Gravity, Twistors & the MHV Formalism

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Based on arXiv:0808.3907 [hep-th] with L. Mason

Hidden Structures in Field Theory Amplitudes

12^{th} September 2008

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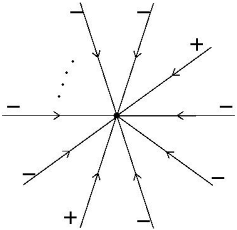

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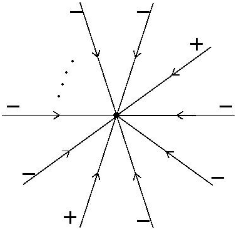

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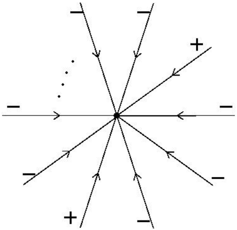

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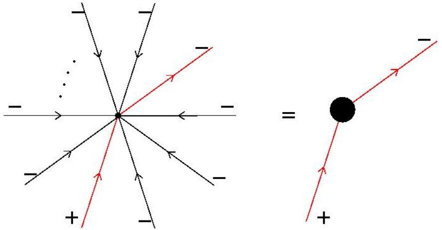

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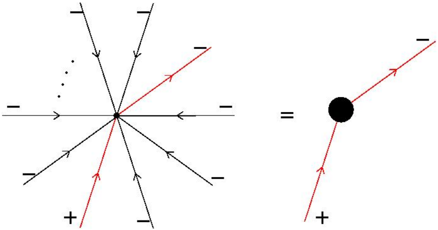

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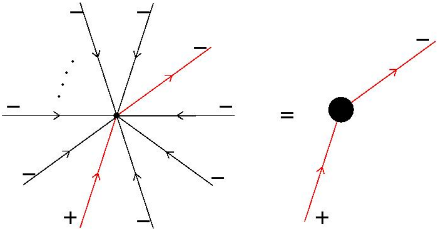

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- Today: focus on gravity

MHV Amplitudes & ASD Spacetimes

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- Can be proved more rigorously using geometric/canonical quantization

Relation to Usual Form of Amplitudes

Claim:

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where BGK amplitude $= \mathrm{i}\kappa^{n-2}/\hbar \times \delta\left(\sum p_i\right)\mathcal{M}_n^{\mathrm{BGK}}$ and

$$\mathcal{M}_n^{\mathrm{BGK}} = [1n]^8 \left\{ \frac{\langle 12\rangle\langle n-2\; n-1\rangle}{[1\; n-1]} \frac{F}{N(n)} \prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}[ij] + \mathrm{P}_{(2,\dots,n-2)} \right\}$$

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(Integral is gauge invariant because of field eqns)

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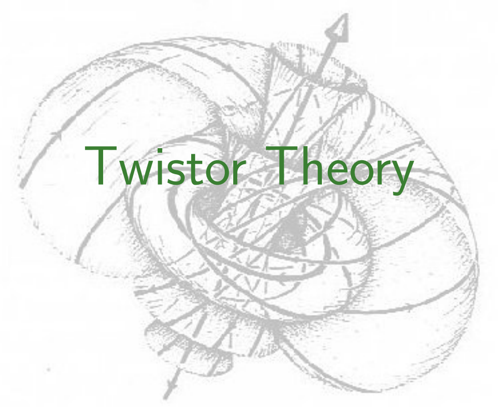
Twistor space is particularly well-suited to exploiting this integrability (integrability in spacetime = holomorphy in twistor space)

Twistor Theory



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- $\epsilon^{\alpha\beta}\partial_\alpha B_\beta = \tilde{h}_{-6}$ ensures $\mathrm{d}\gamma^{\dot\alpha}{}_{\dot\beta} = (\delta R)^{\dot\alpha}{}_{\dot\beta\dot\gamma\dot\delta}\, \mathrm{d}x^{\gamma\dot\gamma} \wedge \mathrm{d}x_\gamma{}^{\dot\delta}$

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- Unknown what $\tilde{h}_{-6}$ deforms $\Rightarrow$ only get ASD spacetime

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- $I(\, , \,) := \epsilon^{\alpha\beta} \frac{\partial}{\partial \omega^\alpha} \wedge \frac{\partial}{\partial \omega^\beta}$

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$$\int_M \Sigma_0 \wedge \gamma \wedge \gamma = \int_{M\times\mathbb{CP}^1\times\mathbb{CP}^1} \mathrm{d}^4x[\pi_n\,\mathrm{d}\pi_n][\pi_1\,\mathrm{d}\pi_1]\; I(B_n,\,\cdot\,)\lrcorner\frac{1}{\bar\partial+\mathcal{L}_V}B_1[\pi_n\,\pi_1]^4$$

Expanding in powers of V gives

$$\sum_{n=3}^{\infty}\int \mathrm{d}^4x\,[\pi_n\,\mathrm{d}\pi_n]\,I(B_n,\,\cdot\,)\lrcorner\left(\frac{1}{\bar\partial}\mathcal{L}_{V_{n-1}}\frac{1}{\bar\partial}\mathcal{L}_{V_{n-2}}\cdots\frac{1}{\bar\partial}\mathcal{L}_{V_3}\frac{1}{\bar\partial}\mathcal{L}_{V_2}\frac{1}{\bar\partial}B_1\,[\pi_n\,\pi_1]^4\right)$$

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- Derivative of δ-fn support $\to$ perturbative description of support on deformed twistor lines

Odds and Ends

ASD background $\Leftrightarrow$ Integrable background complex structure

Twistor Action for MHV Gravity

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$$S = \int_{\mathbb{PT}'} \Omega \wedge B^\alpha \partial_\alpha \left(\bar{\partial} h + \frac{1}{2}\{h, h\} \right)$$

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The spacetime Plebanski action $\int_M \Sigma \wedge \mathrm{d}\Gamma + \int_M \Sigma \wedge \Gamma \wedge \Gamma$ suggests we consider the twistor action

$$S = \int_{\mathbb{PT}'} \Omega \wedge B^\alpha \partial_\alpha \left(\bar{\partial} h + \frac{1}{2}\{h, h\} \right) + \int_{M \times \mathbb{CP}^1 \times \mathbb{CP}^1} \mathrm{d}^4 x \, [\pi_n \, \mathrm{d}\pi_n][\pi_1 \, \mathrm{d}\pi_1] \; I(B_n, \cdot\,) \lrcorner \frac{1}{\bar{\partial} + \mathcal{L}_V} B_1 [\pi_n \, \pi_1]^4$$

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- Can we find a 'connected prescription' for Einstein gravity?