Wilson loops, amplitudes and dual superconformal symmetry

Paul Heslop

Queen Mary, University of London

11th September 2008
NBIA, Copenhagen

based on: Brandhuber, Travaglini, PH. 0707.1153[hep-th]
Brandhuber, Nasti, Spence, Travaglini, PH. arXiv:0805.2763 [hep-th]
Brandhuber, Travaglini, PH. arXiv:0807.4097 [hep-th]

Introduction

- Duality between two a priori completely unrelated objects in $\mathcal{N}$=4 SYM
  1. planar MHV gluon scattering amplitudes
  2. Wilson loops over a curve determined by the external momenta
- New dual superconformal symmetry of the full superamplitude (proved at tree level)
- ($\mathcal{N}$=8 SUGRA IR exponentiation, Wilson loop)

Outline

1. Scattering amplitude/Wilson loop duality
   - One loop MHV amplitudes
   - The Wilson loop
   - Dual conformal invariance
2. Dual superconformal symmetry of the entire S-matrix:
   - at tree level
   - at one loop
3. $\mathcal{N} = 8$ SUGRA

Queen Mary University of London

One loop MHV amplitudes

- Colour-stripped planar $L - loop$ MHV amplitudes $A_n^{(L)}$

L-loop amplitude

$$A_n^{(L)} = A_n^{\text{tree}} \mathcal{M}_n^{(L)}(p_i)$$

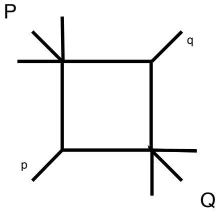
- $\mathcal{M}_n^{(L)}$ is a scalar function of the external momentum p_i only.
- In the first part of this talk we will focus on $\mathcal{M}_n^{(L)}$ for the MHV amplitude
- Amplitudes are infrared divergent: we regularise by dimensional regularisation and work in $d = 4 - 2\epsilon$ dimensions

1-loop n-point MHV gluon amplitude in $\mathcal{N}$=4 SYM.

At one-loop $\mathcal{M}_n^{(1)}$ is a sum of 'two-mass easy' box functions $F^{2\mathrm{m\,e}}$, all with coefficient equal to one:

One loop, n-points

$$\mathcal{M}_n^{(1)} = \sum_{p,q} F^{2\mathrm{m\,e}}(p, q, P, Q; \epsilon) + O(\epsilon) \; .$$ [Bern Dixon Dunbar Kosower 1994]

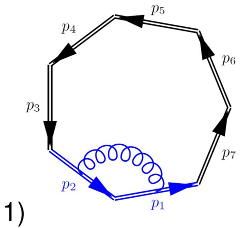


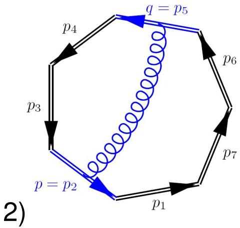
Momenta p and q are null, whereas, in general, $P^2 \neq 0$ and $Q^2 \neq 0$.

$F^{2\mathrm{m\,e}}(p, q, P, Q; \epsilon)$ = complicated hypergeometric functions

One loop Wilson loop calculation

- At strong coupling amplitudes are (T-)dual to certain Wilson loops (see diagram) [Alday Maldacena 2007]
- Lets see what happens at weak coupling
- One loop Wilson loop: take a propagator from any two points on the contour and integrate over the two end-points.
- Thus one must sum over the following two diagrams.





- propagator between points of the same edge vanishes.

(Diagram 1) Cusp diagram

- This gives a simple double integral

$$\sim \frac{1}{2}\frac{(-s)^{-\epsilon}}{\epsilon^2} \qquad (s := (p_1 + p_2)^2)$$

- This is divergent as we approach four dimensions ($\epsilon \to 0$).
- Wilson loop in x-space $\to$ UV divergence
- Here coordinates are momenta $\to$ IR divergence.
- The sum of cusp diagrams gives the divergences of the MHV amplitude

Queen Mary University of London

(Diagram 2) Finite diagram

- WL gives another simple double integral

Diagram 2)

$$\sim \mathcal{F}_\epsilon = \int_0^1 d\tau_p d\tau_q \frac{u}{[-(P^2 + (s - P^2)\tau_p + (t - P^2)\tau_q + u\tau_p\tau_q)]^{1+\epsilon}}$$

$$P = \sum_{i=p+1}^{q-1} p_i,\ Q = \sum_{i=q+1}^{p-1} p_i,\ s := (P+p)^2,\ t := (P+q)^2 \text{ and } u := (p+q)^2$$

- Remarkably, this integral gives the finite part of the two mass easy box function to all orders in ϵ

Indeed there is a different gauge choice in which the cusp diagrams vanish and the 'finite' diagrams give the complete 2me box functions

Queen Mary
University of London

Summary of one-loop calculation

Four-points

The 1-loop Wilson loop calculates the 1 loop MHV scattering amplitude [Drummond Korchemsky Sokatchev 2007] to all orders in ϵ [Brandhuber Travaglini PH 2007]

$n > 4$-points

The 1-loop Wilson loop calculates the 1 loop MHV scattering amplitude to $O(\epsilon^0)$ [Brandhuber Travaglini PH 2007]

(Since for $n > 4$ there exist pentagons at $O(\epsilon)$)

Beyond one loop

- The four- and five-point Wilson loop has been calculated to two loops and to $O(\epsilon^0)$ and agrees with the amplitude and with the BDS conjecture [Drummond Henn Korchemsky Sokatchev 2007]
  (BDS is a conjectured all ordered form of MHV amplitudes [Bern Dixon Smirnov 2005])
- The six-point two loop Wilson loop [Drummond Henn Korchemsky Sokatchev] agrees with the amplitude [Dixon Kosower Roiban Spradlin Vergu Volovich]
- tested numerically (Wilson loop integrals much easier than amplitude integrals)
- 6-point amplitude disagrees with BDS conjecture however

Dual Conformal invariance

- Wilson loops in N=4 are conformally invariant...
- Need to modify to account for dim reg [Drummond Henn Korchemsky Sokatchev 2007]
- Completely fixes the 4 and 5 point Wilson loops to agree with the BDS ansatz
- But not six point where there exist non-trivial conformal invariants

Example (Six-point cross-ratio)

$$\frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}$$

where x_i are the coordinates of the vertices of the Wilson loop $p_i = x_i - x_{i+1}$

- conformal invariance can therefore only fix the amplitude up to an arbitrary function of such cross-ratios

This implies the MHV amplitudes have

dual conformal invariance

(note it acts on x_i not p_i).

A closely related property has been observed in the individual perturbative integrals contributing to four-point amplitudes (true up to five-loops!) by regularising off-shell [Drummond Henn Smirnov Sokatchev 2006]

Outline

1. Scattering amplitude/Wilson loop duality
   - One loop MHV amplitudes
   - The Wilson loop
   - Dual conformal invariance
2. Dual superconformal symmetry of the entire S-matrix:
   - at tree level
   - at one loop
3. $\mathcal{N} = 8$ SUGRA

Dual superconformal symmetry of the entire S-matrix

- So far we have considered MHV amplitudes only
- What about other helicities, other particles?
- What about the MHV tree-level amplitude?
- Conformal transformations?
- Superconformal transformations?

- Need to understand the transformation of spinors λ_i
- enough to consider conformal inversions

$$x_i \longrightarrow \frac{x_i}{x_i^2}$$

(since special conformal transformations $K = IPI$)

$$p_i \; = \; \lambda_i \tilde{\lambda}_i \; = \; x_i - x_{i+1} \; ,$$

- Most general transformation consistent with this [Drummond Henn Smirnov Sokatchev 2008]

$$\lambda_i \rightarrow \frac{(x_i \lambda_i)}{\kappa_i} \qquad\qquad \tilde{\lambda}_i \rightarrow -\frac{\kappa_i}{x_i^2 x_{i+1}^2}(x_i \tilde{\lambda}_i)$$

κ_i arbitrary functions of x_i, x_{i+1}

- MHV tree level: Parke-Taylor with i, j negative helicity the rest positive helicity

$$A_{\text{tree,MHV}}(i,j) := \frac{\langle i\, j\rangle^4}{\langle 12\rangle \cdots \langle n1\rangle}$$

- $\langle 12\rangle \dots \langle n1\rangle$ covariant
- $\langle ij\rangle$ in general not covariant
- Only in the split helicity case, $j = i \pm 1$ do we have conformal invariance of the tree amplitude

However

- Combine amplitudes together into a superamplitude
- The superamplitude is covariant

Queen Mary University of London

- Use Nair's $\mathcal{N}$=4 on-shell superspace

super-wave function

$$\Phi(p,\eta) = A^+(p) + \eta^A \psi_A(p) + \frac{1}{2}\eta^A\eta^B\phi_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C\epsilon_{ABCD}\bar{\psi}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}A^-(p)$$

- all MHV gluon amplitudes together with SUSY-related amplitudes given by a single

superamplitude

$$\mathcal{A}_{\mathrm{MHV}}(1,\ldots,n) = \frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i\tilde{\lambda}_i)\delta^{(8)}(\sum_{i=1}^n \eta_i\lambda_i)}{\langle 12\rangle\cdots\langle n1\rangle}$$

Queen Mary University of London

Dual superspace $(x_i, \theta_i, \lambda_i)$

$$\lambda_i \tilde{\lambda}_i = x_i - x_{i+1} \qquad \eta_i \lambda_i = \theta_i - \theta_{i+1}$$

Transformation of θs under inversions

$$\theta_i \; \longrightarrow \; (x_i^{-1})\theta_i$$

Superamplitude in dual superspace

$$\mathcal{A}_{\text{MHV}}(1, \ldots, n) \; = \frac{\delta^{(4)}(x_1 - x_{n+1})\delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

- The numerator is now invariant under inversions
- hence the MHV superamplitude is covariant under conformal inversions and hence under conformal symmetry

Superconformal symmetry

- We have in fact shown that the super MHV amplitude is invariant under the much larger *super*conformal group
- To see this use

$$S = I\bar{Q}I \qquad \bar{S} = IQI$$

But do we have $Q, \bar{Q}$?

- Q automatic (differences of thetas $\theta_i \to \theta_i + \xi$, similarly to translations)
- $\bar{Q} = \bar{s}$ the standard (ie non-dual) superconformal symmetry (see Ricci's talk).
- s is clearly a symmetry of the tree-level S-matrix. (consistency with other transformations tells us it is a symmetry of the super MHV amplitude also)

The BCFW recursion relation

- We now wish to show covariance of the amplitudes beyond MHV to all amplitudes in the theory
- To do this we use (a supersymmetric form of the) BCFW recursion relation [Arkani-Hamed Cachazo Kaplan, Brandhuber Travaglini PH]
- Shift the momenta p_1, p_2 as in BCFW using a $[12\rangle$ shift

$$\tilde{\lambda}_1 \rightarrow \hat{\tilde{\lambda}}_1 := \tilde{\lambda}_1 + z\tilde{\lambda}_2\,, \qquad \lambda_2 \rightarrow \hat{\lambda}_2 := \lambda_2 - z\lambda_1$$

- Momenta remain on-shell and momentum conservation is preserved by this deformation hence the deformed amplitude $\mathcal{A}(z) := \mathcal{A}(p_1(z), p_2(z) \ldots p_n)$ is itself a well-defined amplitude for all z.

Supersymmetric recursion relation

- Can we consistently deform the superamplitude in this way?
- Notice that in dual space we shift only x_2

$$\hat{p}_1 := x_1 - \hat{x}_2 \,, \qquad \hat{p}_2 := \hat{x}_2 - x_{i+2}$$
$$\hat{x}_2 := x_2 - z\,\lambda_1 \widetilde{\lambda}_2$$

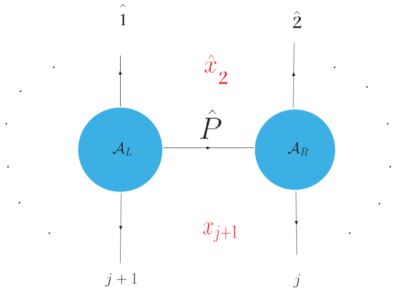


Figure: Generic recursion diagram

- Therefore we shift θ_2 only

$$\hat{\theta}_2 := \theta_2 - z\,\eta_2\lambda_1 \qquad \Longrightarrow \qquad \hat{\eta}_1 = \eta_1 + z\,\eta_2$$

- This shift preserves supersymmetry as well as conservation of momentum
- The z-shifted superamplitude defined thus vanishes as $z \to 0$ (whereas the component amplitudes do not)

$$\mathcal{A}(z) = O(1/z^2) \quad \text{(SUGRA)} \qquad\qquad \mathcal{A}(z) = O(1/z) \quad \text{(SYM)}$$

[Arkani-Hamed Cachazo Kaplan]

(We can define shifted superamplitudes in $\mathcal{N}$=8 SUGRA similarly)

- BCFW argument gives (pole at $z = z_P$ associated with null $\hat{P}$)

Supersymmetric BCFW recursion relation

[Arkani-Hamed Cachazo Kaplan, Brandhuber Travaglini PH]

$$\mathcal{A} = \sum_P \int d^4\eta_{\hat{P}}\; \mathcal{A}_L(z_P)\frac{i}{P^2}\mathcal{A}_R(z_P)$$

3-point anti-MHV superamplitude

- Supersymmetry and Lorentz invariance fix this to [Arkani-Hamed Cachazo Kaplan, Brandhuber Travaglini PH]

$$\mathcal{A}_{\overline{\mathrm{MHV}}}(1,2,3) = \frac{\delta^{(4)}(p_1+p_2+p_3)\delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12]\,[23]\,[31]}$$

- This is supersymmetric (only supersymmetric object which does not have $\delta^{(8)}(\sum_{i=1}^{n}\eta_i\lambda_i)$)

Proof of tree-level covariance

- We can now prove recursively that each amplitude is invariant under conformal inversions and hence superconformal symmetry
- assume, for all $\mathcal{A}_L$ and $\mathcal{A}_R$ the transformation

$$\mathcal{A}_L(1,\ldots,m) \to \mathcal{A}_L(1,2,\ldots,m) \prod_{k=1}^{m} \frac{\kappa_k^2}{x_k^2}$$

Plug into super-BCFW to find

$$\mathcal{A}(1,\ldots,n) \to \mathcal{A}(1,2,\ldots,n) \prod_{k=1}^{n} \frac{\kappa_k^2}{x_k^2}$$

We have shown by induction:

All tree level amplitudes transform covariantly under dual superconformal transformations.

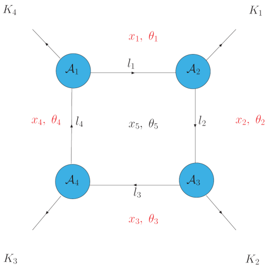
Dual conformal symmetry at one loop

We can play a similar game for the super box coefficients for any one loop amplitude

- All one loop amplitudes written in terms of boxes [Bern Dixon Dunbar Kosower 1994]

$$\mathcal{A}_{1-\text{loop}} = \sum \mathcal{B}_i \, I_i$$

- The box coefficients $\mathcal{B}_i$ can be calculated using quadruple cuts [Britto Cachazo Feng 2005]



- Supersymmetrise: (also done by [Arkani-Hamed Cachazo Kaplan])
  - amplitudes $\rightarrow$ superamplitudes $\implies$ coefficients $\longrightarrow$ supercoefficients
  - Introduce the appropriate fermionic delta functions which impose supermomentum conservation at the four corners of the diagram
- This gives an expression for the supercoefficients

We can show

The supercoefficients are covariant under dual superconformal symmetry

- We have restricted ourselves to the box *coefficients* which are IR finite
- A conjecture concerning dual superconformal properties of one-loop amplitudes themselves:

Conjecture [Drummond Henn Korchemsky Sokatchev]

$\mathcal{A}_n = \mathcal{A}_{n,\mathrm{MHV}}\,\mathcal{R}$ $\qquad \mathcal{R}$ is dual superconformally invariant

Outline

1. Scattering amplitude/Wilson loop duality
   - One loop MHV amplitudes
   - The Wilson loop
   - Dual conformal invariance
2. Dual superconformal symmetry of the entire S-matrix:
   - at tree level
   - at one loop
3. $\mathcal{N} = 8$ SUGRA

A couple of comments:

1. The one loop IR divergent terms in the amplitude exponentiates (shown at 4points 2 loops) [Naculich Nastase Schnitzer, Brandhuber Nasti Spence Travaglini PH]

$$\log(M) = M^{(1)} + O(\epsilon^0)$$

2. We have found an expression for a gravity Wilson loop which reproduces the four-point one loop amplitude in a certain gauge.

$$W[\mathcal{C}] := \left\langle \mathcal{P} \exp \left[i\kappa \oint_{\mathcal{C}} d\tau \; h_{\mu\nu}(x(\tau)) \dot{x}^{\mu}(\tau) \dot{x}^{\nu}(\tau) \right] \right\rangle ,$$

Problem:

- Four-point only
- Gauge non-invariant (although gauge variation localised at the vertices)

Summary and outlook

- Planar MHV amplitudes in $\mathcal{N} = 4$ are equivalent to certain polygonal Wilson loops
- Wilson loop/amplitude duality $\Longrightarrow$ conformal invariance of MHV amplitudes (divided by tree)
- However the full superamplitude (including tree) has superconformal covariance
- Suggests a 'generalised super Wilson loop' to reproduce the full $\mathcal{N}$=4 superamplitude
- This is much better understood in string theory now[Berkovitz Maldacena]
- Superconformal symmetry is the first of an infinite number of conserved charges coming from integrability [Berkovitz Maldacena, Beisert Ricci Tseytlin Wolf]
- Can these be used to fix all amplitudes $n \geq 6$?