D-dimensional Cuts and Extracting Rational Terms

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Introduction

- Accurate predictions for complicated processes essential for the LHC ⇒ NLO computations for $t\bar{t}$+jets, $VV$+jets etc.
- Traditional methods to calculate scattering amplitudes very inefficient

<table>
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On-Shell Methods

- Physical d.o.f. ⇒ efficient evaluation of multi-particle processes.
- Insight into hidden structures:
  - Multi-Loop $\mathcal{N} = 4$ SYM
  - Finiteness of $\mathcal{N} = 8$ Super-Gravity
- Insight from Twistor space: Complex analysis and factorisation

Tree-level: MHV Rules (CSW), On-shell recursion (BCFW)
Recycling Trees Into Loops

- Traditional unitarity methods construct amplitudes from double cuts [Bern,Dixon,Dunbar,Kosower (1994)]

\[
\frac{i}{l^2 - m^2 + i\epsilon} \rightarrow \delta^{(+)}(l^2 - m^2)
\]

- Fit loop amplitudes to a basis of known integral functions [Ellis,Zanderighi], [Denner,Nierste,Scharf] [qcdloop.fnal.gov]

\[
A_n^{(1)} = \sum_{K_4} c_4 + \sum_{K_3} c_3 + \sum_{K_2} c_2 + \sum_{K_1} c_1 + R_a
\]
Recycling Trees Into Loops

- Improve fitting using multiple cuts

\[ e^+ e^- \rightarrow 4 \text{ partons} \]

Quadruple cuts

- Completely determines amplitudes in super-symmetric theories
- Rational terms missed by 4-d cuts:
  - Recursion relations
  - D-dimensional cuts

[Bern, Dixon, Kosower (1997)]
[Britto, Cachazo, Feng (2004)]
[Berger, Bern, Dixon, Forde, Kosower]
[Britto, Feng, Mastrolia]
[Anastasiou, Britto, Feng, Kunszt]
[Ellis, Giele, Kunszt, Melnikov]
Parametrise loop momentum to solve on-shell constraints

\[ l_1 = aK_4^b + bK_1^b + c|K_4^b\rangle[K_1^b] + d|K_1^b\rangle[K_4^b] \]

\[ K_4^b = K_4 - \frac{S_4}{\gamma} K_4 \]
\[ K_1^b = K_1 - \frac{S_1}{\gamma} K_2 \]

\[ S_i = K_i^2 \]
\[ \gamma = K_1 \cdot K_4 \pm \sqrt{(K_1 \cdot K_4)^2 - S_1 S_4} \]

- Solving constraints fixes all four coefficients → 2 solutions

\[ \{ l_1^2 = 0, (l_1 - K_2)^2 = 0, (l_1 - K_{23})^2 = 0, (l_1 + K_1)^2 = 0 \} \]

- Box coefficient given as sum over both solutions

\[ C_4 = \frac{i}{2} \sum_{c=c_{\pm}} A_1 A_2 A_3 A_4 (l_1(c)) \]
Direct Extraction Of Triangle Coefficients

Examine analytic behaviour of triple cut

\[ l_1 = aK_3^b + bK_1^b + t|K_3^b[K_1^b| + \frac{d}{t}|K_3^b[K_1^b| \]

\[ \Rightarrow \int d^4l \Pi \delta(l_i^2) A_1 A_2 A_3 \rightarrow \int dt J_t A_1 A_2 A_3 \]

Consider \( t \) as a complex variable then one can see that

\[ \int dt J_t A_1 A_2 A_3 = \int dt J_t \text{Inf}_t [A_1 A_2 A_3] + \sum_i \frac{\text{Res}_{t=t_i} (A_1 A_2 A_3)}{(t - t_i)} \]

Integrals over non-zero powers of \( t \) vanish

\[ C_3 = - \sum_{\gamma} \text{Inf}_t [A_1 A_2 A_3]|_{t^0} \]

Laurent series around \( t = \infty \)

\[ \text{Inf}_t [X(t)] = x_0 + x_1 t + x_2 t^2 + x_3 t^3 \]
Bubble Coefficients

Here the two unfixed integrals can be parametrised by,

\[ l = yK_1^b + a(1 - y)\chi + t|K_1^b\rangle \langle \chi| + by(1 - y)|\chi\rangle [K_1^b| \]

Decomposition and complex integration similar - non-vanishing integrals

Final Coefficient

\[ C_2 = -i\text{Inf}_t[\text{Inf}_y[A_1A_2]]|t^0,y^i \to Y_i - \frac{1}{2} \sum \sum \text{Inf}_t[A_1A_2A_3(K_3)]|t^i \to T_i \]

\[ K_1 \quad + \sum_{K_3} \quad K_3 \]

\[ l_1 \quad l_2 \quad l_3 \quad l_1 \]

D-dimensional Cuts and Extracting Rational Terms – p.7/19
Massive propagators

- Extension to massive particles in the loops is straightforward
  
  [Britto,Feng,Mastrolia,Yang]
  
  [Kilgore]

- General complex, massive loop momenta:

  \[ l_1 = aK_1^b + bK_2^b + c|K_1^b\rangle[K_2^b] + \frac{ab\gamma - m^2}{\gamma c}|K_2^b\rangle[K_1^b] \]

- Complex analysis is unchanged → New solutions to on-shell constraints
On-shell recursion relations apply to rational terms

Make use of factorisation properties

\[
R_n = \widehat{CR} + \sum_{j=2}^{n-2} \left( A_{j+1}^{(0)} \frac{i}{P_{1,j}^2} R_{n-j+1} + R_{j+1} \frac{i}{P_{1,j}^2} A_{n-j+1}^{(0)} + A_j^{(0)} \frac{iF(P_{1,j})}{P_{1,j}^2} A_{n-j+1}^{(0)} \right)
\]

Multi-gluon amplitudes
Higgs+multi gluon amplitudes

Complex factorisation at one-loop remains to be fully understood

[Bern,Dixon,Kosower (2005)]

[Berger,Bern,Dixon,Forde,Kosower]
[Berger,Dixon,Del Duca]
[SB,Glover,Risager]
[Glover,Mastrolia,Williams]
D-dimensionsal cuts have been used successfully both numerically and analytically

- One-loop gluons amplitudes \( n \leq 20 \) (Rocket)
  
  \[ \text{[Giele,Kunszt,Melnikov]} \]
  
  \[ \text{[Giele,Zanderighi]} \]

- One-loop \( t\bar{t} + n(g) \) amplitudes \( n \leq 3 \)
  
  \[ \text{[Ellis,Giele,Kunszt,Melnikov]} \]
  
  \[ \text{[Bern,Morgan]} \]
  
  \[ \text{[Britto,Feng,Mastrolia,Yang]} \]

- Analytic expressions for \( Hqqg, 5g, 4g[n_S, n_F] \)

- \( 4 - 2\epsilon \)-dimensional cuts are equivalent to massive cuts

\[
\begin{align*}
\ell_{D=4-2\epsilon}^\nu &= \ell_{[4]}^\nu + \ell_{[-2\epsilon]}^\nu \\
\ell_{D}^2 &= 0 \Rightarrow \ell_{[4]}^2 = -\ell_{[-2\epsilon]}^2 = \mu^2
\end{align*}
\]

\[ \text{[Bern,Morgan (1995)]} \]

\[ \text{[Bern,Dixon,Dunbar,Kosower (1997)]} \]

\[ \text{[Ossola,Papadopoulos,Pittau (2008)]} \]
The integral basis in D-dimensions has no rational terms: [Giele et al.]

\[
A_n^{(1), D} = \frac{D - 4}{2} \sum_{K_5} C_{5; K_5} I_{5; K_5}^{D+2} + \sum_{K_4} C_{4; K_4} I_{4; K_4}^{D} \\
+ \frac{D - 4}{2} \sum_{K_4} C_{4; K_4}^{[2]} I_{4; K_4}^{D+2} + \frac{(D - 4)(D - 2)}{2} \sum_{K_4} C_{4; K_4}^{[4]} I_{4; K_4}^{D+4} + \sum_{K_3} C_{3; K_3} I_{3; K_3}^{D} \\
+ \frac{D - 4}{2} \sum_{K_3} C_{3; K_3}^{[2]} I_{3; K_3}^{D+2} + \sum_{K_2} C_{2; K_2} I_{2; K_2}^{D} + \frac{D - 4}{2} \sum_{K_2} C_{2; K_2}^{[2]} I_{2; K_2}^{D+2}
\]

Taking the \( D \to 4 - 2\epsilon \) limit gives us the rational terms

\[
R_n = -\frac{1}{6} \sum_{K_4} C_{4; K_4}^{[4]} - \frac{1}{2} \sum_{K_3} C_{3; K_3}^{[2]} - \frac{1}{6} \sum_{K_2} (K_2^2 - 3(m_1^2 + m_2^2)) C_{2; K_2}^{[2]}
\]
Study analytic properties of the massive quadruple cut

\[ l_1^\sigma = aK_1^b + bK_2^b + c^\sigma |K_1^b\rangle[K_2^b] + \frac{ab\gamma - \mu^2}{\gamma c^\sigma}|K_2^b\rangle[K_1^b] \]

\[
\int d^{-2\epsilon}\mu \int d^4l_1 \prod_{i=1}^{4} \delta(l_i^2 - \mu^2) A_1 A_2 A_3 A_4
\]

\[
= \int d^{-2\epsilon}\mu \sum_{\sigma} \left[ \text{Inf}_{\mu^2} [A_1 A_2 A_3 A_4 (l_1^\sigma)] + \sum_{i} \frac{\text{Res}_{\mu^2 = \mu_i^2} (A_1 A_2 A_3 A_4 (l_1^\sigma))}{\mu^2 - \mu_i^2} \right]
\]

Pentagon contributions drop out of the rational part

\[ C_{4;K_4}^{[4]} = \frac{i}{2} \sum_{K_4} \text{Inf}_{\mu^2} [A_1 A_2 A_3 A_4 (l_1^\sigma)]_{\mu^4} \]

Rational contribution from leading \( \mu^2 \) behaviour at contour boundary
Triangle and Bubble coefficients follow from similar analysis

\[ C_3^{[2]} = \frac{1}{2} \sum_\sigma \text{Inf}_{\mu^2} [\text{Inf}_t [A_1 A_2 A_3 (l_1^\sigma)]] |_{t^0, \mu^2} \]

\[ C_2^{[2]} = -i \text{Inf}_{\mu^2} [\text{Inf}_t [\text{Inf}_y [A_1 A_2 (l_1 (\chi))]]]|_{y^i \rightarrow Y_i, t^0, \mu^2} \]

\[ -\frac{1}{2} \sum_{K_3} \sum_{\sigma = \pm} \text{Inf}_{\mu^2} [\text{Inf}_t [A_1 A_2 A_3 (K_3) (l_1 (y^\sigma, \chi))]] |_{t^i \rightarrow T_i, \mu^2} \]

All on-shell constraints in terms four dimensional momenta

Pentagon contributions isolated
Gluon scattering is well documented → ideal testing ground

Rational terms originate from scalar loops only:

\[ A_n^g = A_n^{N=4} - 4A_n^{N=1} + A_n^{[s]} + \frac{N_f}{N_c} \left( A_n^{N=1} - A_n^{[s]} \right) \]

\[ \Rightarrow R_n^g = \left( 1 - \frac{N_f}{N_c} \right) R_n^{[s]} \]

Compact tree amplitudes from recursion relations [SB,Glover,Khoze,Svrček]

[Forde,Kosower]
Six Gluon All-Plus Amplitude

- Particularly simple: only box contributions

\[ R_6^a(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) = -\frac{1}{6} \sum_{\sigma \in S_n} \left( \right) \]

\[ C_{4;\sigma(1)\sigma(2)\sigma(3)}^{[4]} + C_{4;\sigma(1)\sigma(2)\sigma(3)\sigma(4)}^{[4]} + \frac{1}{2} C_{4;\sigma(1)\sigma(2)\sigma(4)\sigma(5)}^{[4]} \]
Six Gluon All-Plus Amplitude

- Form product of four tree amplitudes

\[ C_{4;123}^{[4]}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) = \frac{2i \langle 5 | l_4 | 4 \rangle \langle 4 | l_1 | 5 \rangle \langle 5 | l_2 | 6 \rangle \mu^2 [1 | l_3 (1 + 2) | 3] \langle 54 \rangle \langle 45 \rangle \langle 56 \rangle \langle 12 \rangle \langle 23 \rangle \langle 1 | l_3 | 1 \rangle \langle 3 | l_4 | 3 \rangle}{\langle 54 \rangle \langle 45 \rangle \langle 56 \rangle \langle 12 \rangle \langle 23 \rangle \langle 1 | l_3 | 1 \rangle \langle 3 | l_4 | 3 \rangle} \]

- Solution for loop momentum around \( \mu \to \infty \):

\[ l_1^\pm \xrightarrow{\mu \to \infty} \pm |\mu| \sqrt{\frac{\langle 5 | 6 | 4 \rangle}{\langle 4 | 6 | 5 \rangle} s_{45} \left( |4 \rangle [5] - \frac{\langle 4 | 6 | 5 \rangle}{\langle 5 | 6 | 4 \rangle} |5 \rangle [4] \right)} \]

- Final coefficient

\[ C_{4;123}^{[4]}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) = \frac{2i \left( s_{45} \langle 6 | 1 + 2 | 3 \rangle [51] [64]^2 - s_{46} \langle 5 | 1 + 2 | 3 \rangle [54]^2 [61] \right) [56]}{\langle 12 \rangle \langle 23 \rangle \text{tr}_5(5, 4, 6, 1) \text{tr}_5(5, 4, 6, 3)} \]
Six Gluon All-Plus Amplitude

Full result:

\[ R_6^g(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) = \sum_{\sigma \in S_n} \left( \right) \]

\[ = \frac{2i \left( s_{45} \langle 6|1 + 2|3\rangle [51][64]^2 - s_{46} \langle 5|1 + 2|3\rangle [54]^2 [61] \right) [56]}{\langle 12\rangle\langle 23\rangle tr_5(5, 4, 6, 1) tr_5(5, 4, 6, 3)} \]

\[ + \frac{2i\langle 5|1 + 2|6\rangle \langle 6|1 + 2|5\rangle [12][43][65]^2}{\langle 12\rangle\langle 34\rangle tr_5(5, 2, 6, 1) tr_5(5, 4, 6, 3)} \]

\[ + \frac{2i \left( \langle 3|1 + 2|3\rangle \langle 6|1 + 2|6\rangle - s_{36}s_{12} \right) [12][54][63]^2}{\langle 12\rangle\langle 45\rangle tr_5(2, 3, 6, 1) tr_5(5, 3, 6, 4)} \]
Numerical Implementations

- Using discrete Fourier projections possible to evaluate rational coefficients directly

\[ \text{Inf}_\mu[A_1 A_2 A_3 A_4] = \frac{i}{2p_\mu} \sum_{\sigma=\pm} \sum_{k=0}^{p_\mu-1} \frac{1}{\mu_k^4} A_1 A_2 A_3 A_4(l_1^\sigma, \mu_k) \]

\[ \mu_k = \mu_\infty \exp \left( 2\pi i \frac{k}{p_\mu} \right) \]

- Accuracy dependent on values of \( p_\mu \) and \( \mu_\infty \).
- Verified results of up to six external gluons
- Further study required to improve speed and accuracy: OPP analysis to remove spurious poles, c.f.

Blackhat [Berger,Bern,Febres-Cordero,Dixon,Forde,Ita,Kosower,Maitre]
Rocket [Giele,Zanderighi;Ellis,Giele,Kunszt,Melnikov]
Conclusions and Outlook

- On-shell methods combined with complex momenta
- Analytic computations are reduced to the analysis of tree amplitudes at the boundary of the complex plane
- Well behaved growth in complexity with number of external legs
- Allows useful insight into hidden structures

No-Triangle Hypothesis in $\mathcal{N} = 8$ Supergravity:
- [Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager]
- [Bjerrum-Bohr, Vanhove]
- [Bern, Carrasco, Forde, Ita, Johansson]
- [Arkani-Hamed, Cachazo, Kaplan]

Multi-loop $\mathcal{N} = 4$ Super-Yang-Mills:
- [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]
- [Cachazo, Spradlin, Volovich]

- Generalisations to massive particles $\bar{t}t + jets, WW + jets$
- Numerical techniques for complicated NLO matrix elements

Combination with real radiation for cross-sections