No-Triangle Hypothesis and Maximal Supergravity

Workshop on Hidden Structures in Field Theory Amplitudes

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Includes work in collaboration with

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Twistor space / New insights

Amplitudes N=4,
N=1, QCD
at NLO, Gravity..

Trees

Twistors

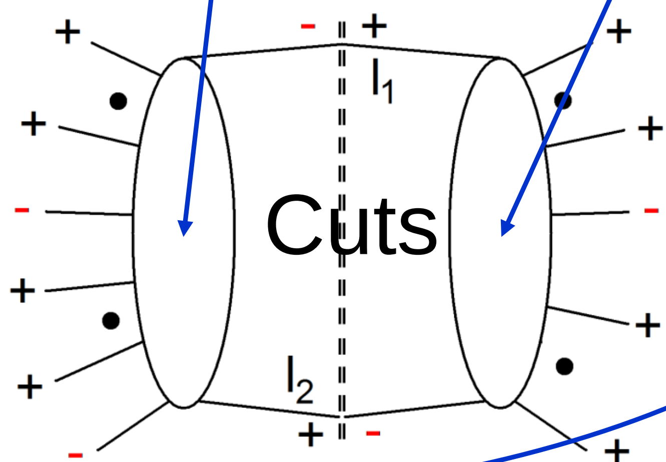
(Witten)

Hidden Beauty!

Loop amplitudes

Simple expressions
for amplitudes

Unitarity



Quantum theory for gravity

- Gravity as a theory of point-like interactions
- Non-renormalisable theory!

Dimensionful

$G_N=1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove higher derivative divergences..

String theory can by introducing new length scale

- Focus: N=8 supergravity – maximal supersymmetry

(Cremmer,Julia, Scherk; Cremmer, Julia)

  – Also cancellations in pure gravity as well..

Calculation of perturbative amplitudes

\# Feynman diagrams:

Factorial Growth!

Momentum vectors :

$(p_i \cdot p_j)$

Generic Feynman amplitudes

Sum over topological different diagrams

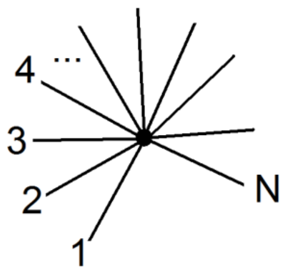
External polarisation tensors :

$(p_i \cdot \varepsilon_j)\ (\varepsilon_i \cdot \varepsilon_j)$

Amplitudes

Specifying external polarisation tensors $(\varepsilon_i , \varepsilon_j)$

Colour ordering



$Tr(T_1 T_2 .. T_n)$

Simplifications

Recursion

Spinor-helicity formalism

Loop amplitudes (Unitarity, Supersymmetric decomposition)

Gravity Amplitudes

Expand Einstein-Hilbert Lagrangian :

Infinitely many vertices

$$\mathcal{L}_{\mathrm{E}H} = \int d^4x \left[\sqrt{-g}R\right]$$

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Vertices: 3pt, 4pt, 5pt,..n-pt

Feynman diagrams: not attractive...!

Complicated expressions for vertices!

45 terms + sym (Sannan)

$$\begin{aligned} V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\ & + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\ & - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\ & + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big], \end{aligned}$$

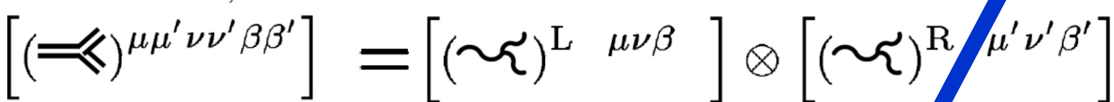
Gravity Amplitudes

KLT relationship (Kawai, Lewellen and Tye)

The KLT relationship relates open and closed strings

Not manifest crossing symmetry

$$A^M_{\text{closed}} \sim \sum_{\Pi,\tilde{\Pi}} e^{i\pi\Phi(\Pi,\tilde{\Pi})} A_M^{\text{left open}}(\Pi) A_M^{\text{right open}}(\tilde{\Pi})$$



(Bern, Carrasco, Johansson)

Better understanding of KLT / organisation of amplitudes

KLT not manifestly crossing symmetric – explicit representation :

$$M_3^{\text{tree}}(1,2,3) = -iA_3^{\text{tree}}(1,2,3)A_3^{\text{tree}}(1,2,3)\,,$$

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)A_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)A_5^{\text{tree}}(3,1,4,2,5)$$

KLT not the simplest form but better than Feynman diagrams

Momentum prefactors cancel double poles

Simplicity of YM amplitudes!!

Helicity states formalism

Spinor products :

$$\langle i\ j\rangle = \epsilon^{mn}\lambda^i_m\lambda^j_n \qquad [i\ j] = \epsilon^{\dot m\dot n}\tilde\lambda^i_{\dot m}\tilde\lambda^j_{\dot n}$$

Different representations of the Lorentz group

$$p_{a\dot a} = \sigma^\mu_{a\dot a}p_\mu$$

$$p^\mu p_\mu = 0 \qquad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a}$$

Momentum parts of amplitudes:

$$q_{a\dot a} = \mu_a\tilde\mu_{\dot a} \quad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a} \quad 2(p\cdot q) = s_{ij} = -\langle\lambda,\mu\rangle[\tilde\lambda,\tilde\mu]$$

Spin-2 polarisation tensors in terms of helicities, (squares of those of YM):

$$\varepsilon^-_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda,\tilde\mu]} \qquad \tilde\varepsilon^+_{a\dot a} = \frac{\mu_a\tilde\lambda_{\dot a}}{\langle\mu,\lambda\rangle} \qquad \begin{matrix}\varepsilon^- & \varepsilon^- \\ \tilde\varepsilon^+ & \tilde\varepsilon^+\end{matrix}$$

(Xu, Zhang, Chang)

Yang-Mills MHV-amplitudes

Tree amplitudes

(n) same helicities vanishes

$A^{tree}(1^+,2^+,3^+,4^+,..) = 0$

First non-trivial example,

(M)aximally

(H)elicity (V)iolating

(MHV) amplitudes

(n-1) same helicities vanishes

$A^{tree}(1^+,2^+,..,j^-,..) = 0$

(n-2) same helicities:

$A^{tree}(1^+,2^+,..,j^-,..,k^-,..) \neq 0$

One single term!!

$A^{tree\ MHV}$ Given by the formula (Parke and Taylor) and proven by (Berends and Giele)

$$i \frac{\langle j\, k \rangle^4}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Gravity MHV amplitudes

- Can be generated from KLT via YM MHV amplitudes.

$$M_4^{\rm tree}(1^-,2^-,3^+,4^+) = i\,\langle 1\,2\rangle^8\,\frac{[1\,2]}{\langle 3\,4\rangle\;N(4)}$$

$$M_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+) = i\,\langle 1\,2\rangle^8\,\frac{\varepsilon(1,2,3,4)}{N(5)}$$

Anti holomorphic Contributions – feature in gravity

$$\varepsilon(i,j,m,n) \equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\,j]\,\langle j\,m\rangle\,[m\,n]\,\langle n\,i\rangle - \langle i\,j\rangle\,[j\,m]\,\langle m\,n\rangle\,[n\,i]$$

- (Berends-Giele-Kuijf) recursion formula

$$\begin{aligned} &M_n^{\rm tree}(1^-,2^-,3^+,\cdots,n^+) \\ &\quad = -i\,\langle 1\,2\rangle^8 \times \left[\frac{[1\,2]\,[n-2\,n-1]}{\langle 1\,n-1\rangle\,N(n)}\Big(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\Big)\prod_{l=3}^{n-3}(-[n|K_{l+1,n-1}|l\rangle) \right. \\ &\qquad\qquad \left. + \mathcal{P}(2,3,\cdots,n-2)\right] \end{aligned}$$

Simplifications from Spinor-Helicity

$$s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

Huge simplifications

$$V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[-\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma})$$
$$+ \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma})$$
$$- P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta})$$
$$+ 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big],$$

45 terms
+ sym

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

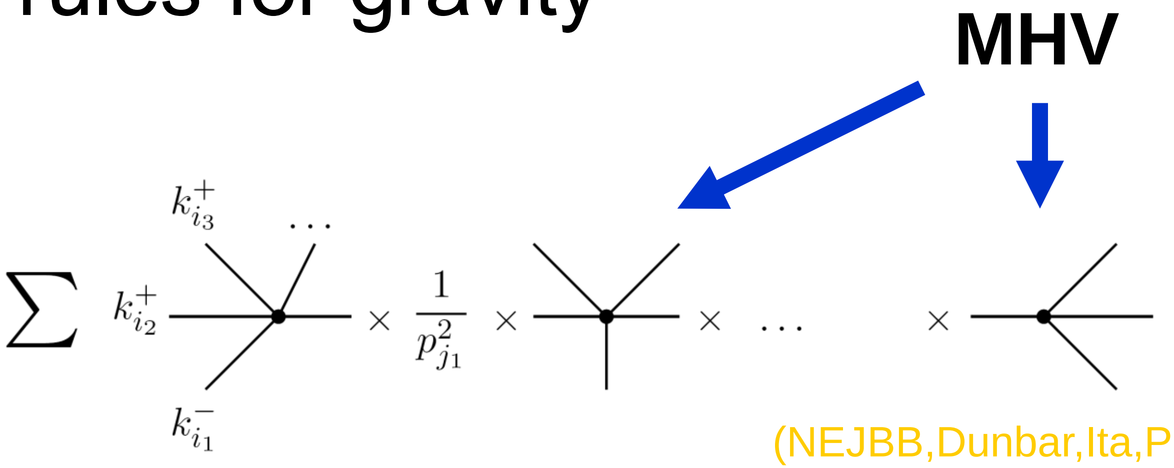
Gravity:

$$\varepsilon^- \ \varepsilon^-$$
$$\tilde{\varepsilon}^+ \ \tilde{\varepsilon}^+$$

$$A_3(1^-, 2^-, 3^+) = -i\frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

Gravity tree properties

MHV rules for gravity



(NEJBB,Dunbar,Ita,Perkins,Risager; Bianchi, Elvang, Freedman; Mason, Skinner; Boels, Larsen, Obers, Vonk)

Recursion

(Bedford, Brandhuber,Spence, Travaglini; Cachazo, Svrtec; NEJBB, Dunbar, Ita; Arkani-Hamed, Kaplan; Hall; Cheung, Arkani-Hamed, Cachazo, Kaplan)

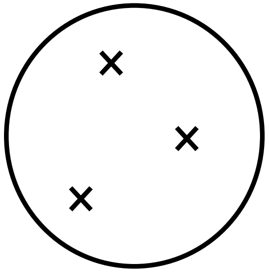
$$\tilde{\lambda}_a \longrightarrow \tilde{\lambda}_a + z\tilde{\lambda}_b$$

$$\lambda_b \longrightarrow \lambda_b - z\lambda_a$$

$$A(0) = -\sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

Gravity scaling behaviour: Unexpected!!

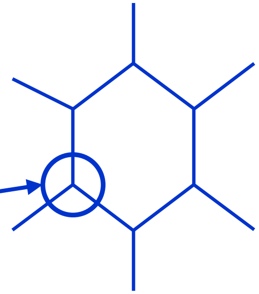
$$A(z) \sim 1/z^2$$



General 1-loop amplitudes

n-pt amplitude

Vertices carry factors of loop momentum



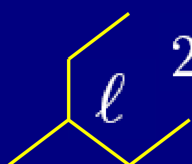
$$\int d^4\ell \frac{F^p(\ell,k,\epsilon)}{\Pi_i p_i^2}$$

p = 2n for gravity

p=n for YM

Propagators

(Passarino-Veltman) reduction



$$2(k\cdot\ell) = (k-\ell)^2 - \ell^2$$

Collapse of a propagator

$$I_r[P^m(l)] \longrightarrow \sum_i I^i_{r-1}[P^{m-1}(l)]$$

$$I^i_4[P^{m'}(l)] \longrightarrow c_i\, I^i_4[1] + \sum_j I^j_3[P^{m'-1}(l)]$$

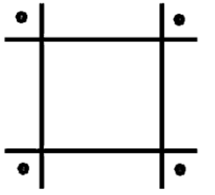
$$M^{1-\text{loop}} = \sum_a c_a I_4^a + \sum_a d_a I_3^a + \sum_a e_a I_2^a + R$$

No-Triangle Hypothesis

Justified suggestion…….

Factorisation suggests this is true for all one-loop amplitudes

$$M^{1\text{-loop}}_{\mathcal{N}=8} = \sum_{i \in \mathcal{C}} c_i \, I_4^i$$



Consequence: N=8 supergravity same one-loop structure as N=4 SYM

Evidence?

Direct evaluation of cuts

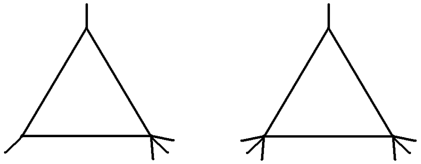
True for 4pt	(Green,Schwarz,Brink)
n-point MHV	(Bern,Dixon,Perelstein,Rozowsky)
6pt NMHV (IR)	(Bern, NEJBB, Dunbar,Ita)
6pt Proof 7pt evidence	(NEJBB, Dunbar,Ita, Perkins, Risager; Bern, Carrasco, Forde, Ita, Johansson)
n-pt proof	(NEJBB, Vanhove; Arkani-Hamed, Cachazo, Kaplan)

No-Triangle Hypothesis by Cuts

Attack different parts of amplitudes 1) .. 2) .. 3) ..

(1) Look at soft divergences (IR)

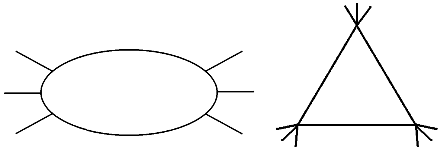
$\rightarrow$ 1m and 2m triangles



Check that boxes gives the correct IR divergences

(2) Explicit unitary cuts

$\rightarrow$ bubble and 3m triangles



(3) Factorisation

$\rightarrow$ rational terms.

(NEJBB, Dunbar,Ita, Perkins, Risager; Arkani-Hamed, Cachazo, Kaplan)

No-triangle hypothesis

(NEJBB, Vanhove)

Generic loop amplitude

$$\int \frac{d^4\bar{\ell}}{(2\pi)^4} \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}}$$

$$M_{n;1} = \mu^{2\epsilon} \int \frac{d^D\ell}{(2\pi)^D} \frac{\prod_j^{2n}(q_{\mu_j}^{(2n,j)}\ell^{\mu_j}) + \prod_j^{2n-1}(q_{\mu_j}^{(2n-1,j)}\ell^{\mu_j}) + \cdots + K}{\ell_1^2 \cdots \ell_n^2}$$

Passarino-Veltman

$$= \sum_a c_a I_4^a + \sum_a d_a I_3^a + \sum_a e_a I_2^a + R$$

Naïve counting!!

$$\int \frac{d^D\ell}{\pi^{\frac{D}{2}}} \prod_{i=1}^n \frac{1}{(\ell - k_1 - \cdots - k_i)^2}$$

→ Tensor integrals derivatives in Q_n

$$\Gamma\left(\frac{D-1}{2}\right) \int_0^\infty \frac{dT}{T} T^{-D/2+n} \int_0^1 d\nu_1 \cdots d\nu_n \frac{1}{n} \left[\sum_{i=1}^n \delta(\nu_i = 1)\right] e^{-T\,Q_n}$$

No-triangle hypothesis

String based formalism natural basis of integrals is

$$G_B(x) = x^2 - |x|\,, \qquad G_F(x) = \text{sign}(x) \qquad Q_n \equiv \sum_{1<i<j<n} (k_i \cdot k_j)\, G_B(\nu_i - \nu_j)$$

$$\boxed{\mathcal{I}_n[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s] \equiv \int_0^1 d^{n-1}\nu\, Q_n^{D/2-n} \prod_{i\in \underline{\mathrm{I}}_r} \partial_{\nu_i} Q_n \prod_{x\in \underline{\mathrm{J}}_s} G_F(x)}$$

$$h_i = \sum_{j=1}^{n-1} c_i^{\,j}\, k_j + q^{\perp}$$

Amplitude takes the form $P(\varepsilon_{ij}, k_i, \nu_i) = P(H_i \cdot K_{[n]}, Y_{ij}\, G_F(\nu_i - \nu_j), (h_i \cdot h_j)\, \delta(\nu_i - \nu_j))$

$$\mathcal{M}_{n;1}^{\mathcal{N}} = \Gamma(n - D/2) \times \int_0^1 d^{n-1}\nu P(\varepsilon_{ij}, k_i, \nu_i)\, Q_n^{D/2-n}$$

Constraint from SUSY

$$\mathcal{M}_{n;1}^{\mathcal{N}} = \sum_{\substack{r+s+u=2n-\mathcal{N} \\ 0\leq u\leq n}} \sum_{l=0}^{u} t_{r,s}^{l}\, \mathcal{I}_{n-l}^{[D+2(u-l)]}[\underline{\mathrm{I}}_r, \underline{\mathrm{J}}_s] \qquad r + s \leq 2n - \mathcal{N}$$

No-triangle hypothesis

Now if we look at integrals

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}] = \int_0^1 d^{n-1}\nu\, Q_n^{D/2-n} \prod_{i\in\underline{\mathrm{I}}_{r+1}} \partial_{\nu_i} Q_n$$

Typical expressions

$$\mathcal{I}_n[\underline{\mathrm{I}}_{r+1}, \underline{\mathrm{J}}_1] = \frac{1}{D/2-n+1}\int_0^1 d^{n-1}\nu\, \partial_{\nu_{i_{r+1}}} Q_n^{D/2-n+1}\, G_F(\nu_{i_{r+1}} - \nu_j) \prod_{i\in\underline{\mathrm{I}}_r} \partial_{\nu_i} Q_n$$

Use

$$Q_n^{D/2-n}\partial_\nu Q_n = (D/2-n+1)^{-1}\partial_\nu Q_n^{D/2-n+1}$$

+ integration by parts

$$\mathcal{I}_n[(\partial Q_n)^r] \rightsquigarrow \mathcal{I}_{n-1}^{\mathrm{mass}}[(\partial Q_n)^{r-2}] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-2}]$$

$$\mathcal{I}_n[(\partial Q_n)^r, G_F] \rightsquigarrow \mathcal{I}_{n-1}^{\mathrm{mass}}[(\partial Q_n)^{r-1}] + \mathcal{I}_{n-1}^{\mathrm{mass}}[(\partial Q_n)^{r-2}, G_F] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-2}, G_F]$$

No-triangle hypothesis

N=8 Maximal Supergravity (r = 2 (n – 4), s = 0) (NEJBB, Vanhove)

$$\mathcal{I}_n[(\partial Q_n)^{2(n-4)}] \rightsquigarrow \mathcal{I}_{n-1}^{\text{mass}}[(\partial Q_n)^{2(n-5)}] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{2(n-5)}]$$

$$\rightsquigarrow \cdots \rightsquigarrow \mathcal{I}_4^{\text{mass}}[\emptyset] + \sum_{m=1}^{n-4} \mathcal{I}_{4+m}^{[D+2m]}[\emptyset] \,.$$

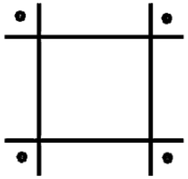
(r = 2 (n – 4) - s, s >0)

$$\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] \rightsquigarrow \mathcal{I}_{n-s}^{\text{mass}}[(\partial Q_n)^{r-s}] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-s}]$$

$$\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] \rightsquigarrow \mathcal{I}_{n-s}^{\text{mass}}[(\partial Q_n)^{r-s}] + \mathcal{I}_n^{[D+2]}[(\partial Q_n)^{r-s}] \rightsquigarrow \cdots$$

$$\rightsquigarrow \mathcal{I}_{n-(r+s)/2}^{\text{mass}}[\emptyset] + \sum_{m=1}^{n-4} \mathcal{I}_{4+m}^{[D+2m]}[\emptyset] \,.$$

$$M_{\mathcal{N}=8}^{\text{1-loop}} = \sum_{i \in \mathcal{C}} c_i \, I_4^i$$



Higher dimensional contributions – vanish by amplitude gauge invariance

Proof of No-triangle hypothesis

No-triangle hypothesis

Generic gravity theories:

N $\leq$ 3 theories constructable from cuts

$$\mathcal{I}_n[(\partial Q_n)^r, (G_F)^s] \to \mathcal{I}^{\text{mass}}_{\mathcal{N}/2}[\emptyset]$$

- Prediction N=4 SUGRA

$$M^{1-\text{loop}} = \sum_a c_a I_4^a + \sum_a d_a I_3^a + \sum_a e_a I_2^a + \cancel{R}$$

- Prediction pure gravity

$$M^{1-\text{loop}} = \sum_a c_a I_4^a + \sum_a d_a I_3^a + \sum_a e_a I_2^a + R$$

No-triangle for multiloops

- No-triangle hypothesis 1-loop
  - Consequences for powercounting arguments above one-loop..

$D < 10/L + 2$

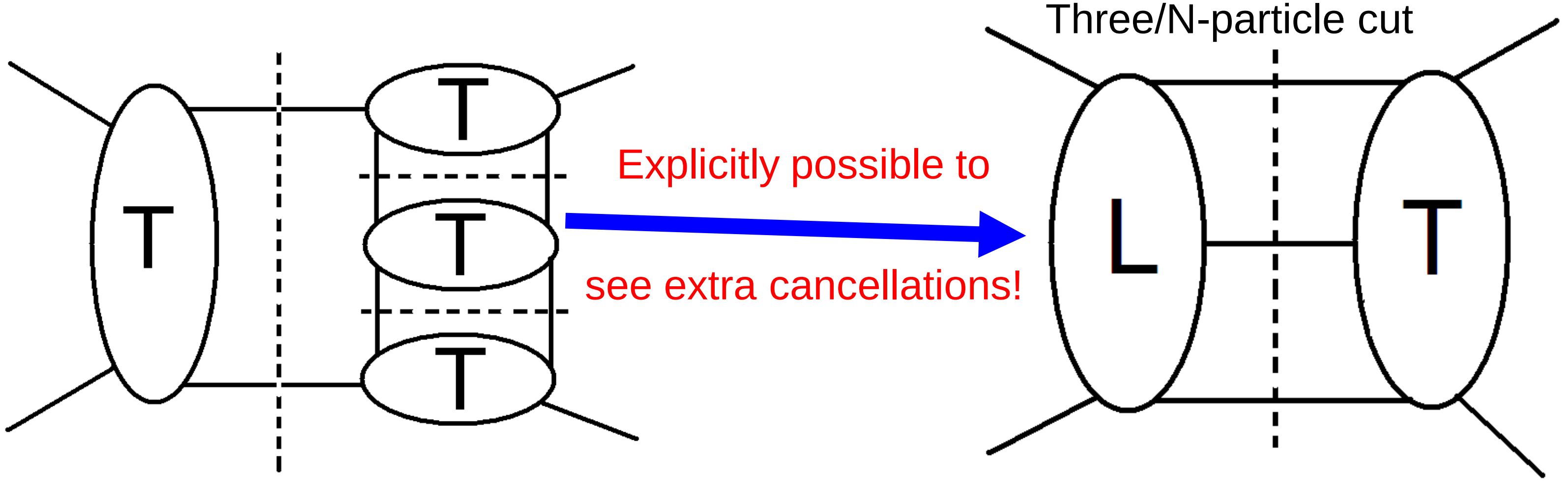
Two-particle cut might miss certain cancellations

Possible to obtain YM bound??

$D < 6/L + 4$ for gravity???

Bound might be too conservative!!

Iterated two-particle cut



(Bern, Dixon, Perelstein, Rozowsky; Bern, Dixon, Roiban)

Conclusions

- Graviton amplitudes $\leftrightarrow$ much benefit from recent progress

(..twistor / helicity structure, hidden simplicity, string based formalism..)

- Gravity $\leftrightarrow$ much simpler – than Lagrangian / power counting

indicate (no-triangle property $\leftrightarrow$ extra simplicity..)

- Unordered amplitudes might be even simpler than

ordered amplitudes (due to lack of boundary terms..)

Consequences at higher loop order

Finiteness??!

- String based / helicity formalism is very helpful

  – however better ways to deal with gravity amplitudes still important to focus on..