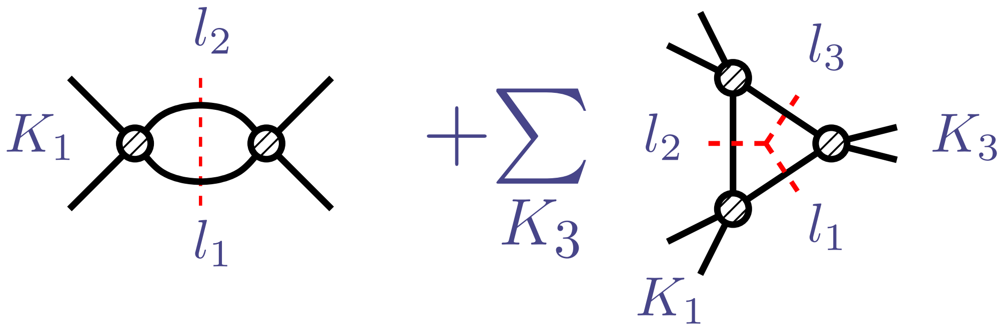
Bubble Coefficients

- Here the two unfixed integrals can be parametrised by,

$$l = yK_1^\flat + a(1-y)\chi + t|K_1^\flat\rangle[\chi| + by(1-y)|\chi\rangle[K_1^\flat|$$

- Decomposition and complex integration similar - non-vanishing integrals
- Final Coefficient

$$C_2 = -i\text{Inf}_t[\text{Inf}_y[A_1 A_2]]|_{t^0, y^i \to Y_i} - \frac{1}{2}\sum_{\{K_3\}}\sum_{y_\pm}\text{Inf}_t[A_1 A_2 A_3(K_3)]|_{t^i \to T_i}$$



Massive propagators

- Extension to massive particles in the loops is straightforward

[Britto,Feng,Mastrolia,Yang]

[Kilgore]

- General complex, massive loop momenta:

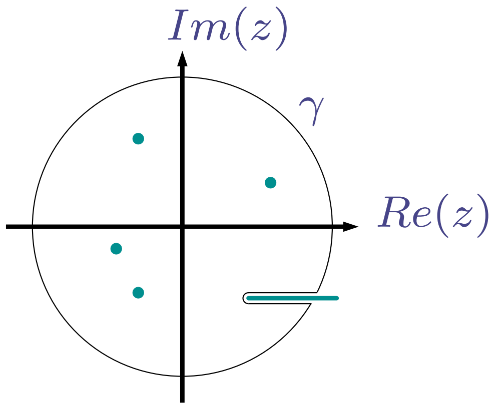
$$l_1 = aK_1^\flat + bK_2^\flat + c|K_1^\flat\rangle[K_2^\flat| + \frac{ab\gamma - m^2}{\gamma c}|K_2^\flat\rangle[K_1^\flat|$$

- Complex analysis is unchanged $\rightarrow$ New solutions to on-shell constraints

Extracting Rational Terms

- On-shell recursion relations apply to rational terms [Bern,Dixon,Kosower (2005)]
- Make use of factorisation properties

$$R_n = \widehat{CR} + \sum_{j=2}^{n-2} \left(A_{j+1}^{(0)} \frac{i}{P_{1,j}^2} R_{n-j+1} + R_{j+1} \frac{i}{P_{1,j}^2} A_{n-j+1}^{(0)} + A_{j+1}^{0} \frac{i\mathcal{F}(P_{1,j})}{P_{1,j}^2} A_{n-j+1}^{(0)} \right)$$



  - Multi-gluon amplitudes [Berger,Bern,Dixon,Forde,Kosower]
  - Higgs+multi gluon amplitudes [Berger,Dixon,Del Duca] [SB,Glover,Risager] [Glover,Mastrolia,Williams]
- Complex factorisation at one-loop remains to be fully understood

Extracting Rational Terms

- D-dimensional cuts have been used successfully both numerically and analytically
  - One-loop gluons amplitudes $n \leq 20$ (`Rocket`) [Giele,Kunszt,Melnikov] [Giele,Zanderighi]
  - One-loop $t\bar{t} + n(g)$ amplitudes $n \leq 3$ [Ellis,Giele,Kunszt,Melnikov]
  - Analytic expressions for $Hqqg, 5g, 4g[n_S, n_F]$ [Bern,Morgan] [Britto,Feng,Mastrolia,Yang]
- $4 - 2\epsilon$-dimensional cuts are equivalent to massive cuts

$$l^{\nu}_{[D=4-2\epsilon]} = l^{\nu}_{[4]} + l^{\nu}_{[-2\epsilon]}$$

$$l^2_{[D]} = 0 \Rightarrow l^2_{[4]} = -l^2_{[-2\epsilon]} = \mu^2$$

[Bern,Morgan (1995)]
[Bern,Dixon,Dunbar,Kosower (1997)]
[Ossola,Papadopoulos,Pittau (2008)]

$$\int d^D l = \int d^{-\epsilon}(\mu^2) \int d^D l_{[4]}$$

D-dimensional Integral Basis

- The integral basis in D-dimensions has no rational terms: [Giele et al.]

$$A_n^{(1),D} = \frac{D-4}{2}\sum_{K_5} C_{5;K_5} I_{5;K_5}^{D+2} + \sum_{K_4} C_{4;K_4} I_{4;K_4}^{D}$$
$$+ \frac{D-4}{2}\sum_{K_4} C_{4;K_4}^{[2]} I_{4;K_4}^{D+2} + \frac{(D-4)(D-2)}{2}\sum_{K_4} C_{4;K_4}^{[4]} I_{4;K_4}^{D+4} + \sum_{K_3} C_{3;K_3} I_{3;K_3}^{D}$$
$$+ \frac{D-4}{2}\sum_{K_3} C_{3;K_3}^{[2]} I_{3;K_3}^{D+2} + \sum_{K_2} C_{2;K_2} I_{2;K_2}^{D} + \frac{D-4}{2}\sum_{K_2} C_{2;K_2}^{[2]} I_{2;K_2}^{D+2}$$

- Taking the $D \to 4-2\epsilon$ limit gives us the rational terms

$$R_n = -\frac{1}{6}\sum_{K_4} C_{4:K_4}^{[4]} - \frac{1}{2}\sum_{K_3} C_{3:K_3}^{[2]} - \frac{1}{6}\sum_{K_2}\left(K_2^2 - 3(m_1^2+m_2^2)\right) C_{2:K_2}^{[2]}$$

Rational Terms From The Large Mass Limit

- Study analytic properties of the massive quadruple cut [SB (2008)]

$$l_1^\sigma = aK_1^\flat + bK_2^\flat + c^\sigma|K_1^\flat\rangle[K_2^\flat| + \frac{ab\gamma - \mu^2}{\gamma c^\sigma}|K_2^\flat\rangle[K_1^\flat|$$

$$\int d^{-2\epsilon}\mu \int d^4 l_1 \prod_{i=1}^{4} \delta(l_i^2 - \mu^2) A_1 A_2 A_3 A_4$$

$$= \int d^{-2\epsilon}\mu \sum_\sigma \left[\mathrm{Inf}_{\mu^2}[A_1 A_2 A_3 A_4(l_1^\sigma)] + \sum_i \frac{\mathrm{Res}_{\mu^2=\mu_i^2}(A_1 A_2 A_3 A_4(l_1^\sigma))}{\mu^2 - \mu_i^2} \right]$$

- Pentagon contributions drop out of the rational part

$$C_{4;K_4}^{[4]} = \frac{i}{2} \sum_{K_4} \mathrm{Inf}_{\mu^2}[A_1 A_2 A_3 A_4(l_1^\sigma)]|_{\mu^4}$$

- Rational contribution from leading μ^2 behaviour at contour boundary

Rational Terms From The Large Mass Limit

- Triangle and Bubble coefficients follow from similar analysis

$$C_3^{[2]} = \frac{1}{2}\sum_{\sigma} \mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[A_1A_2A_3(l_1^\sigma)]]|_{t^0,\mu^2}$$

$$C_2^{[2]} = -i\mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[\mathrm{Inf}_y[A_1A_2(l_1(\chi))]]]|_{y^i\to Y_i,t^0,\mu^2}$$
$$-\frac{1}{2}\sum_{K_3}\sum_{\sigma=\pm} \mathrm{Inf}_{\mu^2}[\mathrm{Inf}_t[A_1A_2A_3(K_3)(l_1(y^\sigma,\chi)]]|_{t^i\to T_i,\mu^2}$$

- All on-shell constraints in terms four dimensional momenta
- Pentagon contributions isolated

Application To Gluon Scattering

- Gluon scattering is well documented $\longrightarrow$ ideal testing ground
- Rational terms originate from scalar loops only:

$$A_n^g = A_n^{\mathcal{N}=4} - 4A_n^{\mathcal{N}=1} + A_n^{[s]} + \frac{N_f}{N_c}\left(A_n^{\mathcal{N}=1} - A_n^{[s]}\right)$$

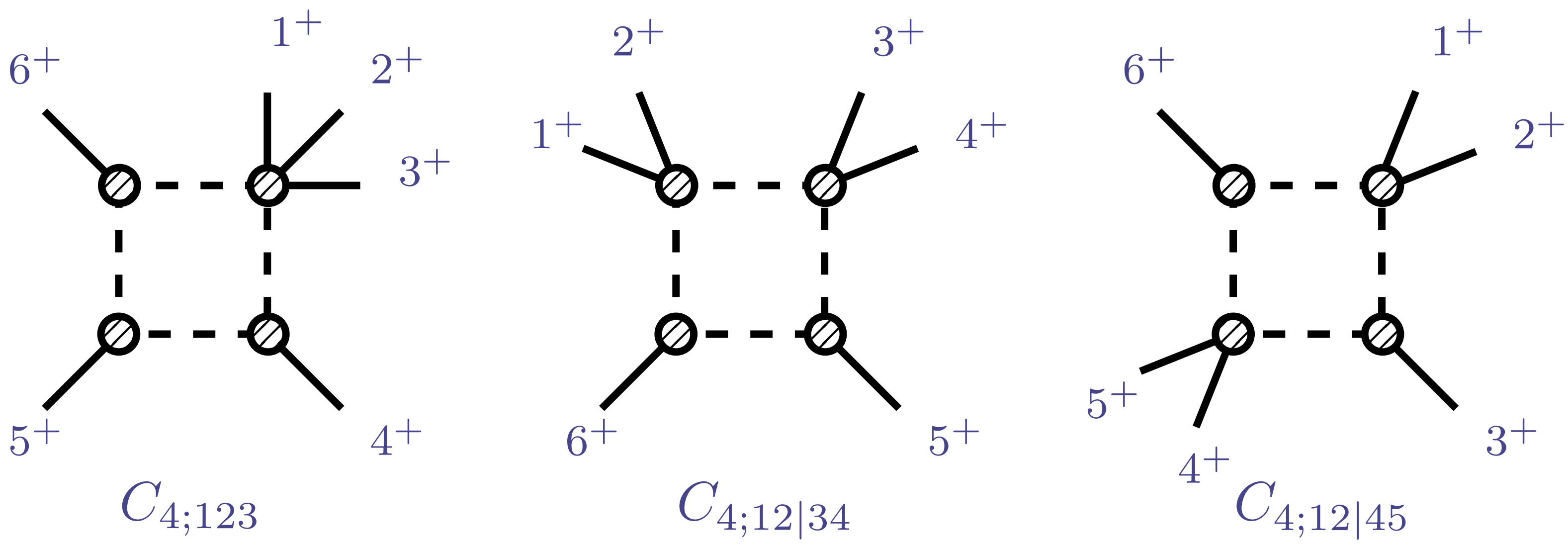
$$\Rightarrow R_n^g = \left(1 - \frac{N_f}{N_c}\right) R_n^{[s]}$$

- Compact tree amplitudes from recursion relations [SB,Glover,Khoze,Svrček] [Forde,Kosower]

Six Gluon All-Plus Amplitude

- Particularly simple: only box contributions

$$R_6^g(1^+,2^+,3^+,4^+,5^+,6^+) = -\frac{1}{6}\sum_{\sigma\in S_n}\Bigg(C^{[4]}_{4;\sigma(1)\sigma(2)\sigma(3)} + C^{[4]}_{4;\sigma(1)\sigma(2)|\sigma(3)\sigma(4)} + \frac{1}{2}C^{[4]}_{4;\sigma(1)\sigma(2)|\sigma(4)\sigma(5)}\Bigg)$$



Six Gluon All-Plus Amplitude

- Form product of four tree amplitudes

$$C_{4;123}^{[4]}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{2i\langle 5|l_4|4]\langle 4|l_1|5]\langle 5|l_2|6]\mu^2[1|l_3(1+2)|3]}{\langle 54\rangle\langle 45\rangle\langle 56\rangle\langle 12\rangle\langle 23\rangle\langle 1|l_3|1]\langle 3|l_4|3]}$$

- Solution for loop momentum around $\mu \to \infty$:

$$l_1^{\pm} \stackrel{\mu\to\infty}{\to} \pm|\mu|\sqrt{\frac{\langle 5|6|4]}{\langle 4|6|5]s_{45}}}\left(|4\rangle[5| - \frac{\langle 4|6|5]}{\langle 5|6|4]}|5\rangle[4|\right)$$

- Final coefficient

$$C_{4;123}^{[4]}(1^+,2^+,3^+,4^+,5^+,6^+) = \frac{2i\left(s_{45}\langle 6|1+2|3][51][64]^2 - s_{46}\langle 5|1+2|3][54]^2[61]\right)[56]}{\langle 12\rangle\langle 23\rangle \mathrm{tr}_5(5,4,6,1)\mathrm{tr}_5(5,4,6,3)}$$

Six Gluon All-Plus Amplitude

- Full result:

$$R_6^g(1^+,2^+,3^+,4^+,5^+,6^+) = \sum_{\sigma\in S_n}\Bigg(\frac{2i\left(s_{45}\langle 6|1+2|3][51][64]^2 - s_{46}\langle 5|1+2|3][54]^2[61]\right)[56]}{\langle 12\rangle\langle 23\rangle \mathrm{tr}_5(5,4,6,1)\mathrm{tr}_5(5,4,6,3)} + \frac{2i\langle 5|1+2|6]\langle 6|1+2|5][12][43][65]^2}{\langle 12\rangle\langle 34\rangle \mathrm{tr}_5(5,2,6,1)\mathrm{tr}_5(5,4,6,3)} + \frac{2i\left(\langle 3|1+2|3]\langle 6|1+2|6] - s_{36}s_{12}\right)[12][54][63]^2}{\langle 12\rangle\langle 45\rangle \mathrm{tr}_5(2,3,6,1)\mathrm{tr}_5(5,3,6,4)}\Bigg)$$

Numerical Implementations

- Using discrete Fourier projections possible to evaluate rational coefficients directly

$$\mathrm{Inf}_\mu[A_1A_2A_3A_4] = \frac{i}{2p_\mu}\sum_{\sigma=\pm}\sum_{k=0}^{p_\mu-1}\frac{1}{\mu_k^4}A_1A_2A_3A_4(l_1^\sigma,\mu_k)$$

$$\mu_k = \mu_\infty \exp\left(2\pi i\frac{k}{p_\mu}\right)$$

- Accuracy dependent on values of p_μ and μ_∞.
- Verified results of up to six external gluons
- Further study required to improve speed and accuracy: OPP analysis to remove spurious poles, c.f.
  `Blackhat` [Berger,Bern,Febres-Cordero,Dixon,Forde,Ita,Kosower,Maitre]
  `Rocket` [Giele,Zanderighi;Ellis,Giele,Kunszt,Melnikov]

Conclusions and Outlook

- On-shell methods combined with complex momenta
- Analytic computations are reduced to the analysis of tree amplitudes at the boundary of the complex plane
- Well behaved growth in complexity with number of external legs
- Allows useful insight into hidden structures

No-Triangle Hypothesis in $\mathcal{N}=8$ Supergravity: [Bjerrum-Bohr,Dunbar,Ita,Perkins,Risager]

[Bjerrum-Bohr,Vanhove]

[Bern,Carrasco,Forde,Ita,Johansson]

[Arkani-Hamed,Cachazo,Kaplan]

Multi-loop $\mathcal{N}=4$ Super-Yang-Mills: [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

[Cachazo,Spradlin,Volovich]

- Generalisations to massive particles $t\bar{t}+jets, WW+jets$
- Numerical techniques for complicated NLO matrix elements

Combination with real radiation for cross-sections