Multiloop Gluon Amplitudes

and

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• In maximally supersymmetric $\mathcal{N} = 4$ super-Yang Mills theory, the problem of computing any one-loop amplitude can be reduced to that of computing tree amplitudes. (1990s–2004 [Bern, Dixon, Kosower; Britto, Cachazo, Feng; . . .])

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$\Rightarrow$ The key to such simplification is that although one-loop amplitudes have many poles and branch cuts with a complicated structure of intersections, they are completely determined by their highest codimension singularities, called the leading singularity.

Background: On-Shell Methods

Any L-loop scattering amplitude can, in principle, be obtained by summing over all Feynman diagrams:

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p, \ell)$$

$p =$ external momenta

$\ell =$ loop momenta

However, in practice this is a hopeless exercise due to the enormously large number of Feynman diagrams and their complexity in Yang-Mills theory.

Background: On-Shell Methods

$$\mathcal{A}^{(L)}(p) = \int d\ell_1 \cdots d\ell_L \sum_j F_j(p, \ell)$$

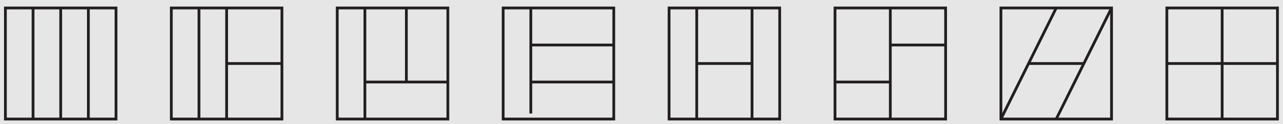
Rather, calculations typically proceed by first finding a representation of the amplitude in terms of a relatively simple basis of integrals $\{I_i\}$:

$$\mathcal{A}^{(L)}(p) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p, \ell)$$

where the coefficients $c_i(p)$ are computed by other means, such as the unitarity-based method [Bern, Dixon, Kosower, 1990s] or maximal cuts [Bern, Carrasco, Johansson, Kosower, 2007].

An Example

For example, unitarity based methods were used to express the four-loop four-particle amplitude in $\mathcal{N} = 4$ super-Yang-Mills as the sum of the following eight integrals (a couple of numerator factors are suppressed):



[Bern, Czakon, Dixon, Kosower, Smirnov, 2006]

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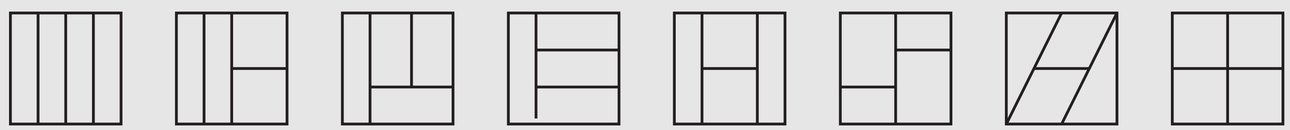


[Bern, Czakon, Dixon, Kosower, Smirnov, 2006]

Caution: These are not Feynman diagrams. Well, they are, but if we were drawing all Feynman diagrams there would be enormously too many. The result of BCDKS demonstrates, in this example, that all Feynman diagrams of other topologies add up to zero!

In searching for a representation

$$\int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell)$$

we are free to impose, if we desire, that the equality of both sides holds at the level of the integrand:

$$\sum_j F_j(p,\ell) = \sum_i c_i(p) I_i(p,\ell)$$

$$\sum_j F_j(p,\ell) = \sum_i c_i(p) I_i(p,\ell) \tag{1}$$

Here's a failsafe, but horribly inefficient, algorithm for finding such a representation:

1. Make a guess for a suitable basis $\{I_i\}$ of integrals.
2. Choose, at random, n values of the loop momenta $\{\ell_i\}$ and evaluate equation (1) at these values; this gives you n linear equations for the $c_i(p)$.
3. Solve the linear equations!
   - If there is no solution, you need to use a larger basis of integrals.
   - If there is ambiguity (the solution is not unique), then either you need to make n larger (more equations), or you are using an overcomplete basis.

Why is this method so terrible? We don't want to go anywhere NEAR a Feynman diagram!

$$\int d\ell_1 \cdots d\ell_L \sum_j F_j(p,\ell) = \sum_i c_i(p) \int d\ell_1 \cdots d\ell_L \, I_i(p,\ell)$$

On-shell methods follow essentially the same strategy, except that instead of choosing random values for the loop momenta $\{\ell_i\}$, you cleverly choose the loop momenta to take values which put various internal propagators on-shell.

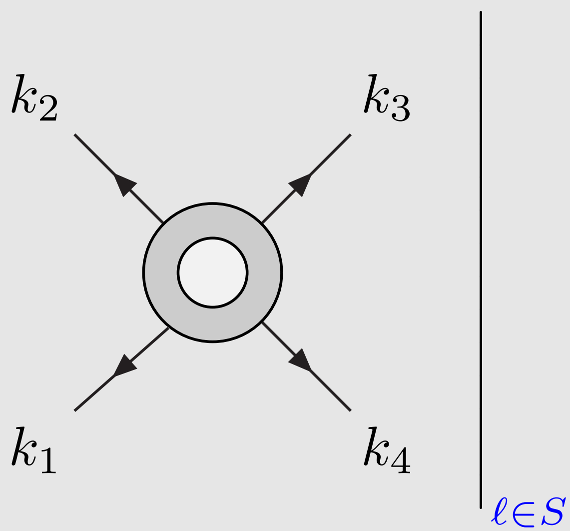
Then both sides of equation (2) are singular, so you just compare the coefficients of various singularities (i.e., the residues at the poles), on both sides of (2) to get your linear equations...

One Loop: BCF

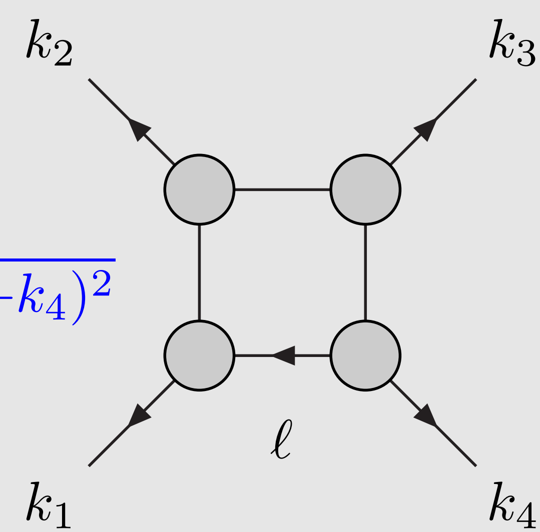
For example, at one-loop, if we choose ℓ from the set

$$S = \{\ell \in \mathbb{C}^4 : \ell^2 = 0, (\ell - k_1)^2 = 0, (\ell - k_1 - k_2)^2 = 0, (\ell + k_4)^2 = 0\}$$

(generically S consists of two isolated points) then the sum over all one-loop Feynman diagrams reduces to a product of four tree amplitudes:



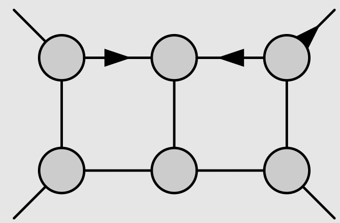
$$= \frac{1}{\ell^2(\ell-k_1)^2(\ell-k_{12})^2(\ell+k_4)^2}$$



It is crucial that ℓ be allowed to be complex. [Britto, Cachazo, Feng (2005)]

Higher Loops: Maximal Cuts

The natural generalization to higher loops is to restrict the loop momentum to the locus where all propagators are simultaneously on-shell. These have been called maximal cuts [Bern, Carrasco, Johansson, Kosower, 2007].



For example, for this topology there is a one-complex-dimensional subset S of $\mathbb{C}^8$ where all seven visible propagators go simultaneously on-shell. (Actually, two disconnected one-dimensional subspaces...)

Cachazo and Buchbinder (2005) observed that for four-particle amplitudes, the product of seven on-shell tree amplitudes is constant along the locus S. BCJK observed, and exploited, the fact that this situation apparently persists at least through five loops!

Leading Singularities

In the previous two loop-example, the maximal cut method makes use of a dimension-1 singularity in $\mathbb{C}^8$.

The ‘worst’ singularities in an L-loop amplitude are the isolated singularities, which live at discrete points in $\mathbb{C}^{4L}$— called leading singularities.

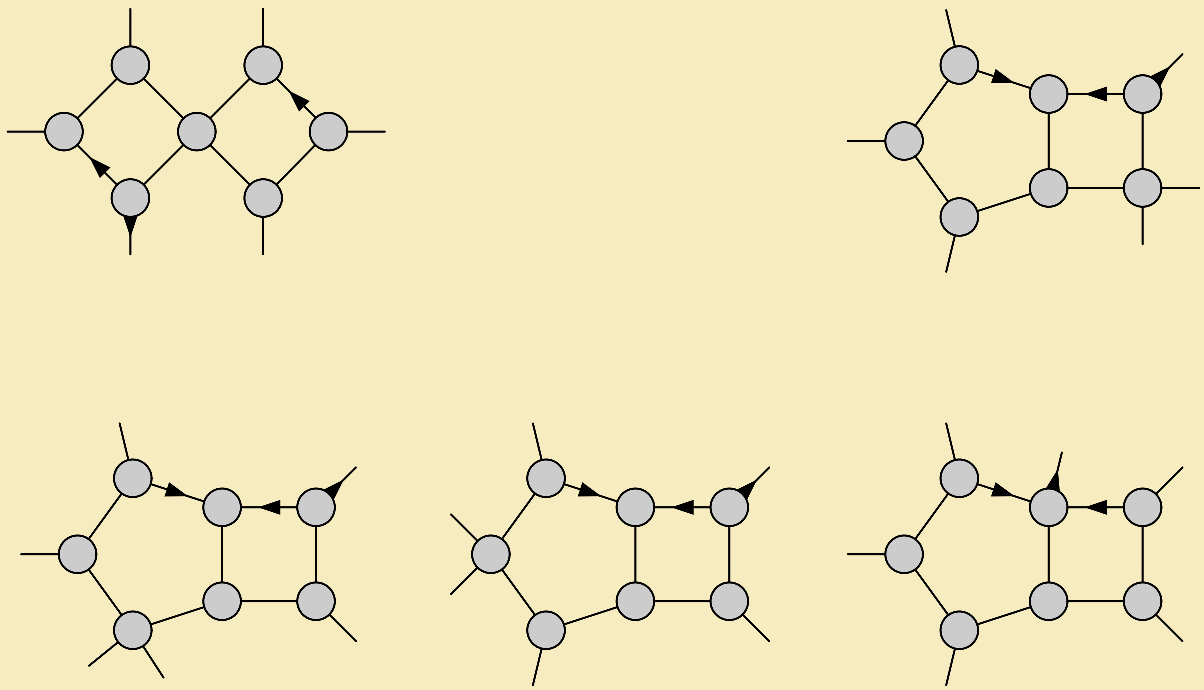
The leading singularity conjecture for $\mathcal{N} = 4$ YM or $\mathcal{N} = 8$ supergravity, postulates than an arbitrary amplitude is completely determined by its residues at all possible leading singularities. [Arkani-Hamed, Cachazo, Kaplan (2008)]

The leading singularity method involves finding a linear combination of integrals that has the same residues as the amplitude you're interested in. [Cachazo (2008)]

Now I’ll go briefly through two examples.

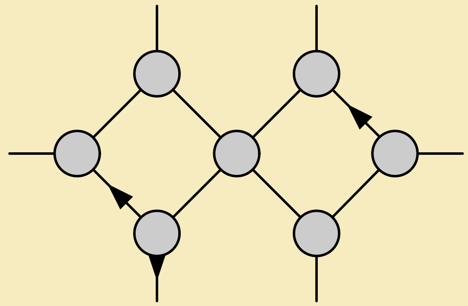
Example 1: Two-Loop Six-Particle Amplitude

There are five obvious topologies (where eight different propagators can go simultaneously on-shell) associated with isolated singularities in $\mathbb{C}^8$



(Actually each diagram here represents four distinct isolated singularities—there are in each case four solutions for the on-shell loop momenta.)

For example, if we look at the first diagram:



(A)

it represents the sum over the subset of all Feynman diagrams which contain all eight of the indicated propagators.

This set of Feynman diagrams has isolated poles at

$$\begin{aligned} S &= \{(\ell_1, \ell_2) \in \mathbb{C}^8 : \ell_1^2 = 0, (\ell_1 + p_1)^2 = 0, (\ell_1 - p_2)^2 = 0, \\ &\qquad (\ell_1 - p_2 - p_3)^2 = 0, \ell_2^2 = 0, (\ell_2 - p_4)^2 = 0, \\ &\qquad (\ell_2 + p_5)^2 = 0, (\ell_2 + p_5 + p_6)^2 = 0\} \end{aligned}$$

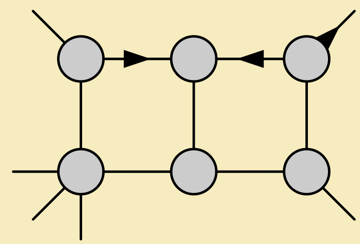
For generic external momenta p_i this consists of four distinct points in $\mathbb{C}^8$.

At each of these four points, the amplitude has an isolated order-8 pole.

To calculate the residue at this pole (i.e., the result of integrating over the corresponding contour Γ) is simple: just take the product of seven on-shell tree-level amplitudes, at each of the grey circles, and evaluate this product at the corresponding solution (ℓ_1, ℓ_2).

There are other, more subtle leading singularities:

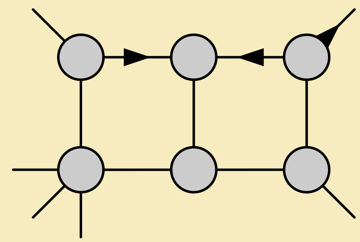
To see how these arise, consider the topology:



(F)

Although it looks like there is only a pole of order 7, not 8, there is another hidden singularity.

To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:



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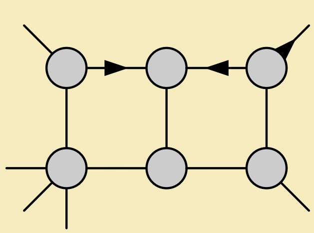
To expose it, consider a contour integral which computes the residue at either of the two singularities of the right-hand box:

$$\int_\Gamma d^4\ell_2 \frac{1}{\ell_2^2(\ell_1+k_1)^2(\ell_1+k_1+k_2)^2(\ell_1+\ell_2)^2} = \frac{1}{2}\frac{1}{(k_1+k_2)^2(\ell_2-k_1)^2}$$

where the right-hand side is just the Jacobian evaluated at the location of the singularity. Now this Jacobian has itself another singularity $1/(\ell_2-k_1)^2$.

The conclusion is that there do exist isolated poles of order 8 in such topologies. The residues at these poles can be computed by integrating over appropriate contours Γ.

There are a total of 8 different topologies of this type:



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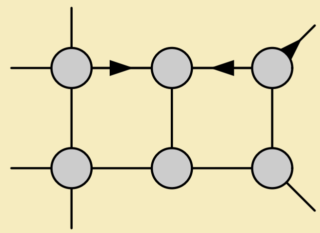
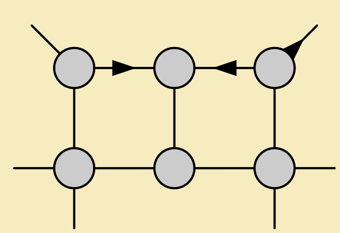
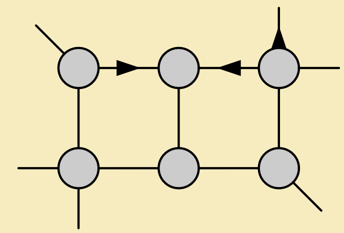


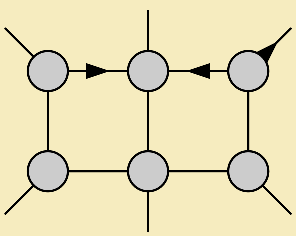
(G)



(H)



(I)



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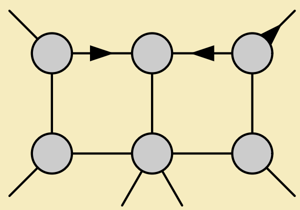
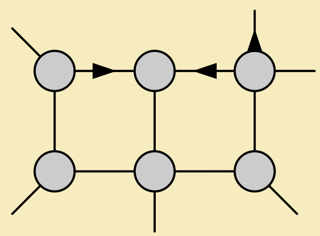
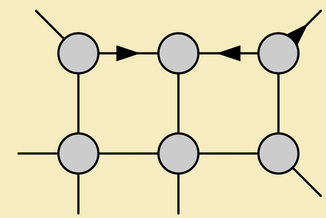


(K)



(L)



(M)

Contructing a Basis of Integrals

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We begin with a set that just contains the 13 scalar integrals appropriate to the 13 different topologies shown on the previous slides.

It turns out that with just this set of integrals, the linear equations have no solution, so we must add additional integrals to the set $\{I_i\}$

There is a systematic procedure to do this, which ends when one is able to solve all of the equations...

Contructing a Basis of Integrals

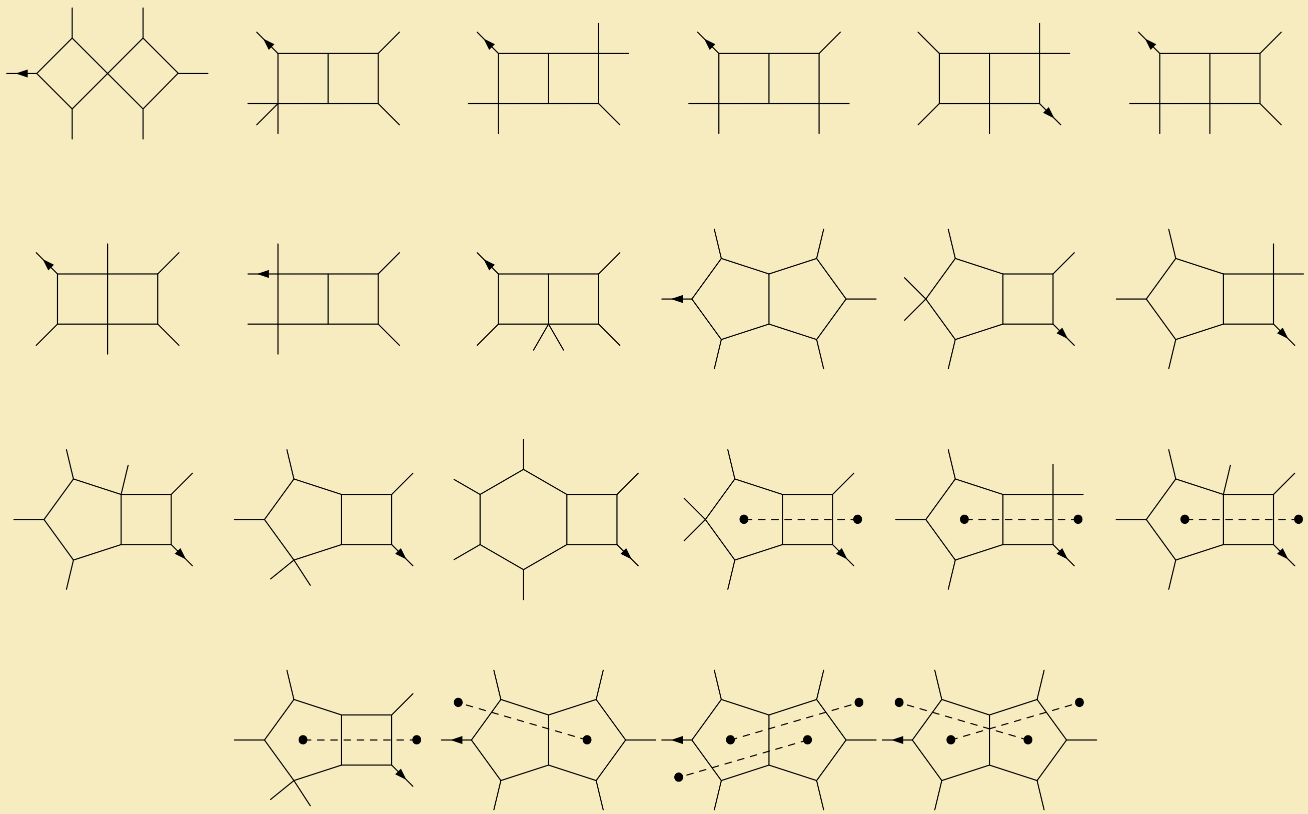
It can happen that when this procedure finishes, one ends up with a set of integrals $\{I_i\}$ that is overcomplete.

This happens because loop integrals for 6 or more external particles can frequently be expressed as linear combinations of other integrals. [van Neerven and Vermaseren, 1984].

If this happens, then the equations do not have a unique solution: given any solution $\{c_i\}$, one can add any set of coefficients $\{\tilde{c}_i\}$ that is actually zero due to a reduction identity.

Result 1

We find a representation of the 2-loop six-particle MHV amplitude in terms of



(Several of these can actually be set to zero using reduction identities).

Why?

Interest in this amplitude stemmed from at least two sources:

1. The ABDK/BDS conjecture for planar n-point MHV amplitudes:

$$M_n^{(2)}(\epsilon) - \frac{1}{2}(M_n^{(1)}(\epsilon))^2 + (\zeta(2) + \zeta(3)\epsilon + \zeta(4)\epsilon^2)M_n^{(1)}(2\epsilon) + \frac{\pi^4}{72} = \mathcal{O}(\epsilon)$$

which we found fails beginning at $n = 6$ [Bern, Dixon, Kosower, Roiban, MS, Volovich, Vergu].

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2. However, the apparent equality between amplitudes and light-like Wilson loops [Drummond, Henn, Korchemsky, Sokachev; Brandhuber, Heslop, Travaglini] survives at $n = 6$!

$\Longrightarrow$ Numerical ccomparisons indicate that precisely the same function appears for the corresponding IR-subtracted lightlike hexagon Wilson loop.

Parity-Even vs. Parity-Odd

Actually in 0803.1465 we only looked at the parity-even of the MHV amplitude; this means

$$M_6^{(2)}\Big|_{\rm even} = \frac{1}{2}\left(\frac{A^{(2)}_{6,\rm MHV}}{A^{(\rm tree)}_{6,\rm MHV}} + \frac{A^{(2)}_{6,\overline{\rm MHV}}}{A^{(\rm tree)}_{6,\overline{\rm MHV}}}\right)$$

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It seems to be widely believed that the parity-odd part drops out of the ABDK/BDS ansatz; i.e. that the remainder function $R(x_i)$ is purely even.

Indeed in 0805.4832 we checked that this appears to work out, but really our main motivation for writing 0805.4832 was to check that the leading singularity method works in this case!

2-loop Check of the Leading Singularity Method

Remember that it is still a **conjecture** that amplitudes in $\mathcal{N} = 4$ Yang-Mills are uniquely determined by their leading singularities, so we could easily have failed.

What form would 'failure' take?

In our calculation we looked at the residues of the amplitude at a total of 396 leading singularities (isolated points in $\mathbb{C}^8$ where the integrand has poles of order 8. (Really only 13 + various permutations)

We built an ansatz for the amplitude in terms of 189 different scalar integrals. (Really only 22 + various permutations)

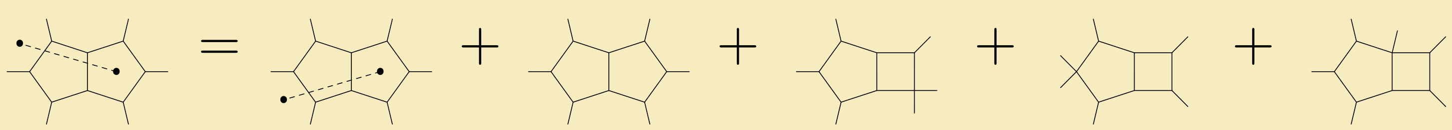
We found a 30-parameter family of solutions...

2-loop Check of the Leading Singularity Method

At this point we had to entertain two options:

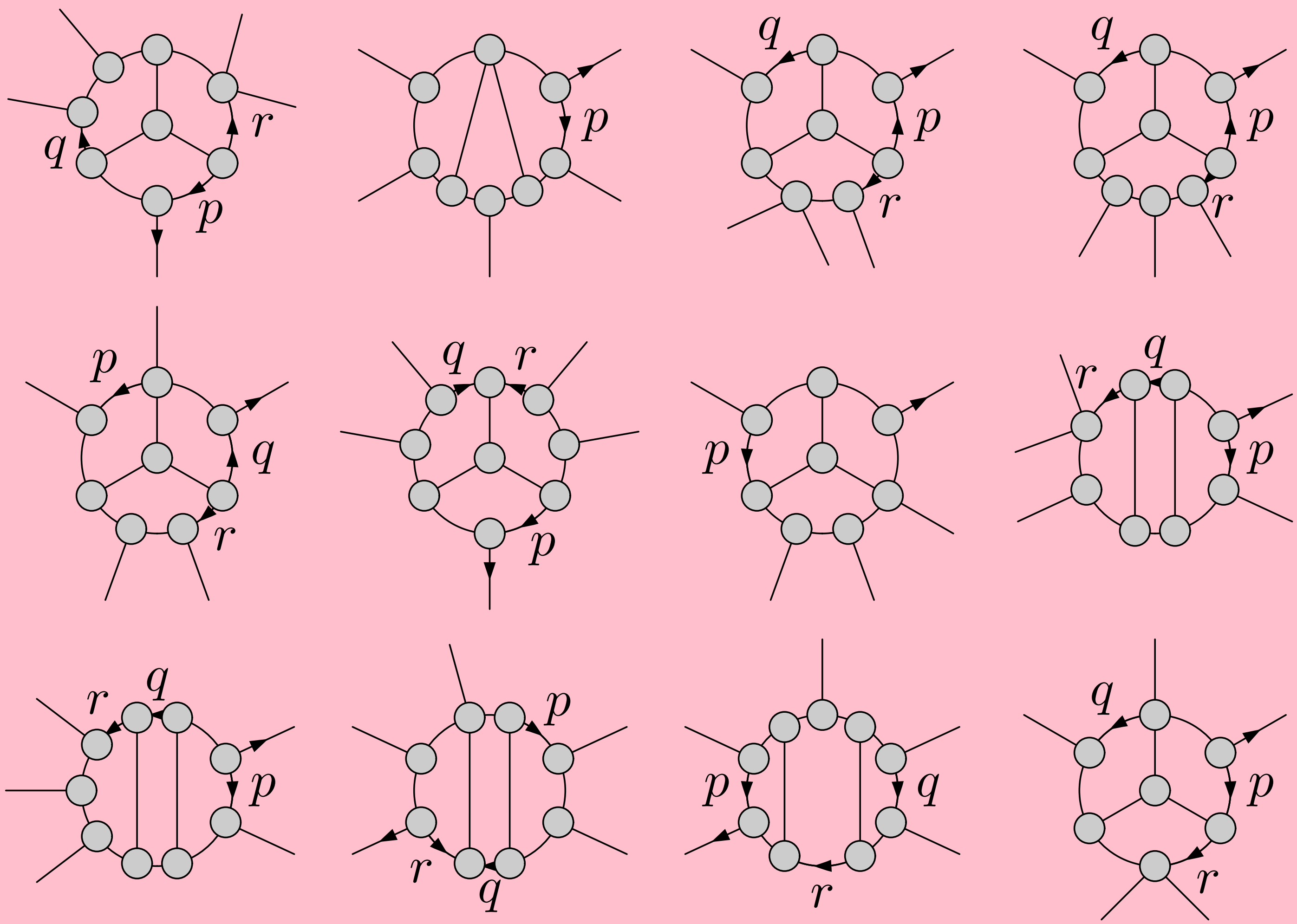
1. Knowledge of just the leading singularities is **not** enough information to completely fix the amplitude, or
2. Our basis of scalar integrals was overcomplete.

The majority of time on this project was spent proving (2) — specifically, we found precisely 30 independent linear relations amongst our basis, thereby establishing that the amplitude was indeed uniquely determined by its leading singularities.



Example 2: Three-Loop Five-Particle Amplitude

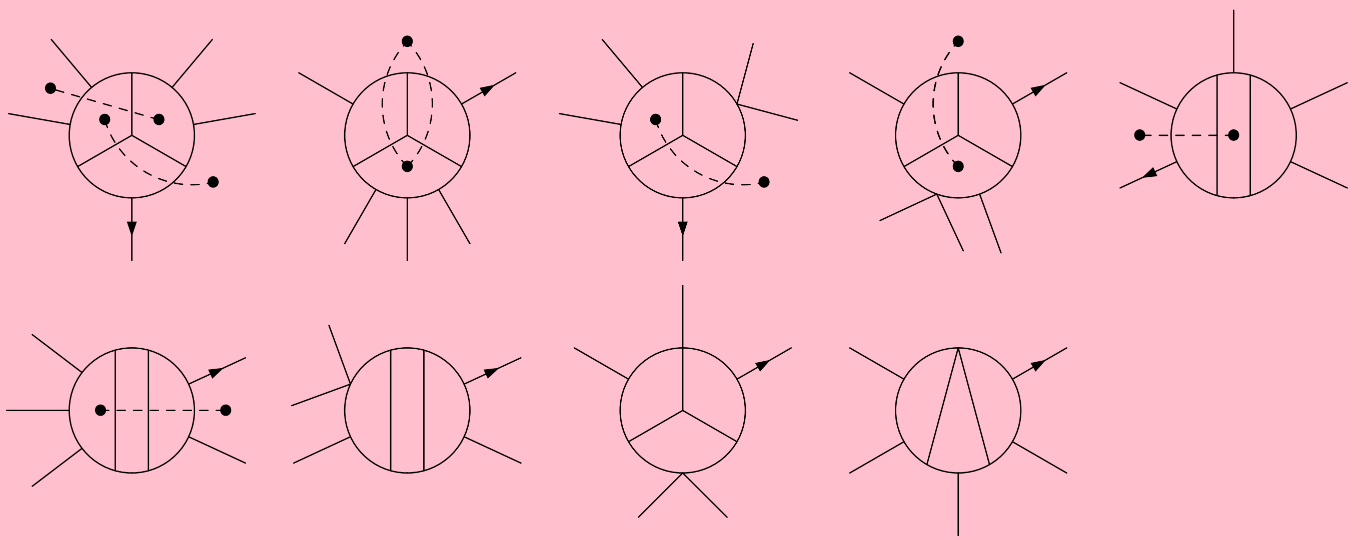
There are 12 topologies associated to leading singularities:



Result 2: Three-Loop Five-Particle Amplitude

It is easy to solve the resulting linear equations by hand (!) to find a representation of the amplitude in terms of a simple basis of integrals. [Spradlin, Volovich, Wen (2008)]

The explicit expressions for all coefficients are shown in our paper. The parity-even part of the amplitude involves 9 dual conformal integrals:

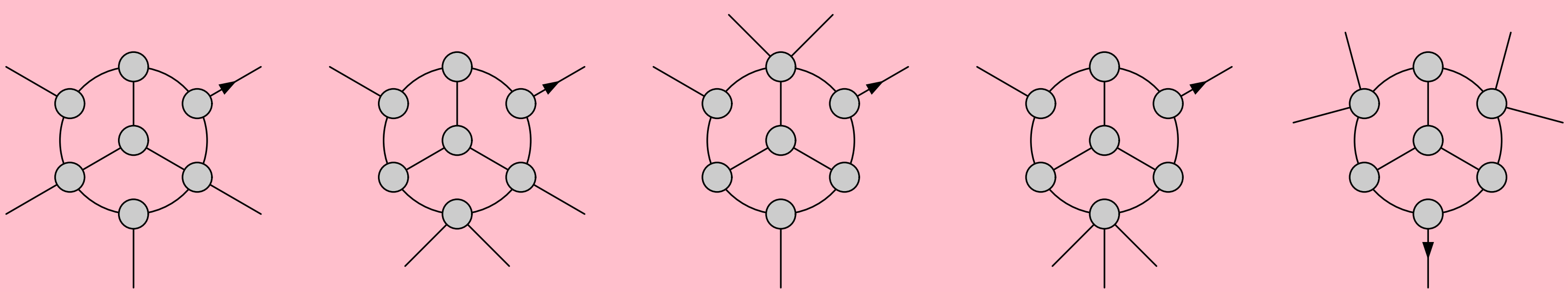


Three-Loop Test of the Leading Singularity Method

In this case there is no independent calculation to compare our result to. Therefore, since the leading singularity conjecture is still a conjecture, my lawyers have advised me to refer to our result for the 3-loop 5-particle amplitude as a conjecture.

What form could 'failure' take?

In this example there is a different kind of possible failure: there are five topologies with **no leading singularities**!



If the amplitude contains any of these, the leading singularity method will never notice.

BDS at Three Loops

The one, two and three loop "obstructions" are

$$
\begin{aligned}
M^{(1)} &= -\frac{5}{2}\frac{1}{\epsilon^2} + \frac{5\pi^2}{8} + \frac{179\zeta(3)}{24}\epsilon + \frac{97\pi^4}{1440}\epsilon^2 - \left(\frac{51\pi^2\zeta(3)}{32} - \frac{137\zeta(5)}{8}\right)\epsilon^3 - \cdots \\
M^{(2)} &= \frac{25}{8}\frac{1}{\epsilon^4} - \frac{35\pi^2}{24}\frac{1}{\epsilon^2} - \frac{865\zeta(3)}{48}\frac{1}{\epsilon} - \frac{97\pi^4}{1152} + \cdots \\
M^{(3)} &= -\frac{125}{48}\frac{1}{\epsilon^6} + \frac{325\pi^2}{192}\frac{1}{\epsilon^4} + \frac{4175\zeta(3)}{192}\frac{1}{\epsilon^3} + \frac{499\pi^4}{10368}\frac{1}{\epsilon^2} + \cdots
\end{aligned}
$$

These obstructions satisfy the expected BDS relation

$$
M^{(3)}(\epsilon) = -\frac{1}{3}(M^{(1)}(\epsilon))^3 + M^{(1)}(\epsilon)M^{(2)}(\epsilon) + f^{(3)}M^{(1)}(3\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)
$$

with

$$
f^{(3)} = \frac{11\pi^4}{180} + \left(\frac{5\pi^2\zeta(3)}{6} + 6\zeta(5)\right)\epsilon + a\epsilon^2, \qquad C^{(3)} = b.
$$

where we found $a = 85.263$, $b = 17.8241$.

On $\mathcal{O}(\epsilon)$ Terms

It is well-known that any one-loop amplitude can be expressed as a linear combination of scalar box integrals.

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The problem is dimensional regularization: if the integrals were well-defined in precisely four dimensions, then indeed it would seem that quadruple cuts could be used to express everything in terms of box integrals.

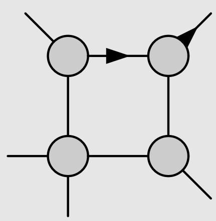
However in $D = 4-2\epsilon$ there are additional contributions, beginning at $\mathcal{O}(\epsilon)$, which cannot be expressed in terms of box integrals (for $n > 4$ particles).

For example, the 1-loop 5-point amplitude (to all orders in ϵ) can be expressed as a linear combination of box integrals and the massless pentagon integral.

On $\mathcal{O}(\epsilon)$ Terms

Although the BCF quadruple cut method misses the pentagon, the full exact result was obtained by Cachazo (2008) using leading singularities!

For this amplitude there are 10 isolated points in $\mathbb{C}^4$ where the integrand has an order-4 pole. Diagramatically:



and four cyclic permutations

This diagram represents the set

$$S = \{\ell \in \mathbb{C}^4 : \ell^2 = 0, (\ell - k_1)^2 = 0, (\ell - k_1 - k_2)^2 = 0, (\ell + k_5)^2 = 0\}$$

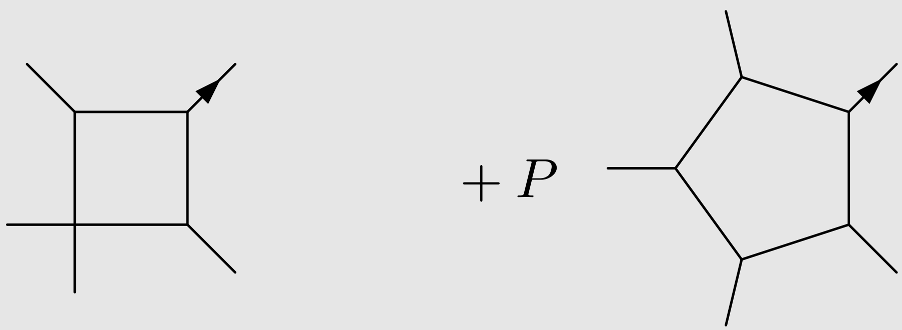
which consists of two distinct points $\{\ell^{(1)}, \ell^{(2)}\}$ (for generic external momenta).

The residue of the amplitude at any singularity is obtained by multiplying tree amplitudes, summed over all allowed internal helicities:

$$\sum_h A^{\rm tree} A^{\rm tree} A^{\rm tree} A^{\rm tree} \Big|_{l^{(1)}} = 0,$$

$$\sum_h A^{\rm tree} A^{\rm tree} A^{\rm tree} A^{\rm tree} \Big|_{l^{(2)}} = A_5^{\rm tree}$$

By comparing to the ansatz

B  $+\, P$

for some coefficients B and P, we find 2 equations

$$B + \frac{P}{(\ell^{(1)} + k_5 + k_4)^2} = 0, \qquad B + \frac{P}{(\ell^{(2)} + k_5 + k_4)^2} = A_5^{\text{tree}}$$

which determine the two coefficients. [Cachazo (2008)]

On $\mathcal{O}(\epsilon)$ Terms

I am inclined to believe that the leading singularity method may be enough to provide correct, all-orders-in-$\mathcal{O}(\epsilon)$ expressions for $n = 4, 5$ particle amplitudes, but starting at $n = 6$ there are terms that it misses for sure.

The terms that is misses are those that are $\mathcal{O}(\epsilon)$ in the integrand — not necessarily those that are $\mathcal{O}(\epsilon)$ after performing the integral.

These so-called "μ-terms" vanish whenever the loop momenta live in $D = 4$, i.e. they are proportional to the -2ϵ-dimensional components of ℓ. After integration, such terms can be finite, can vanish, or can even be IR divergent!

The leading singularity method in $D = 4$ will never see these terms. However (1) it is possible that they could be extracted by considering higher integer dimension $5, 6, \ldots$, and (2) in all known cases they apparently drop out of the log of the amplitude (through $\mathcal{O}(1)$) [BDKRSVV (2008)] .

Conclusion

The motivation for the work I have described has been:

- To check the ABDK/BDS conjecture in the first place it could have failed (once you accept dual conformal symmetry): at 2-loops for 6-particles.
- Although we found a discrepancy with the ABDK/BDS conjecture, happily a possibly even more exciting result turned out to be true: equality of the amplitude with the hexagon Wilson loop, despite no apparent symmetry requiring this equality.
- We have tested the applicability (and found some current limitations of) the leading singularity method.