A New Identity for Gauge Theory Amplitudes

Hidden Structures workshop, NBI

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arXive:0805.3993 [hep-ph]

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arXiv:0808.4112 [hep-th]

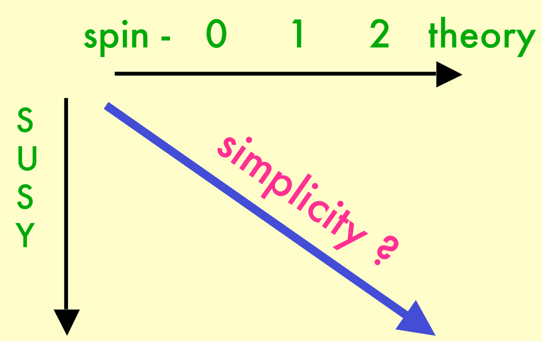
Outline

- Motivation
  - Simplicity in scattering amplitudes
  - UV superfiniteness of $\mathcal{N} = 8$ SUGRA
  - A hidden structure in cuts
- A surprising new identity at tree level
- New relations between partial amplitudes
- Beautiful reformulation of Kawai-Lewellen-Tye relations
- Outlook & Summary

Motivation - simplicity vs. complexity

- Physical theories - gravity and gauge theories - have surprisingly simple on-shell scattering amplitudes
- Feynman rules are much more complex - gravity in particular
- Adding SUSY increases complexity of Lagrangian & Feynman rules - yet scattering amplitudes becomes simpler
- Simplest theories ? ← Arkani-Hamed, Cachazo, Kaplan
  - $\mathcal{N} = 4$ SYM - solvable (in 't Hooft limit) ?
  - $\mathcal{N} = 8$ SUGRA - perturbatively finite ?

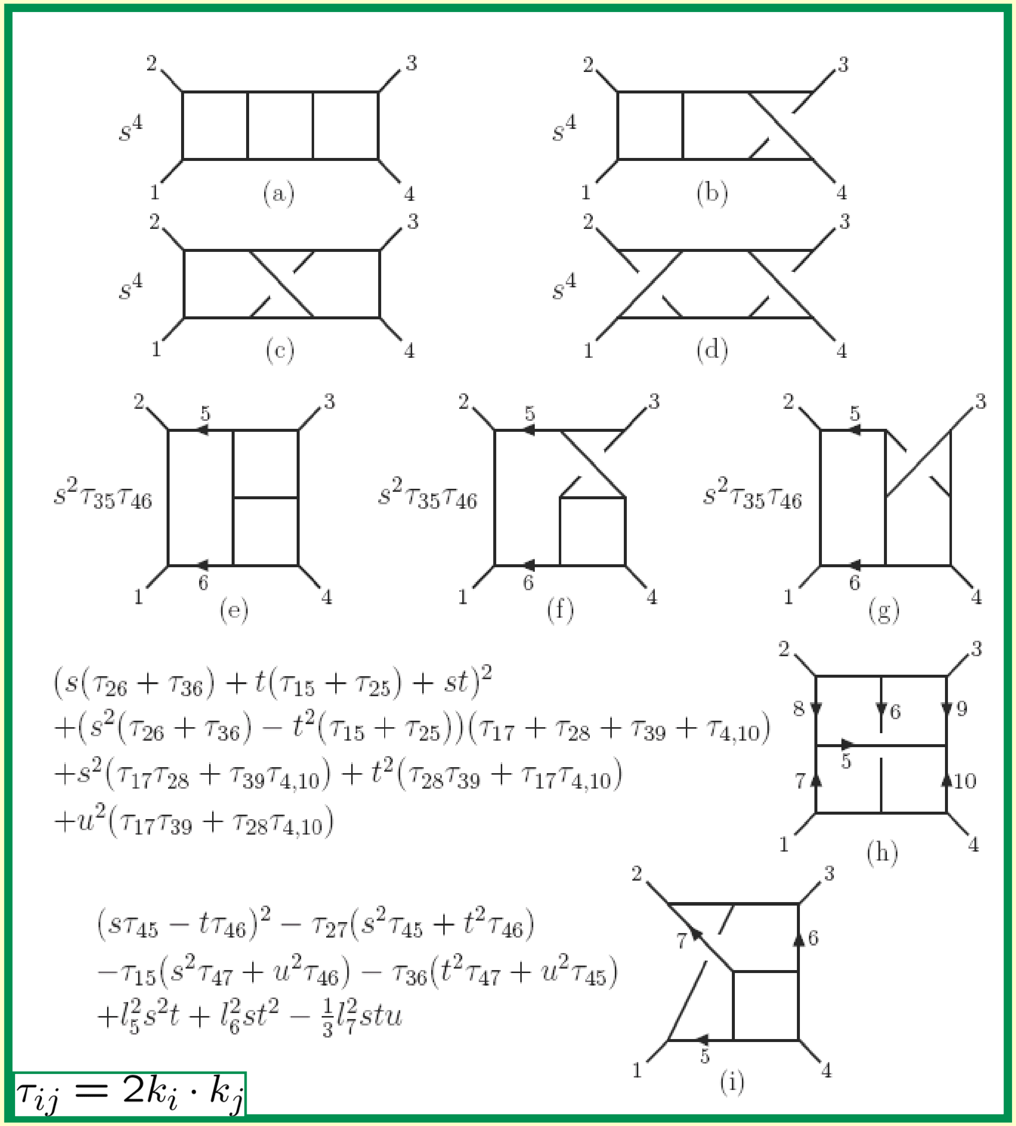
see talks by Arkani-Hamed, Benincasa, Bjerrum-Bohr, Heslop, Ricci, Skinner, Spradlin & Vanhove



$\mathcal{N}=8$ supergravity "superfinite"

Bern, Carrasco, Dixon, HJ, Kosower, Roiban; hep-th/0702112
Bern, Carrasco, Dixon, HJ, Roiban arXiv:0808.4112 [hep-th]

$\mathcal{N}=8$ supergravity amplitude *manifestly* has diagram-by-diagram power counting of $\mathcal{N}=4$ SYM!



UV "superfinite" for $D = 4$, $L = 3$

$$D < \frac{6}{L} + 4$$

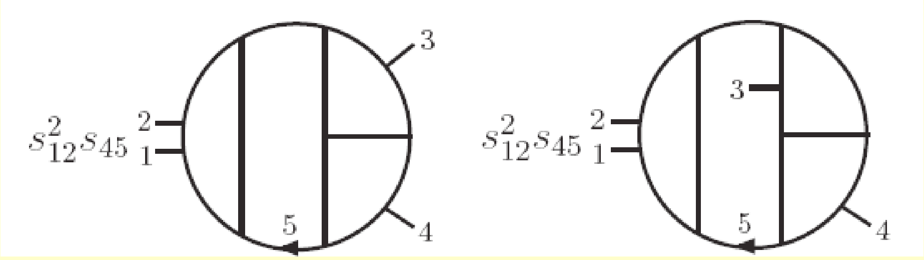
After integration $\Rightarrow$
$D = 6$ divergence demonstrated
$\Rightarrow$ bound is saturated

Four-loop calculation in progress

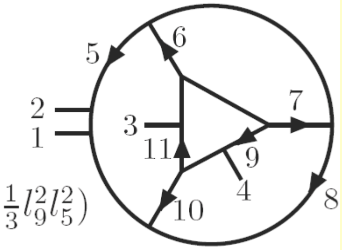
$\mathcal{N}=4$ super-Yang-Mills case is complete
$\mathcal{N}=8$ supergravity still in progress

Bern, Carrasco, Dixon, HJ, Roiban (2008)

Some $\mathcal{N}=4$ SYM contributions…

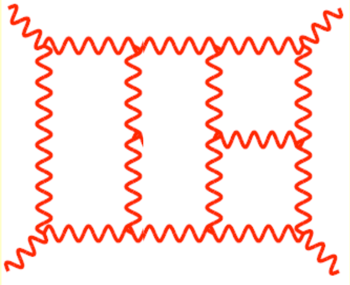


$$s_{12}^2 s_{98} - s_{12}s_{35}s_{67} + \tfrac{1}{3}l_9^2 s_{12}(s_{35}-s_{12}) + s_{12}(l_5^2 s_{4,10} - l_5^2 l_{11}^2 - \tfrac{1}{3}l_9^2 l_5^2)$$



50 distinct planar and non-planar diagrammatic topologies

one Feynman diagram…

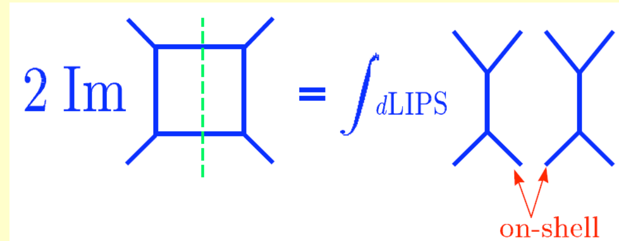


…~ 10^{26} terms in pure gravity

How we do it - the Unitarity Method

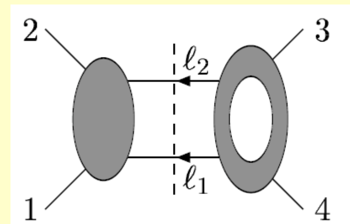
TIME

optical theorem



unitarity method

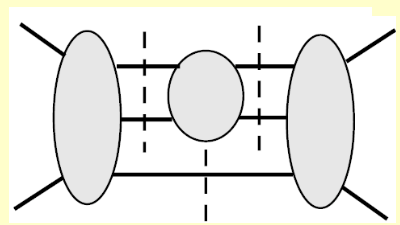
Bern, Dixon, Dunbar and Kosower (1994)



see Badger's talk

generalized unitarity

Bern, Dixon and Kosower

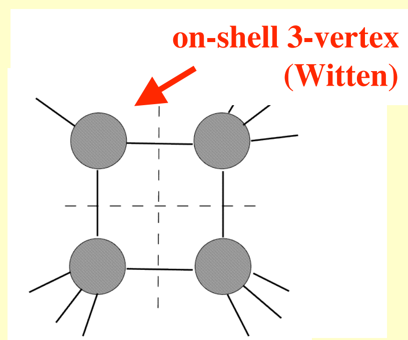


quadruple cut
(leading singularity)

Britto, Cachazo, Feng;
Buchbinder, Cachazo (2004)

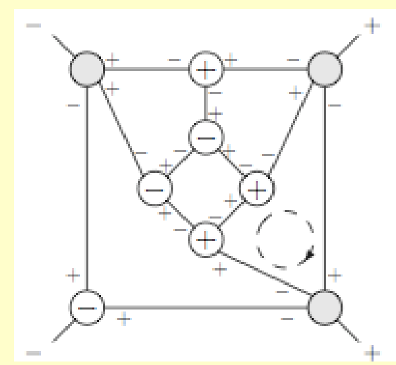
Cachazo and Skinner
Cachazo, Spradlin, Volovich
(2008)

see Benincasa's talk



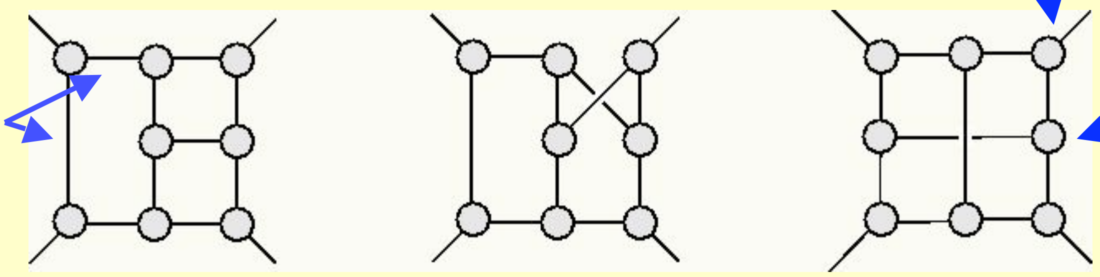
maximal cut

Bern, Carrasco, HJ
and Kosower (2007)

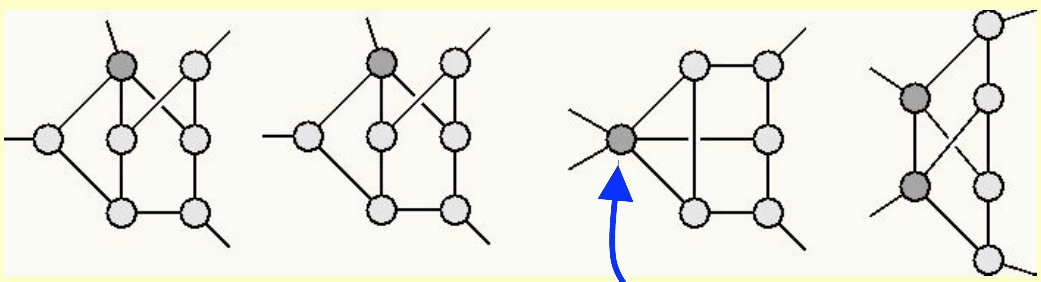


Maximal cuts - a *systematic* approach for any theory

- put maximum number of propagator on-shell → simplifies calculation



- systematically release cut conditions → great control of missing terms



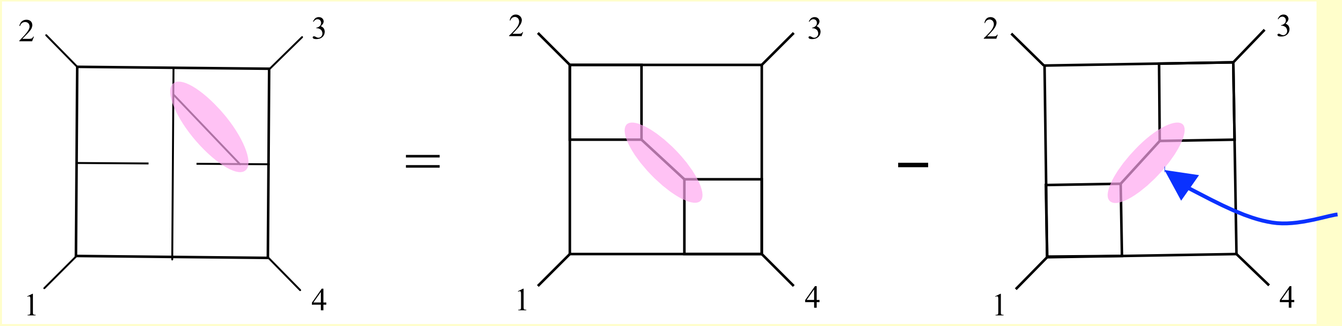
Preferred technique for 4-loop calculation — exposes a ***hidden structure***

A hidden structure emerges

maximal cuts with one 4pt blob involves 3 diagrams ...

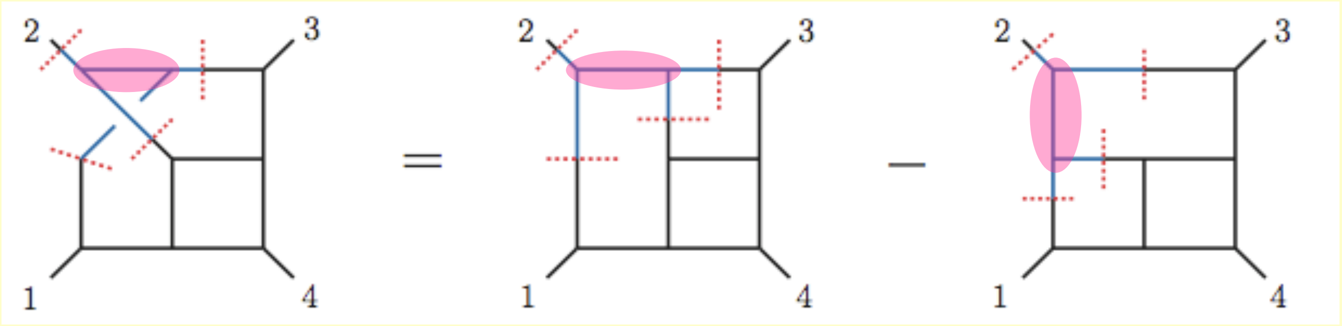
surprise ! ..... these diagrams are not independent

Yang-Mills
4 loops



4pt blob
with off-shell
internal
momenta

Yang-Mills
3 loops



What is going on ? → 4pt blob (tree amplitude) has a *hidden structure*

Gauge theory at tree-level

Color decomposition

- Modern decomposition

$$\mathcal{A}_n^{\rm tree}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} {\rm Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\rm tree}(1,2,\ldots,n)$$

gauge invariant

- Alternative decomposition, 4pt example

$$\mathcal{A}_4^{\rm tree}(1,2,3,4) = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

- Map

$$\tilde{f}^{abc} \equiv i\sqrt{2} f^{abc} = {\rm Tr}([T^a, T^b]T^c)$$

color structures

$$A_4^{\rm tree}(1,2,3,4) \equiv \frac{n_s}{s} + \frac{n_t}{t},$$

$$A_4^{\rm tree}(1,3,4,2) \equiv -\frac{n_u}{u} - \frac{n_s}{s}$$

$$A_4^{\rm tree}(1,4,2,3) \equiv -\frac{n_t}{t} + \frac{n_u}{u}$$

kinematic structures

color factors

$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$

$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

kinematic numerators

$$n_s, n_t, n_u$$

absorbs 4-pt contact terms
– but gauge dependent!

A surprising 4pt identity

$$\mathcal{A}_4^{\text{tree}}(1,2,3,4) = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

color factors

$$c_u \equiv \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_3 a_1}$$

$$c_s \equiv \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}$$

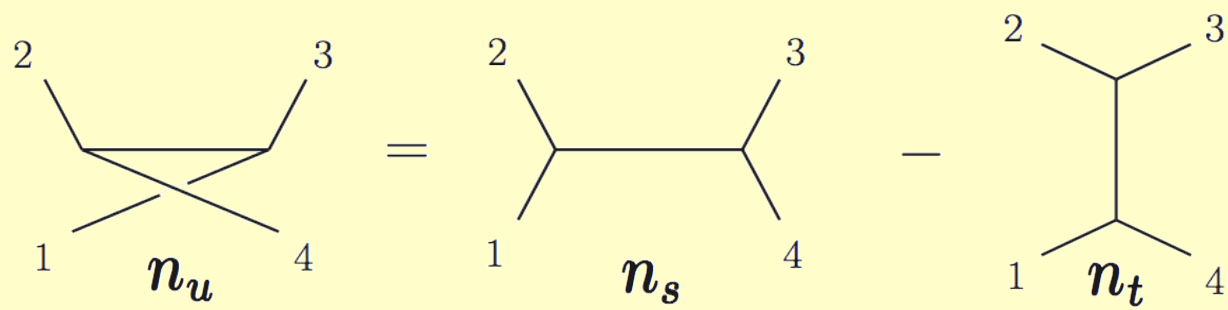
$$c_t \equiv \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}$$

- Jacobi identity for color

$$c_u = c_s - c_t$$

- And a Jacobi identity for kinematics

$$n_u = n_s - n_t$$



Zhu; Goebel, Halzen, Leveille noticed this 4pt property in 1980

- Kinematic numerators gauge dependent - but 4pt identity is gauge invariant

$$-n'_s + n'_t + n'_u = -n_s + n_t + n_u - \alpha(k_i, \varepsilon_i)(s+t+u) = 0$$

~ gauge parameter

Similar identity at higher points

- Decomposing 5pt amplitude in terms of 15 cubic diagrams

$$\mathcal{A}_5^{\text{tree}} = g^3\Big(\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} + \frac{n_6c_6}{s_{14}s_{25}} + \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} + \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}} + \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\Big),$$

kinematic numerator (n_6)

color factor (c_6)

propagators ($s_{14}s_{25}$)

$$s_{ij} = (k_i + k_j)^2$$

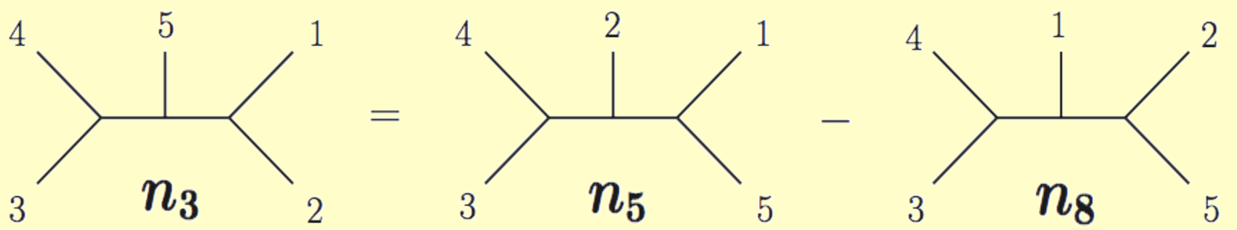
- Equivalent to partial amplitudes

$$A_5^{\text{tree}}(1,2,3,4,5) \equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}$$

etc…

- Kinematic Jacobi identity holds…

$$n_3 - n_5 + n_8 = 0 \quad \Leftrightarrow \quad c_3 - c_5 + c_8 = 0$$



$$c_3 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_5c}\tilde{f}^{ca_1a_2}$$
$$c_5 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_2c}\tilde{f}^{ca_1a_5}$$
$$c_8 \equiv \tilde{f}^{a_3a_4b}\tilde{f}^{ba_1c}\tilde{f}^{ca_2a_5}$$

…but is no longer gauge invariant!

not gauge invariant...yet physical

- In a general theory we can solve the 15 n_i's at 5 pts
  - 9 independent kinematic Jacobi identities
  - plus 2 constraints:

$$n_5 \equiv s_{51}s_{34}\Big(A_5^{\text{tree}}(1,2,3,4,5) - \frac{n_1}{s_{12}s_{45}} - \frac{n_2}{s_{23}s_{51}} - \frac{n_3}{s_{34}s_{12}} - \frac{n_4}{s_{45}s_{23}}\Big)$$

$$n_6 \equiv s_{14}s_{25}\Big(A_5^{\text{tree}}(1,4,3,2,5) - \frac{n_5}{s_{43}s_{51}} - \frac{n_7}{s_{32}s_{14}} - \frac{n_8}{s_{25}s_{43}} - \frac{n_2}{s_{51}s_{32}}\Big)$$

  - $\Rightarrow$ 4 undetermined n_i's (pure gauge transformations)

$$A_5^{\text{tree}}(1,3,4,2,5) = \frac{-s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5) + s_{14}(s_{24}+s_{25})A_5^{\text{tree}}(1,4,3,2,5)}{s_{13}s_{24}}$$

$$A_5^{\text{tree}}(1,2,4,3,5) = \frac{-s_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5) + s_{45}(s_{12}+s_{24})A_5^{\text{tree}}(1,2,3,4,5)}{s_{24}s_{35}}$$

etc...

- Any 5pt tree is a linear combination of two basis amplitudes

$$A_5(\ldots\ldots) = \alpha\, A_5(1,2,3,4,5) + \beta\, A_5(1,4,3,2,5)$$

true for any gauge theory and in *D*-dimensions

Tree level n-points – a conjecture

- A gauge theory tree amplitude can be expanded in purely cubic diagrams

full amplitude $\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = g^{n-2} \sum_i \frac{n_i c_i}{(\prod_j p_j^2)_i}$

partial amplitude $A_n^{\text{tree}}(1,2,3,\ldots,n) = \sum_j \frac{n_j}{(\prod_m p_m^2)_j}$

color factors $c_i = \tilde{f}^{abc} \tilde{f}^{cde} \ldots \tilde{f}^{xyz}$

- Jacobi identity true for both color and kinematics...

$$c_\alpha = c_\beta - c_\gamma \quad \Leftrightarrow \quad n_\alpha = n_\beta - n_\gamma$$

...as long as gauge invariance is enforced for $(n - 3)!$ partial amplitudes

$$A_n^{\text{tree}}(\mathcal{P}_i\{1,2,3,\ldots,n\}) = \left[\sum_j \frac{n_j}{(\prod_m p_m^2)_j}\right]_i$$

$\Rightarrow$ only $(n - 3)!$ linearly independent partial amplitudes

Checked through 8 pts!

All-n formula – partial amplitude relations

- General relations for gauge theory partial amplitudes

$$A_n^{\text{tree}}(1,2,\{\alpha\},3,\{\beta\}) = \sum_{\{\sigma\}_j \in \text{POP}(\{\alpha\},\{\beta\})} A_n^{\text{tree}}(1,2,3,\{\sigma\}_j) \prod_{k=4}^{m} \frac{\mathcal{F}(3,\{\sigma\}_j,1|k)}{s_{2,4,\ldots,k}}$$

where

$$\{\alpha\} \equiv \{4,5,\ldots,m-1,m\}, \qquad \{\beta\} \equiv \{m+1,m+2,\ldots,n-1,n\}$$

and

$$\mathcal{F}(3,\sigma_1,\sigma_2,\ldots,\sigma_{n-3},1|k) \equiv \mathcal{F}(\{\rho\}|k) = \left\{\begin{array}{ll} \sum_{l=t_k}^{n-1} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} < t_k \\ -\sum_{l=1}^{t_k} \mathcal{G}(k,\rho_l) & \text{if } t_{k-1} > t_k \end{array}\right\} + \left\{\begin{array}{ll} s_{2,4,\ldots,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\ -s_{2,4,\ldots,k} & \text{if } t_{k-1} > t_k > t_{k+1} \\ 0 & \text{else} \end{array}\right\}$$

and

$$\mathcal{G}(i,j) = \left\{\begin{array}{ll} s_{i,j} & \text{if } i < j \text{ or } j = 1,3 \\ 0 & \text{else} \end{array}\right\}$$

and t_k is the position of leg k in the set $\{\rho\}$

$$A_n(\ldots\ldots) = \alpha_1 A_n(1,2,..,n) + \alpha_2 A_n(2,1,..,n) + \ldots + \alpha_{(n-3)!} A_n(3,2,..,n)$$

Examples

4pts

$$A_4^{\rm tree}(1,2,\{4\},3) = \frac{A_4^{\rm tree}(1,2,3,4)s_{14}}{s_{24}}$$

5pts

$$A_5^{\rm tree}(1,2,\{4\},3,\{5\}) = \frac{A_5^{\rm tree}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\rm tree}(1,2,3,5,4)s_{14}}{s_{24}},$$

$$A_5^{\rm tree}(1,2,\{4,5\},3) = \frac{-A_5^{\rm tree}(1,2,3,4,5)s_{34}s_{15} - A_5^{\rm tree}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}}$$

6pts

$$\begin{aligned} A_6^{\rm tree}(1,2,\{4,5\},3,\{6\}) = & -\frac{A_6^{\rm tree}(1,2,3,4,5,6)s_{34}(s_{15}+s_{56})}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,4,6,5)s_{34}s_{15}}{s_{24}s_{245}} \\ & - \frac{A_6^{\rm tree}(1,2,3,6,4,5)(s_{34}+s_{46})s_{15}}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,5,4,6)(s_{14}+s_{46})(s_{245}+s_{35})}{s_{24}s_{245}} \\ & - \frac{A_6^{\rm tree}(1,2,3,5,6,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}} - \frac{A_6^{\rm tree}(1,2,3,6,5,4)s_{14}(s_{245}+s_{35}+s_{56})}{s_{24}s_{245}} \end{aligned}$$

Gravity & Gauge theory relations

KLT relations

Kawai, Lewellen and Tye

Originally a string theory tree level identity

closed string ~ (left-mover open string) × (right-mover open string)

Field theory limit ⇒ gravity theory ~ (gauge theory) × (gauge theory)

$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\,\widetilde{A}_4^{\rm tree}(1,2,4,3)$$

$$M_5^{\rm tree}(1,2,3,4,5) = is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)\,\widetilde{A}_5^{\rm tree}(2,1,4,3,5) + is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\,\widetilde{A}_5^{\rm tree}(3,1,4,2,5)$$

gravity states are direct products of gauge theory states $|1\rangle_{\rm grav} = |1\rangle_{\rm gauge} \otimes |1\rangle_{\rm gauge}$

New identity + KLT

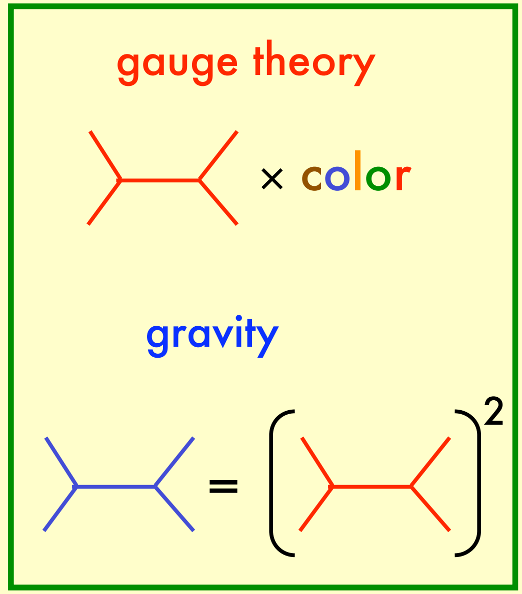
Feeding the new identity through KLT gives...

$$n_u = n_s - n_t$$

$$+ \quad M_4^{\text{tree}}(1,2,3,4) = -i s_{12} A_4^{\text{tree}}(1,2,3,4)\, \widetilde{A}_4^{\text{tree}}(1,2,4,3)$$

$$= \quad -iM_4^{\text{tree}}(1,2,3,4) = \frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u}$$

... a beautiful "numerator squaring" relationship



Compare to gauge theory...

$$\frac{1}{g^2}\mathcal{A}_4^{\text{tree}}(1,2,3,4) = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

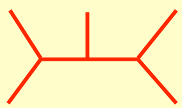
Unlike KLT this "squaring" relationship is between local objects n_i and is manifestly crossing (Bose) symmetric

Holds at all-*n* tree level

- At 5 points

true given that n_i and $\tilde{n}_i$ satisfy kinematic Jacobi identities

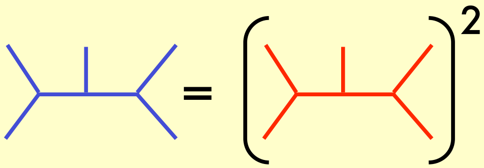
gauge theory



× color

$$\mathcal{A}_5^{\rm tree} = g^3\Big(\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} + \frac{n_6c_6}{s_{14}s_{25}} + \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} + \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}} + \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\Big),$$

gravity



$$\mathcal{M}_5^{\rm tree} = i\left(\frac{\kappa}{2}\right)^3\Big(\frac{n_1\tilde{n}_1}{s_{12}s_{45}} + \frac{n_2\tilde{n}_2}{s_{23}s_{51}} + \frac{n_3\tilde{n}_3}{s_{34}s_{12}} + \frac{n_4\tilde{n}_4}{s_{45}s_{23}} + \frac{n_5\tilde{n}_5}{s_{51}s_{34}} + \frac{n_6\tilde{n}_6}{s_{14}s_{25}} + \frac{n_7\tilde{n}_7}{s_{32}s_{14}} + \frac{n_8\tilde{n}_8}{s_{25}s_{43}} + \frac{n_9\tilde{n}_9}{s_{13}s_{25}} + \frac{n_{10}\tilde{n}_{10}}{s_{42}s_{13}} + \frac{n_{11}\tilde{n}_{11}}{s_{51}s_{42}} + \frac{n_{12}\tilde{n}_{12}}{s_{12}s_{35}} + \frac{n_{13}\tilde{n}_{13}}{s_{35}s_{24}} + \frac{n_{14}\tilde{n}_{14}}{s_{14}s_{35}} + \frac{n_{15}\tilde{n}_{15}}{s_{13}s_{45}}\Big),$$

- At *n* points

$$\mathcal{A}_n^{\rm tree}(1,2,3,\ldots,n) = g^{n-2}\sum_i \frac{n_ic_i}{(\prod_j p_j^2)_i}$$

$$\mathcal{M}_n^{\rm tree}(1,2,3,\ldots,n) = i\left(\frac{\kappa}{2}\right)^{n-2}\sum_i \frac{n_i\tilde{n}_i}{(\prod_j p_j^2)_i}$$

Checked trough 8 points !

Outlook - beyond on-shell & tree-level ?

- Gauge theory
  - Generalization to loop-level kinematic Jacobi identity ?
  - Find special gauge where Feynman rules manifestly obeys the identity
  - Lagrangian understanding highly desirable

- Gravity
  - Gravity Feynman rules = $(\text{gauge theory Feynman rules})^2$ possible ?
  - Might clarify the KLT relations in terms of the Lagrangians
  - Possible matching off-shell and non-perturbative physics between gravity and gauge theory

Summary

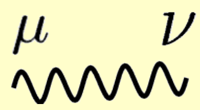
- We uncovered a new Jacobi-like identity for tree diagrams in gauge theory amplitudes
- The identity is gauge dependent - presumably there is a special gauge where it is manifest - nonetheless the identity relates and constrains physical information
- The identity imply new relations for gauge invariant partial amplitudes
- Combine with KLT to uncover a new local and manifestly crossing (Bose) symmetric "squaring" relationship between gravity and gauge theories
- **It is possible that this identity holds on the Lagrangian level - can deepen our understanding of gauge theory and gravity**

Extra slides

Pure Yang-Mills

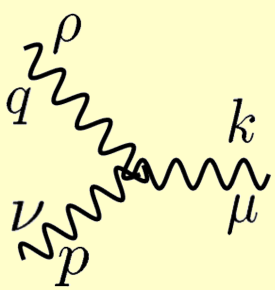
$$\mathcal{L}_{\rm YM} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu}$$

Feynman gauge propagator :

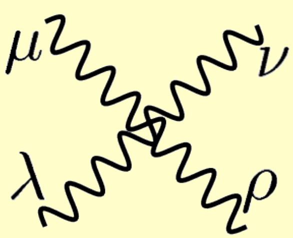


$$= \frac{-i\eta_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$$

vertices:



$$= -g f^{abc}\left[\eta_{\mu\nu}(k-p)_\rho + \eta_{\rho\mu}(q-k)_\nu + \eta_{\nu\rho}(p-q)_\mu\right]$$



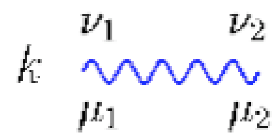
$$= \begin{array}{l} -ig^2 f^{abe} f^{ecd}(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}) \\ -ig^2 f^{ace} f^{edb}(\eta_{\mu\sigma}\eta_{\rho\nu} - \eta_{\mu\nu}\eta_{\rho\sigma}) \\ -ig^2 f^{ade} f^{ebc}(\eta_{\mu\nu}\eta_{\sigma\rho} - \eta_{\mu\rho}\eta_{\sigma\nu}) \end{array}$$

Pure Einstein gravity

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

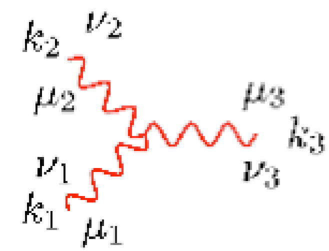
de Donder gauge propagator :

$$P_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\left[\eta_{\mu\nu}\eta_{\alpha\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\alpha}\eta_{\nu\beta}\right]\frac{i}{k^2 + i\epsilon}$$



cubic vertex:

$$\begin{aligned} &G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\ &\quad \text{sym}[\, -\tfrac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\ &\qquad + P_6(k_1 \cdot k_2 \eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\ &\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\ &\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})] \end{aligned}$$



After symmetrization ~ 100 terms!

Gauge theory amplitude properties

- Tree level, adjoint representation

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

- Well-known partial amplitude properties

$$A_n^{\text{tree}}(1,2,\ldots,n) = A_n^{\text{tree}}(2,\ldots,n,1)$$ cyclic symmetry

$$A_n^{\text{tree}}(1,2,\ldots,n) = (-1)^n A_n^{\text{tree}}(n,\ldots,2,1)$$ reflection symmetry

} $(n - 1)!$

$$\sum_{\sigma\in\text{cyclic}} A_n^{\text{tree}}(1,\sigma(2,3,\ldots,n)) = 0$$ "photon"-decoupling identity

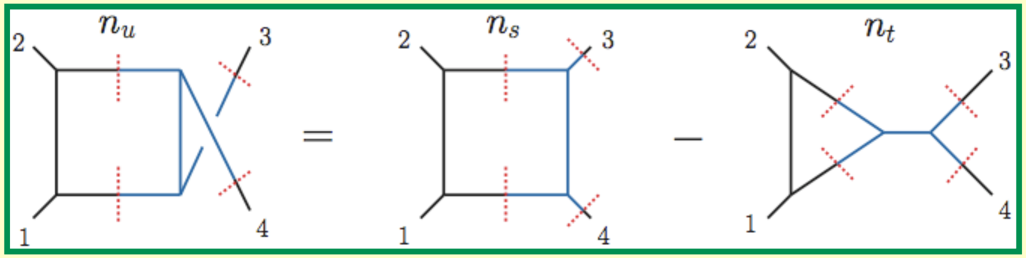
$$A_n^{\text{tree}}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\{\sigma\}_i\in\text{OP}(\{\alpha\},\{\beta^T\})} A_n^{\text{tree}}(1,\{\sigma\}_i,n)$$ Kleiss-Kuijf relations

} $(n - 2)!$

- *New relations reduce independent basis to* **$(n - 3)!$**

A peek at loop level

- Identity holds in unitarity cuts



- Some known examples point at loop-level identity… but not checked in general