Operator Growth and Complexity in Krylov space

(What We Talk About When We Talk About "Complexity")

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Outline

- Introduction/Motivation
- Krylov basis, operator growth and “Complexity” of operators and states
- Examples
- Conclusions/Questions

Based on:

“Quantum chaos and the complexity of spread of states” with V. Balasubramanian, J.M. Magan, Q. Wu, Phys. Rev. D. 106 (2022) 4, 046007

“Geometry of Krylov Complexity” with J.M. Magan, D. Patramanis Phys. Rev. Res. **4**, 013041

w.i.p with D. Patramanis and Sinong Liu.





General Problem

Unitary evolution of operators and states (QM or QFT):

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \qquad\qquad i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle$$

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \qquad\qquad |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

Generically, a “simple” operator $\mathcal{O}(0)$ “grows” and becomes “complex” (in operator space)

Similarly, a “simple” reference quantum state $|\Psi(0)\rangle$ “spreads” and becomes “complex” (in Hilbert space)

How to quantify this “Complexity”?

Motivation/Intuition:

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} = \mathcal{O}(0) + it[H,\mathcal{O}(0)] + \frac{(it)^2}{2}[H,[H,\mathcal{O}(0)]] + ...$$

E.g.

$$H = \sum_i (Z_i \cdot Z_{i+1} + B_x X_i + B_z Z_i) \qquad \mathcal{O}(0) = X_1$$

2

$$\mathcal{O}(t) = X_1 - 2t(Y_1 \cdot Z_2 + B_z Y_1)$$

6

$$-2t^2(B_x Y_1 \cdot Y_2 - B_x B_z Z_1 - B_x Z_1 \cdot Z_2 + 2B_z X_1 \cdot Z_2 + B_z^2 X_1 + X_1 \cdot Z_2^2)$$

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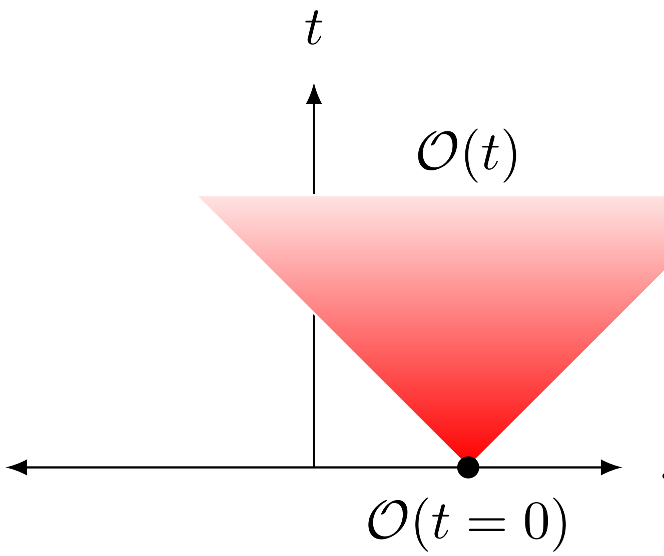
$$+t^3(..........)$$



[S.Datta talk]

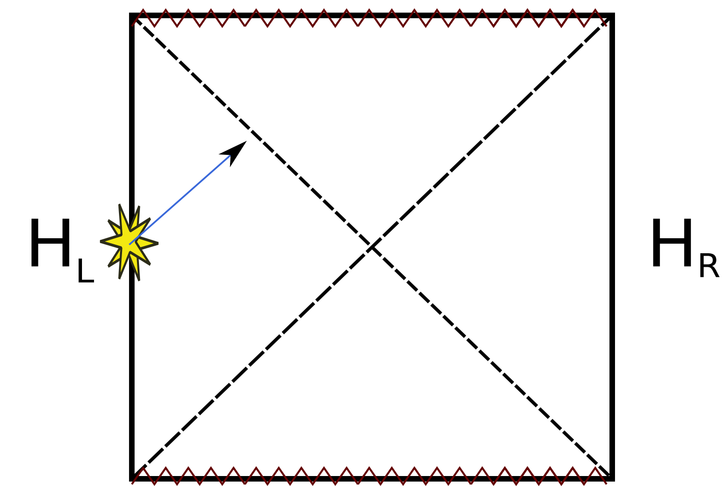
Common lore: the more “chaotic” H, the faster the operator grows.

How to quantify this: A universal definition of the operator size/complexity?

Motivation: Operator Growth and Holography (BH & QChaos)

1. Butterfly effect

$$|\psi'\rangle = e^{-iH_L t_w} O_L(x) e^{iH_L t_w} |\psi\rangle$$



[Shenker,Stanford'13]

[Roberts,Stanford'14]

[+ Susskind]

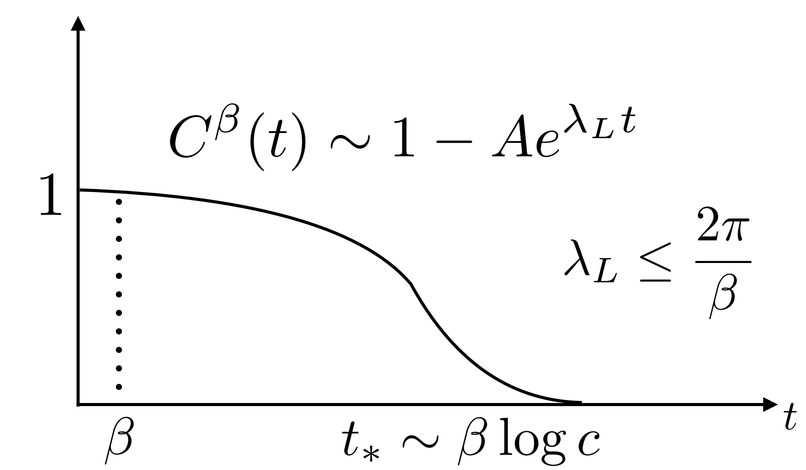
2. Growth of "Precursors": $W(t) = U^\dagger(t) W U(t)$

3. Goals: Universal/working definition of the "Operator Size"? Operator Complexity? Quantum Chaos?

Partial answers" from **O**ut-of-**T**ime **O**rdered **C**orrelators:

$$C^\beta(t) = \frac{\langle W(t) V W(t) V\rangle_\beta}{\langle W(t) W(t)\rangle_\beta \langle V V\rangle_\beta}$$

"Maximal chaos" (OTOC) for Einstein BH dual in the bulk.



[Maldacena,Shenker,Stanford'15]

Related:

Many-body QChaos?, Thermalisation (or lack thereof)

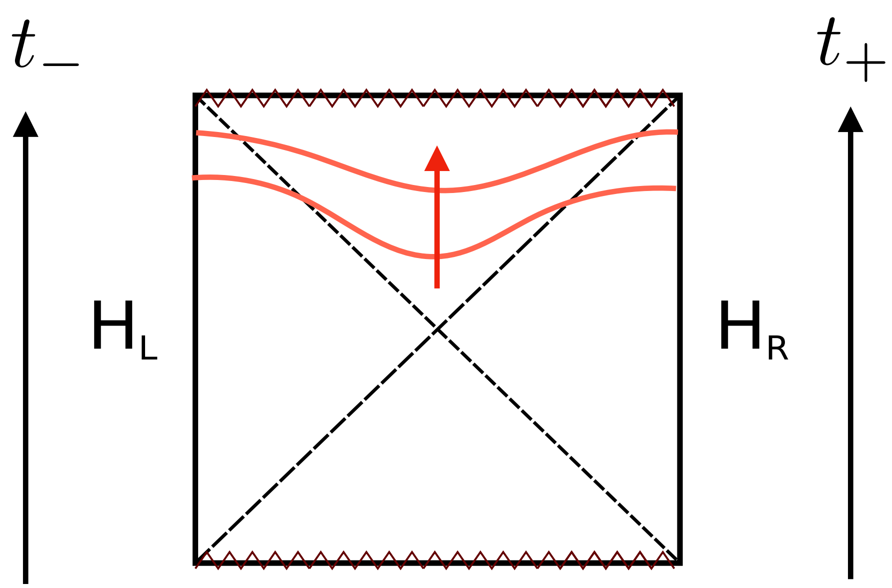
"Central Dogma": Black Hole = Strongly Interacting Qubits

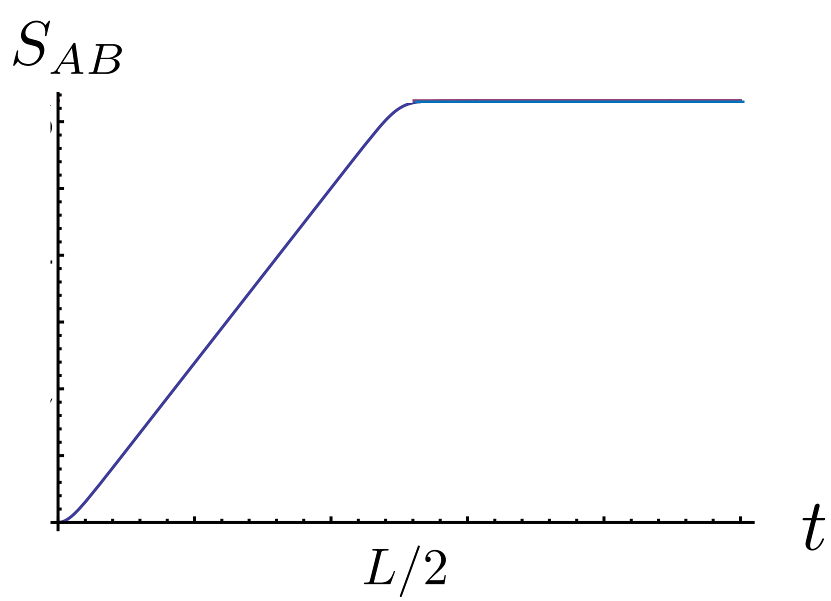
Motivation: Complexity in Holography?

[Hartman&Maldacena '13] (2d CFT)

Time-evolved Thermofield-double state

$$|\Psi_\beta(t)\rangle = e^{-i(H_L+H_R)t}\frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n,n\rangle$$





BH (ERB) continues to grow with t but entanglement entropy saturates ("not enough")

What is the "CFT dual" of this (ERB) growth? "Complexity" of the TFD state? [Susskind,'14]

Is there some useful universal notion of complexity (number)? Unexplored in QFT…

Attempts and Hopes for “Complexity”

States (Formation,Evolution):

Geometric Approaches (“Nielsen”)

AdS/TN (Path Integral Complexity)

“Distance measures” (Inf. metric)

….

Operators (Growth,Chaos)

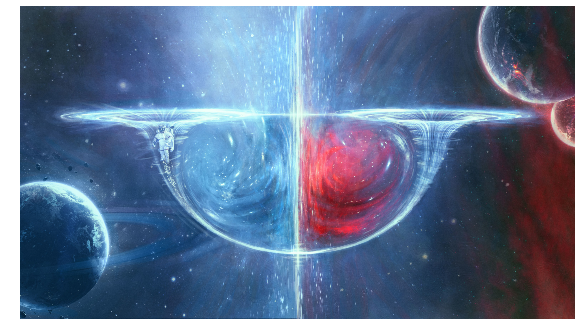
Operator Size/Complexity?

“Operator Size” in SYK

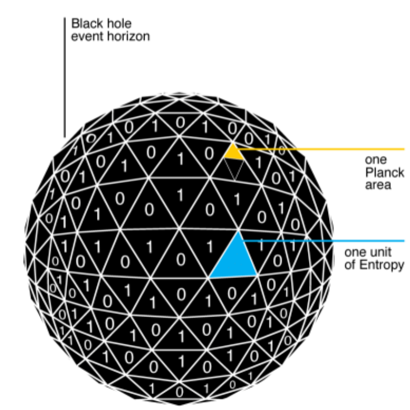
OTOC

….

Growth of the ERB,
late time physics of BH,
singularity?



?

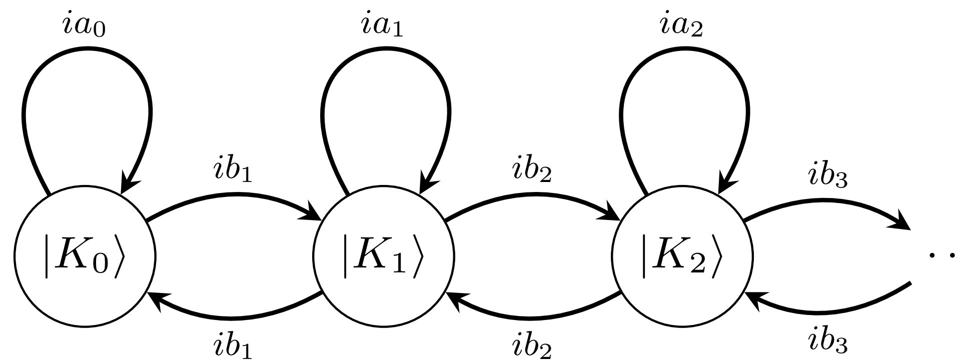


Near (behind?)
horizon of BH….

Goal: Better understanding $\mathcal{H}_{bulk} \simeq \mathcal{H}_{bdr}$

Universal framework for “Complexity”?

[Balasubramanian, PC, Magan, Wu ’22]



Today: discuss a notion(s) of “complexity” based on the so-called Krylov basis

that can be universally defined (and computed) in systems from QM to QFT

and show some recent results for both, operators and states.

References (Operator Growth):

[Qi, Streicher ’18] [Roberts,Stanford,Streicher ’18]

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19] [Barbon, Rabinovici, Shir, Sinha ’19] [Dymarsky, Gorsky ’19]

[Rabinovici, Sanchez-Garrido, Shir, Sonner ’20] [Magan, Simon’20] [Jian, Swingle, Xian ’21]

[Kar, Lamprou, Rozali, Sully ’21] [Dymarsky, Smolkin ’21] [Rabinovici, Sanchez-Garrido, Shir, Sonner ’21’22]…

Basic Idea

Given

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi(0)\rangle \qquad\qquad \mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \equiv e^{i\mathcal{L}t}\mathcal{O}(0)$$

$\mathcal{L} = [H, \cdot]$

More generally we can think about quantum circuits (circuit H and circuit t)

We can expand them in a certain basis (Krylov basis):

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle \qquad\qquad |\mathcal{O}(t)) = e^{i\mathcal{L}t}\,|\mathcal{O}_0) = \sum_n \phi_n(t)\,|\mathcal{O}_n)$$

Unitarity: Probability distribution

$$p_n(t) = |\phi_n(t)|^2 \qquad\qquad \sum_n |\phi_n(t)|^2 = 1$$

Use this probability to characterise the evolution/growth and “complexity”

Krylov Basis

[Recursion Method: Viswanath,Muller '63]

Unitary evolution/Q-circuit

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\Psi_n\rangle$$

Goal: Given states

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

construct an orthonormal basis $|K_n\rangle$ recursively (Lanczos algorithm, G-S):

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \qquad |K_n\rangle = b_n^{-1}|A_n\rangle$$

with "Lanczos coefficients":

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}$$

Such that $b_0 = 0$ and $|K_0\rangle = |\Psi_0\rangle$

Krylov Basis details

[Recursion Method: Viswanath,Muller '63]

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \qquad |K_n\rangle = b_n^{-1}|A_n\rangle$$

Start from normalized $|K_0\rangle = |\psi_0\rangle$

n=0

$$|A_1\rangle = (H - a_0)\,|K_0\rangle \qquad |K_1\rangle = b_1^{-1}(H - a_0)\,|K_0\rangle$$

$$0 = \langle K_0\,|K_1\rangle = b_1^{-1}(\langle K_0|\,H\,|K_0\rangle - a_0) \qquad a_0 = \langle K_0|\,H\,|K_0\rangle$$

$$1 = \langle K_1|K_1\rangle = b_1^{-2}\,\langle A_1|A_1\rangle \qquad b_1 = \langle A_1|A_1\rangle^{1/2}$$

n=1

$$|A_2\rangle = (H - a_1)\,|K_1\rangle - b_1\,|K_0\rangle \qquad |K_2\rangle = b_2^{-1}\,|A_2\rangle$$

$$0 = \langle K_1|K_2\rangle = b_2^{-1}\,[(\langle K_1|H|K_1\rangle - a_1) - b_1\,\langle K_1|K_0\rangle] \qquad a_1 = \langle K_1|H|K_1\rangle$$

$$0 = \langle K_0|K_2\rangle = \langle K_0|H|K_1\rangle - b_1 = \langle K_0|(H - a_0)|K_1\rangle - b_1$$

$$= b_1\,\langle K_1|K_1\rangle - b_1 = 0$$

$$1 = \langle K_2|K_2\rangle = b_2^{-2}\langle A_2|A_2\rangle \qquad b_2 = \langle A_2|A_2\rangle^{1/2}$$

Krylov Basis

[Recursion Method: Viswanath,Muller ’63]

In the Krylov basis, the Hamiltonian becomes tri-diagonal

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle \qquad \langle K_m|\, H\, |K_n\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Expanding our state in the Krylov basis

“Hessenberg form”

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle \qquad \sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

By construction, we have a Schrödinger equation for the coefficients (amplitudes)

$$i\partial_t\,|\Psi(t)\rangle = \sum_n i\partial_t\phi_n(t)\,|K_n\rangle$$

$$i\partial_t\,|\Psi(t)\rangle = H\,|\Psi(t)\rangle = \sum_n \phi_n(t) H\,|K_n\rangle = \sum_n \left[a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)\right]|K_n\rangle$$

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t) \qquad \phi_n(0) = \delta_{n,0}$$

Lanczos coef. from return amplitude

[Recursion Method: Viswanath,Muller '63]

[Balasubramanian, PC, Magan, Wu '22]

Lanczos coeff. are encoded in the "return amplitude" (auto-correlator, Loschmidt amp.)

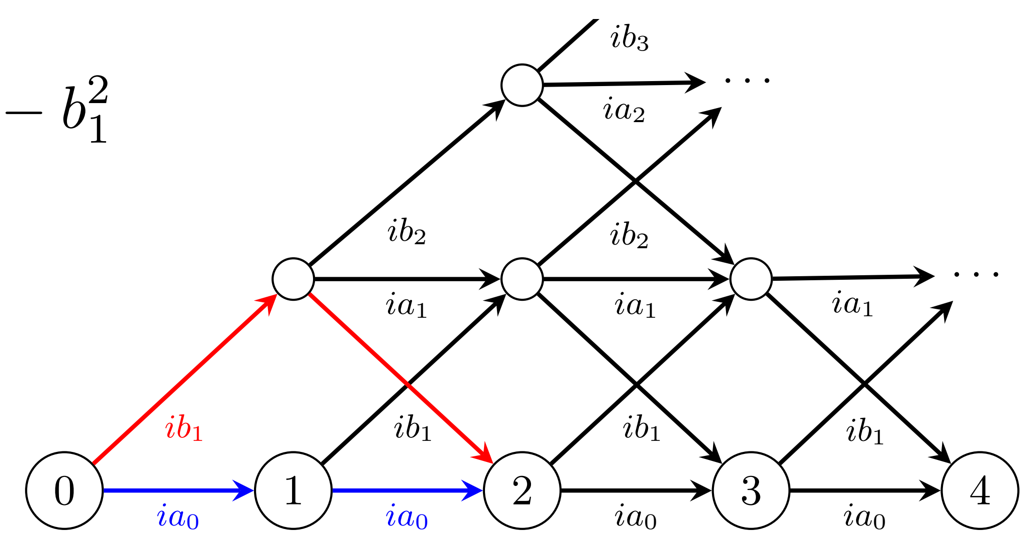
$$\boxed{S(t) \equiv \langle \Psi(t)|\Psi(0)\rangle = \langle \Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)}$$

Moments

$$\mu_n = \frac{d^n}{dt^n} S(t)\Big|_{t=0} = \langle \psi(0)|\frac{d^n}{dt^n}e^{iHt}|\psi(0)\rangle\Big|_{t=0} = \langle K_0|(iH)^n|K_0\rangle$$

Knowing moments allows to find Lanczos coefficients (algorithm)

e.g. $\langle K_0|\,(iH)\,|K_0\rangle = ia_0$ $\qquad$ $\langle K_0|\,(iH)^2\,|K_0\rangle = -a_0^2 - b_1^2$



(Comment) Inverse relations:

$$a_0 = -i\mu_1, \qquad b_1^2 = \mu_1^2 - \mu_2$$

Interesting for modular H: entanglement, capacity of ent.,…

Physics of Lanczos coeff?
[Balasubramanian, Magan, Wu '22]

Operator Growth in the Krylov Basis

[Recursion Method: Viswanath,Muller '63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]

Heisenberg evolution

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \qquad \mathcal{O}(t) = e^{iHt}\, \mathcal{O}(0)\, e^{-iHt}$$

Formally, we can write the operator as

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [H, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [H, [H, \mathcal{O}]], \ldots$$

Liouvillian (super)operator

$$\mathcal{L} = [H, \cdot], \qquad \mathcal{O}(t) \equiv e^{i\mathcal{L}t}\mathcal{O} \qquad \tilde{\mathcal{O}}_n \equiv \mathcal{L}^n \mathcal{O}$$

Given $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \ldots\}$ we need a basis (GNS) $|\mathcal{O})$ $\mathcal{L}\,|\mathcal{O}) = |[H, \mathcal{O}])$

We must pick an inner product (freedom):

$$(A|B)^g_\beta = \int_0^\beta g(\lambda)\, \langle e^{\lambda H} A^\dagger e^{-\lambda H} B \rangle_\beta \, d\lambda \qquad \langle A \rangle_\beta = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} A\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

$$g(\lambda) \geq 0, \quad g(\beta - \lambda) = g(\lambda), \quad \frac{1}{\beta}\int_0^\beta d\lambda g(\lambda) = 1.$$

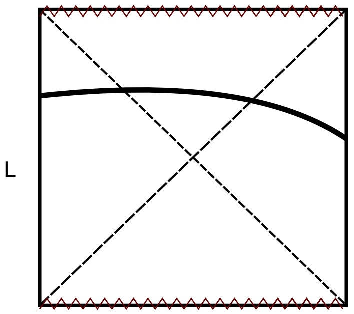
Operator Growth in the Krylov basis

[Recursion Method: Viswanath,Muller ’63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]

The most common: Wightman

$$(A|B) = \langle e^{H\beta/2} A^{\dagger} e^{-H\beta/2} B \rangle_{\beta} \qquad g(\lambda) = \delta(\lambda - \beta/2)$$



Then we follow the Lanczos algorithm

$$|A_{n+1}) = \mathcal{L}\,|\mathcal{O}_n) - b_n\,|\mathcal{O}_{n-1}) \qquad |\mathcal{O}_n) = b_n^{-1}\,|A_n) \qquad b_n = (A_n|A_n)^{1/2}$$

$$|\mathcal{O}_0) = |\mathcal{O}(0)) = |\mathcal{O}) \qquad b_0 = 0$$

Most of the inner products will involve Tr() so we don’t need $a_n = 0$

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

Schrödinger equation:

$$\boxed{\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0}}$$

Lanczos coefficients are encoded in the return amplitude

$$S(t) = (\mathcal{O}_0|\mathcal{O}(t)) = \left(\mathcal{O}_0|e^{i\mathcal{L}t}|\mathcal{O}_0\right) = \varphi_0(t) = \frac{1}{Z}\sum_{n,m} |\langle n|\mathcal{O}|m\rangle|^2 e^{-\left(\frac{\beta}{2}-it\right)E_n} e^{-\left(\frac{\beta}{2}+it\right)E_m}$$

Krylov Basis Summary

States

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$i\partial_t \phi_n(t) = a_n \phi_n(t) + b_n \phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$S(t) \equiv \langle \Psi(t)|\Psi(0)\rangle = \langle \Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

Operators

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

$$\sum_n |\varphi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$S(t) = (\mathcal{O}(0)|\mathcal{O}(t)) = (\mathcal{O}_0|e^{i\mathcal{L}t}|\mathcal{O}_0) = \varphi_0(t)$$

Connections:

$$|\Psi(t)\rangle = \mathcal{O}(-t)\,|\Psi(0)\rangle$$

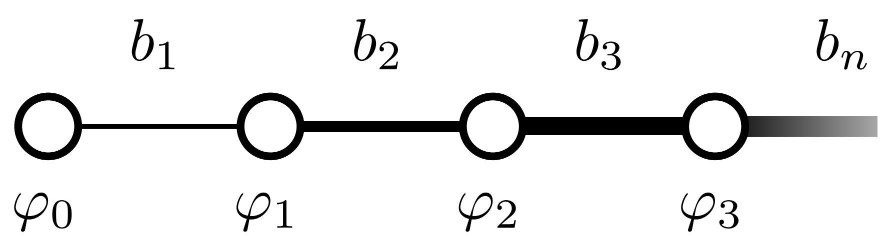
E.g. Wightman $|\psi(t)\rangle = \rho_\beta^{1/4}\mathcal{O}_L(t)\rho_\beta^{-1/4}|\psi_\beta\rangle$

Questions?

Krylov/Spread Complexity

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]
[Balasubramanian, PC, Magan, Wu ’22]

The physics of the growth/evolution <=> motion of a particle on a chain



$$\sum_n |\varphi_n(t)|^2 = 1$$

The further in the chain the particle is, the more “complex” state in the Krylov basis needs to be employed (to represent the state or the operator)

A natural (working) definition of “complexity” as an average position on the chain:

$$\mathcal{C}_\Psi(t) = \sum_n n|\phi_n(t)|^2 \qquad K_\mathcal{O} = \sum_n n|\varphi_n(t)|^2$$

Evolution can be also characterised with other QI/Probability tools:

K-entropy $S_K = -\sum_n p_n \log p_n$ K-variance, K-capacity, $C_K = e^{S_K}$ …

[Barbon, Rabinovici, Shir, Sinha ’19] [PC, Datta ’21] [Patramanis ’21]. ….

Comment

[Balasubramanian, PC, Magan, Wu '22]

Starting from the state: $|\psi(t)\rangle \; = \; e^{-iHt}|\psi(0)\rangle$

Complexity = "Spread in Hilbert space"

Take a basis: $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, \cdots\}$ and a "cost function" (a family, $c_n = n$)

$$C_{\mathcal{B}}(t) = \sum_n c_n |\langle\psi(t)|B_n\rangle|^2 \equiv \sum_n c_n \, p_{\mathcal{B}}(n,t)$$

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t)$$

minimum (finite t) for the Krylov basis!

Intuition: For discrete time evolution, assume n=N-1 vectors equal to the Krylov basis. Then in the next step:

$$|\psi_N\rangle = p_{\perp}|K_N\rangle + p_{||}|\chi_{||}\rangle$$

Ex.1 q-SYK and “Universal Operator Growth Hypothesis”

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]

[Roberts,Stanford, Streicher ’18]

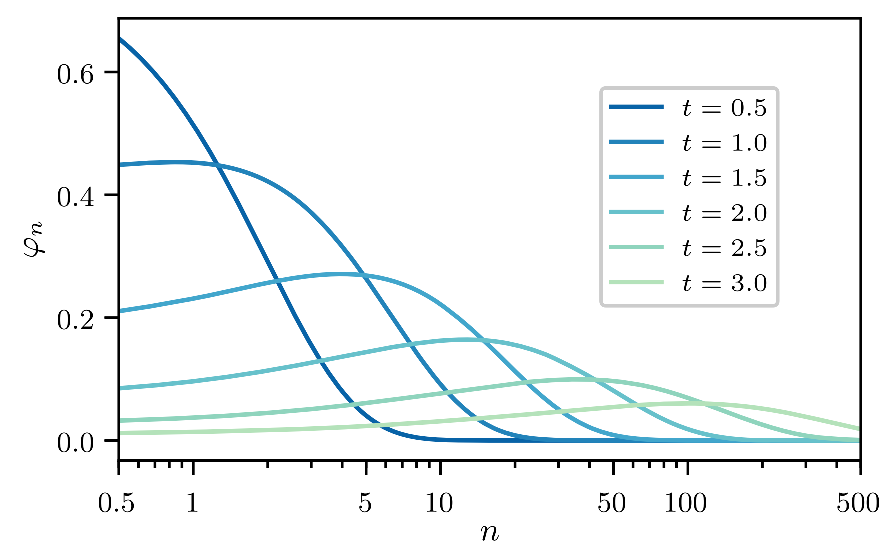
$$H^{(q)}_{\mathrm{SYK}} = i^{q/2} \sum_{1\le i_1<i_2<\cdots<i_q\le N} J_{i_1\ldots i_q}\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_q} \qquad \mathcal{O} = \sqrt{2}\gamma_1$$

At large q this growth is represented by return amplitude (low T, $\eta \sim 2/q$)

$$S(t) = (\mathcal{O}(t)|\mathcal{O}(0)) \simeq \frac{1}{\cosh^\eta\left(\frac{\pi t}{\beta}\right)} \sim e^{-\frac{\eta\pi}{\beta}t}$$

Lanczos coefficients and amplitudes (a_n=0)

$$b_n = \frac{\pi}{\beta}\sqrt{n(\eta+n-1)} \qquad \varphi_n(t) = \sqrt{\frac{\Gamma(\eta+n)}{n!\Gamma(\eta)}}\frac{\tanh^n(\frac{\pi t}{\beta})}{\cosh^\eta(\frac{\pi t}{\beta})}$$



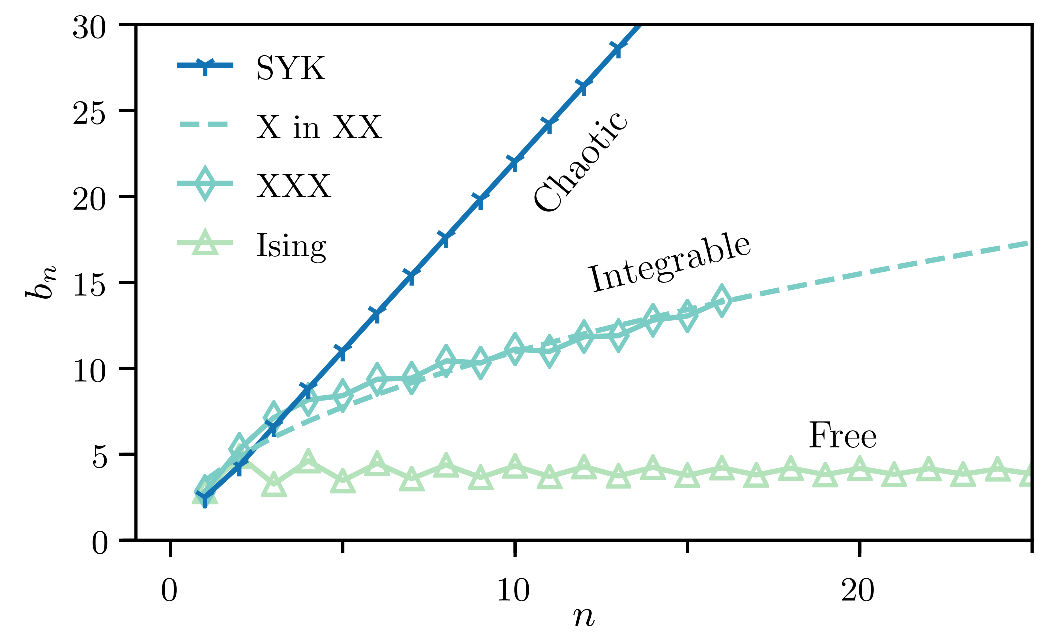
$$K_{\mathcal{O}} = \eta\sinh^2\left(\frac{\pi t}{\beta}\right) \sim \frac{\eta}{4}e^{\frac{2\pi}{\beta}t}$$

Ex.1 q-SYK and “Universal Operator Growth Hypothesis”

[Parker et al. ’19]

Hypothesis: “Maximal growth of Lanczos coefficients”

$$b_n \leq \alpha n + O(1)$$



Claim: Saturated for “maximally chaotic” systems (OTOC)

Saturation <=> exponential growth of Krylov Complexity

$$b_n = \alpha n + O(1) \quad \Rightarrow \quad K_{\mathcal{O}} \sim e^{\lambda t} \quad \lambda = 2\alpha$$

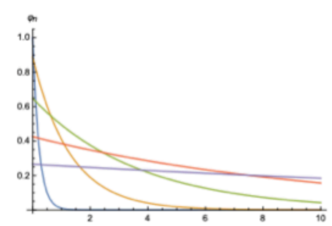
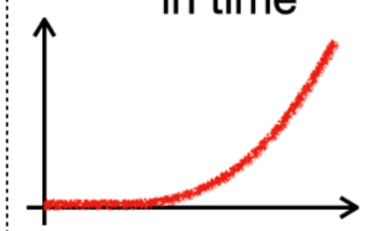
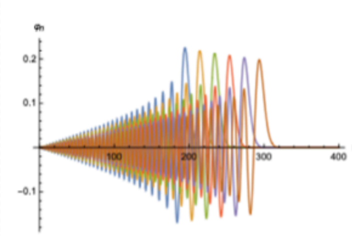
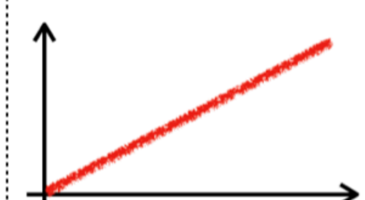
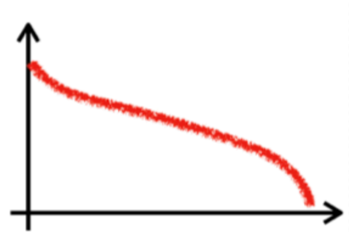
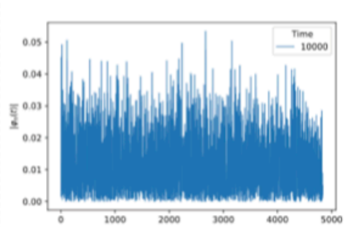
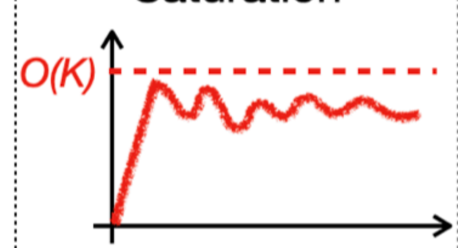
Ex.2 Extensive studies of operator growth

[Barbon, Rabinovici, Shir, Sinha '19]
[Rabinovici, Sanchez-Garrido, Shir, Sonner '21'22]

Continuum limit: $x = \epsilon n, \quad \varphi(x,t) = \varphi_n(t), \quad v(x) = 2\epsilon b_n = 2\epsilon b(\epsilon n)$

$$\partial_t \varphi(x,t) + v(x)\partial_x \varphi(x,t) + \frac{1}{2}v'(x)\varphi(x,t) = 0 \qquad \text{(cont. eq for } p = |\varphi|^2\text{)}$$

Numerics (Operator growth in XXZ chain + Integrability breaking terms, RMT)

n	**Lanczos coefficients**	**wavefunction**	**K-complexity**	**time scales**
$1 \ll n < S$	Linear growth in *n* $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in *n* $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

S=#dof

Fig from: [Rabinovici, Sanchez-Garrido, Shir, Sonner '22]

[PC, Magan, Patramanis '21]
[Balasubramanian, PC, Magan, Wu '22]

Ex.3 Symmetry approach: e.g. SL(2,R)

Take the SL(2,R) algebra

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \qquad [L_1, L_{-1}] = 2L_0$$

And the representation (basis)

$$|h,n\rangle = \sqrt{\frac{\Gamma(2h)}{n!\Gamma(2h+n)}} L_{-1}^n |h\rangle$$

$$\begin{aligned} L_0 |h,n\rangle &= (h+n) |h,n\rangle , \\ L_{-1} |h,n\rangle &= \sqrt{(n+1)(2h+n)} |h,n+1\rangle , \\ L_1 |h,n\rangle &= \sqrt{n(2h+n-1)} |h,n-1\rangle , \end{aligned}$$

Consider a class of models/states where the (state/operator) evolution in the Krylov space can be represented by

$$H = \gamma L_0 + \alpha(L_{-1} + L_1) \qquad |K_n\rangle = |h,n\rangle$$

Then we can immediately read-off the Lanczos coefficients

$$a_n = \gamma(n+h) \qquad b_n = \alpha\sqrt{n(2h+n-1)}$$

Related: Toda system, Orthogonal polynomials

[Dymarsky, Gorsky '19] [Muck, Yang '22]

Ex.3 Symmetry approach: SL(2,R)

[PC, J.M.Magan, D.Patramanis ’21]
[Balasubramanian, PC, Magan, Wu ’22]

The state evolution can be represented as a generalised coherent state (“driven CFT”)

$$|\Psi(t)\rangle = e^{-iHt}\,|h\rangle = e^{AL_{-1}}e^{BL_0}e^{CL_1}\,|h\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

These “amplitudes” solve the Schrödinger equation with SL(2,R) Lanczos coeff.

We can also derive a general result for complexity for this symmetry setup

$$C_\Psi(t) = \sum_{n=0}^{\infty} n p_n = \frac{2h}{1-\frac{\gamma^2}{4\alpha^2}}\sinh^2\left(\alpha t\sqrt{1-\frac{\gamma^2}{4\alpha^2}}\right)$$

Exponential growth for: $\gamma < 2\alpha$

Oscillating: $\gamma > 2\alpha$

Quadratic for: $\gamma = 2\alpha \;\; (a_n \sim 2\alpha n, \quad b_n \sim \alpha n)$

Complexity and Variation of L_0 :

$$C_\Psi(t) = \langle\Psi(t)|L_0|\Psi(t)\rangle - \langle\Psi(0)|L_0|\Psi(0)\rangle$$

Application to SYK(~2d CFT)

[PC, J.M.Magan, D.Patramanis '21]

$$\varphi_n(t) = \sqrt{\frac{\Gamma(\eta+n)}{n!\Gamma(\eta)}} \frac{\tanh^n(\frac{\pi t}{\beta})}{\cosh^\eta(\frac{\pi t}{\beta})} \qquad b_n = \frac{\pi}{\beta}\sqrt{n(\eta+n-1)}$$

Displacement operator and generalised coherent states

[Perelomov'72]

$$D(\xi) = e^{\xi L_{-1} - \bar{\xi} L_1} \qquad |z,h\rangle \equiv D(\xi)\,|h\rangle \qquad \xi = \tfrac{1}{2}\rho e^{i\phi} \qquad z = \tanh\left(\frac{\rho}{2}\right) e^{i\phi}$$

$$|z,h\rangle = \sum_{n=0}^{\infty} e^{in\phi} \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}} \frac{\tanh^n(\rho/2)}{\cosh^{2h}(\rho/2)} |k,h\rangle$$

"Trajectory in Phase Space": $\rho = 2\alpha t, \qquad \phi = \pi/2 \qquad \alpha = \pi/\beta$

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \qquad \mathcal{L} = \alpha\,(L_{-1} + L_1) \qquad |K_n\rangle = |h = \eta/2, n\rangle$$

Relation to geometric complexity?

[PC, J.M.Magan, D.Patramanis ’21] [PC, Datta ’21]
[Miyaji et al. ’15]

With coherent states we can associate a natural “information metric”

$$ds^2_{FS} = \langle dz|dz\rangle - \langle dz|z\rangle\langle z|dz\rangle \qquad \text{(Fubini-Study)}$$

E.g. for SL(2,R) this becomes a hyperbolic disc metric

$$ds^2_{FS} = \frac{2h dz d\bar{z}}{(1-z\bar{z})^2} = \frac{h}{2}\left(d\rho^2 + \sinh^2(\rho) d\phi^2\right) \qquad R = -\frac{4}{h}$$

Operator growth is a geodesic in this manifold (phase space): $\rho = 2\alpha t, \qquad \phi = \pi/2$

Observe a universal relation between the Volume and Krylov complexity

$$V_t = \int_0^{2\alpha t} d\rho \int_0^{2\pi} d\phi \sqrt{g} = 2\pi h \sinh^2(\alpha t) = \pi K_{\mathcal{O}}$$

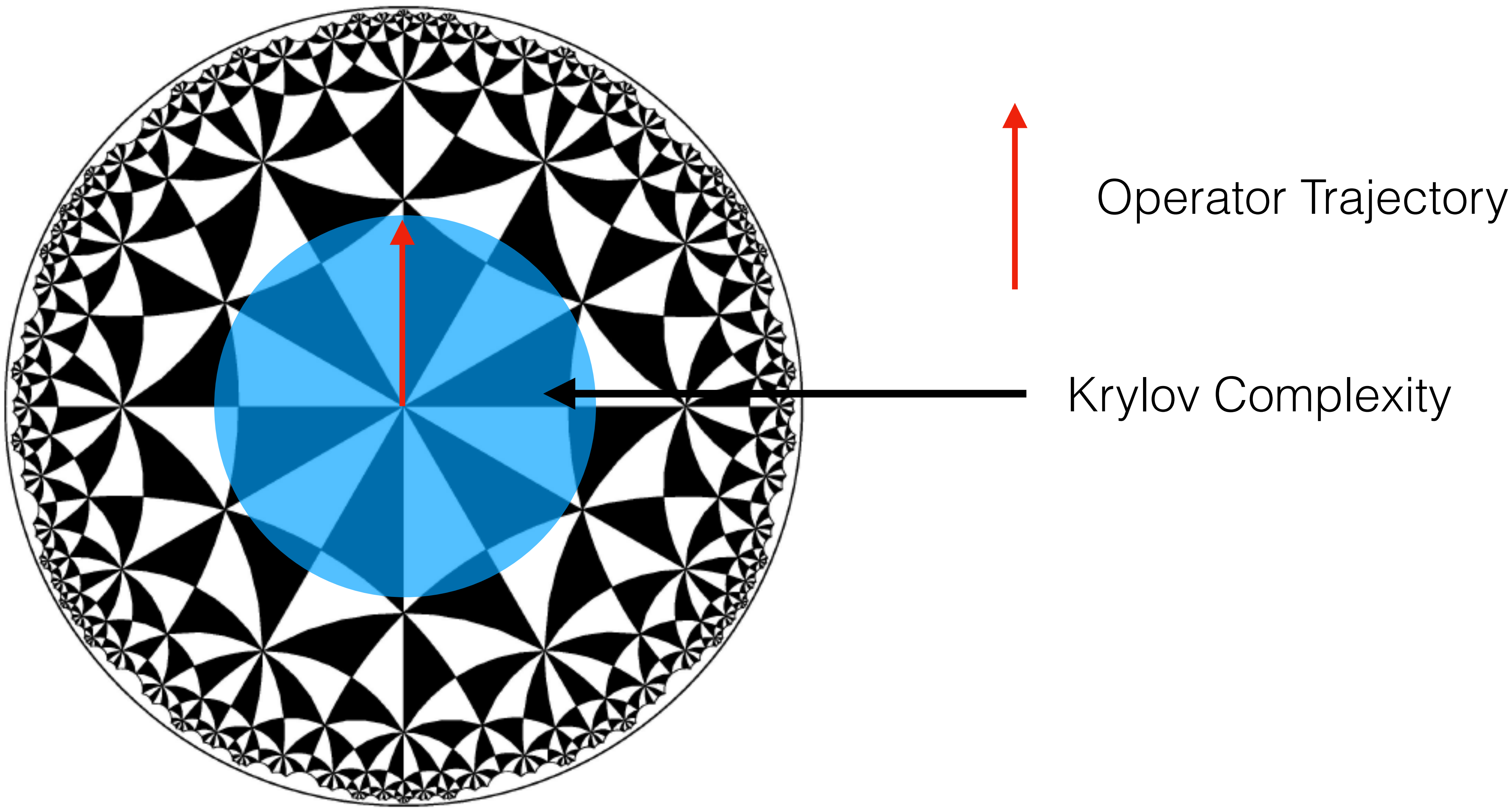
In all the symmetry examples that we studied (SU(2),HW)

Krylov complexity operator and L_0 (Symmetry generator)

$$L_0 = i\partial_\phi$$

$$L_{-1} = -ie^{-i\phi}\left[\coth(\rho)\partial_\phi + i\partial_\rho\right]$$
$$L_1 = -ie^{i\phi}\left[\coth(\rho)\partial_\phi - i\partial_\rho\right].$$

Cartoon:



Phase Space “Information Geometry”

Ex.4 Complexity for evolution of the TFD

[Balasubramanian, PC, Magan, Wu ’22]

Consider the TFD state

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n,n\rangle \qquad Z(\beta) = \sum_n e^{-\beta E_n}$$

and its time evolution

[Hartman,Maldacena ’13]

$$|\psi_\beta(t)\rangle = e^{-iHt}|\psi_\beta\rangle \qquad H = H_L + H_R \qquad H = H_{L/R}$$

Goal: expand this state in the Krylov basis and compute complexity.

Lanczos coefficients from the moments of

$$S(t) = \langle\Psi_\beta(t)|\Psi_\beta\rangle = \frac{Z(\beta - it)}{Z(\beta)}$$

(~SFF)! [Polchinski et al. ’16]

Non-universal, can be extracted once we know Z (also in some limits).

See [Balasubramanian, PC, Magan, Wu ’22]

Ex.5 Evolution of TFD for 2 HO

$$H_L = H_R = \omega(\hat{n} + \frac{1}{2}), \qquad E_n = \omega(n + \frac{1}{2}) \qquad Z(\beta) = \frac{1}{2\sinh\left(\frac{\beta\omega}{2}\right)}$$

return amplitude

$$\mathcal{S}(t) = \frac{\sinh\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{(\beta - it)\omega}{2}\right)}$$

Lanczos coefficients

$$a_n = \gamma(n + \frac{1}{2}), \qquad b_n = \alpha n, \qquad \gamma = \frac{\omega}{\tanh\left(\frac{\omega\beta}{2}\right)}, \qquad \alpha = \frac{\omega}{2\sinh\left(\frac{\omega\beta}{2}\right)}$$

Amplitudes (solutions of Schrodinger eq.)

$$\phi_n(t) = \frac{\sinh\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{(\beta+it)\omega}{2}\right)}\left(\frac{-i\sin\left(\frac{\omega t}{2}\right)}{\sinh\left(\frac{(\beta+it)\omega}{2}\right)}\right)^n$$

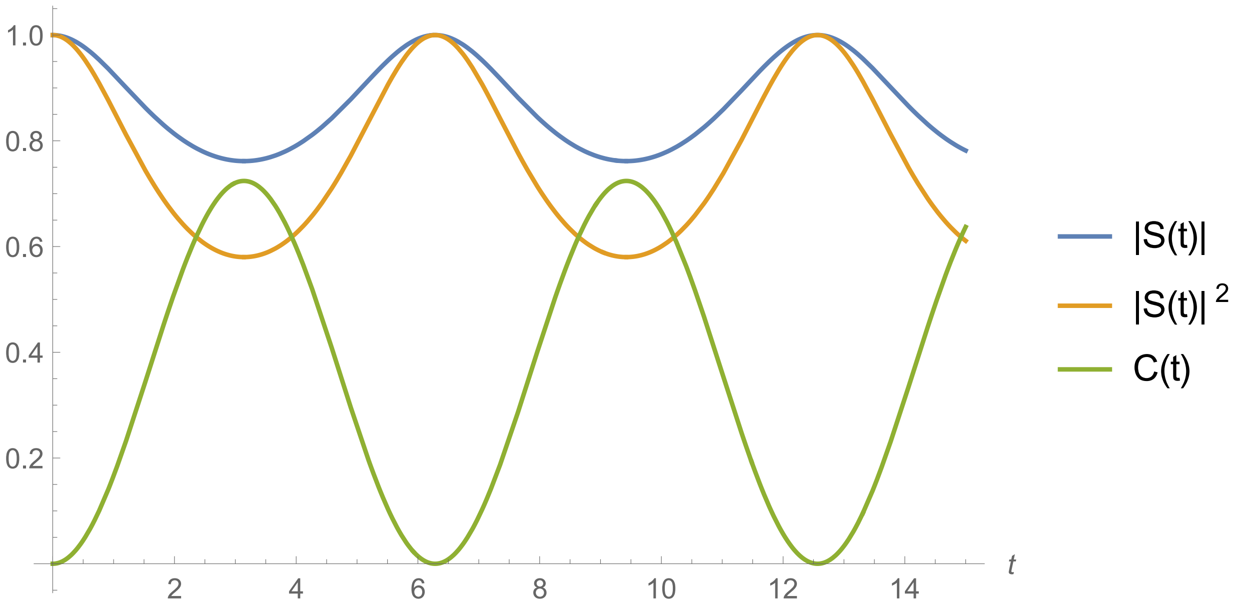
Spread/Krylov complexity

$$\mathcal{C}_\Psi(t) = \frac{\sin^2(\omega t/2)}{\sinh^2(\beta\omega/2)}$$

Ex.5 Evolution of TFD for 2 HO

$$\mathcal{S}(t) = \frac{\sinh\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{(\beta - it)\omega}{2}\right)}$$

$$\mathcal{C}_\Psi(t) = \frac{\sin^2(\omega t/2)}{\sinh^2(\beta\omega/2)}$$



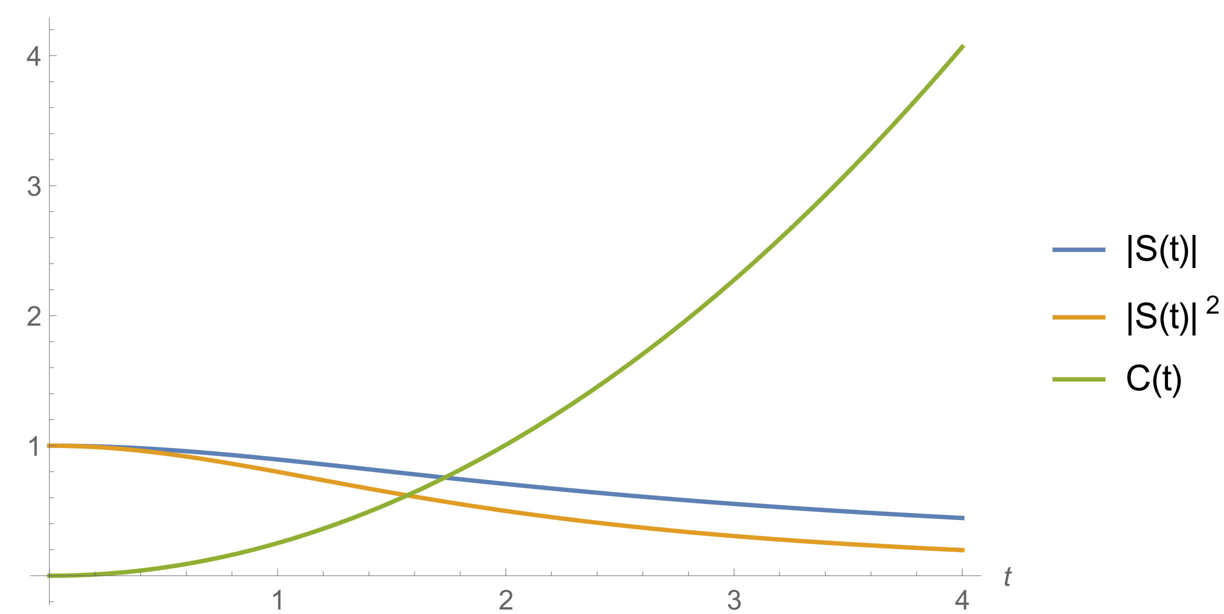
“Inverted” h.o.

$\omega \to i\omega$

$$\mathcal{S}(t) = \frac{-i\sin\left(\frac{\beta\omega}{2}\right)}{\sinh\left(\frac{\omega(t - i\beta)}{2}\right)}$$

$$\mathcal{C}_\Psi(t) = \frac{\sinh^2(\omega t/2)}{\sin^2(\beta\omega/2)} \sim e^{\omega(t - t_s)}$$

$$t_s = \frac{2}{\omega}\log(2\sin\frac{\beta\omega}{2})$$



Ex.6 Evolution of the TFD for RMT

[Balasubramanian, PC, Magan, Wu '22]

Late Times: "Black Holes and RM"

[Polchinski et al. '16]

Consider a random Hamiltonian (NxN, Hermitian matrix, GUE,…)

$$H = \begin{pmatrix} -0.625778 + 0.i & 0.0534572 - 0.238692i & -0.106837 + 0.170713i \\ 0.0534572 + 0.238692i & 0.518485 + 0.i & 0.995288 - 0.0813202i \\ -0.106837 - 0.170713i & 0.995288 + 0.0813202i & -0.589891 + 0.i \end{pmatrix}$$

We can easily diagonalise it, compute SFF, moments, Lanczos, etc.

We want to put it into the tri-diagonal form

and exponentiate

$$\begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

There exist very efficient algorithms/libraries (Python or Mathematica) to put a matrix into this form (Hessenberg). So we can also read off Lanczos coeff. this way.

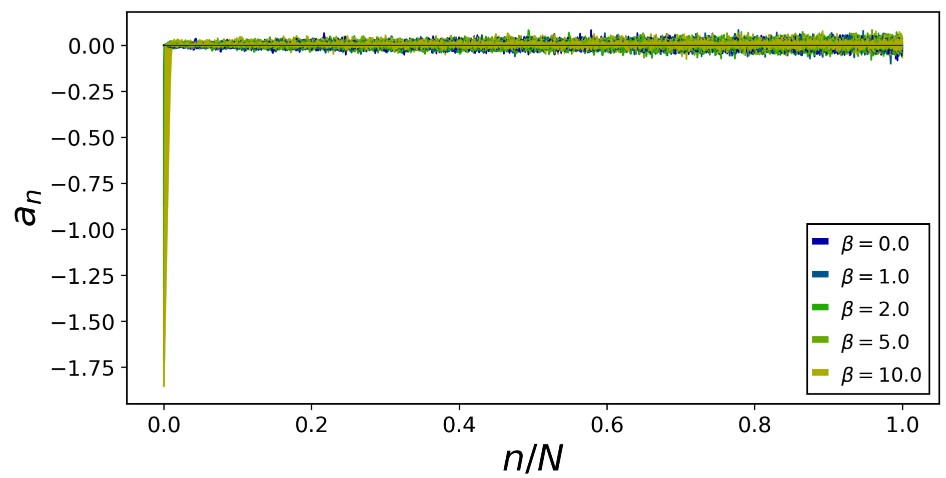
We also need to "rotate" a TFD into vec: {1,0,0,….}

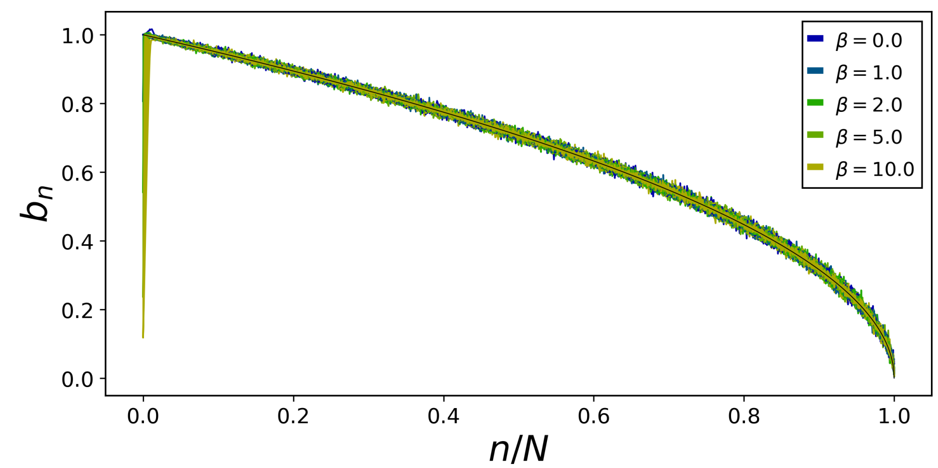
Then applying exp(-iHt) to the initial state gives all the $\phi_n(t)$

Ex.3 Evolution of the TFD for RMT

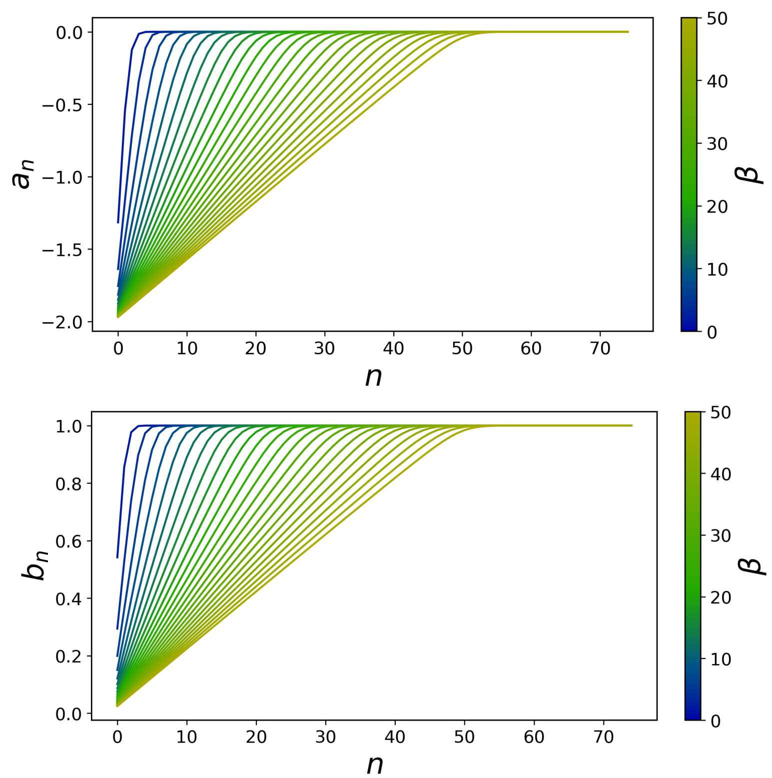
[Balasubramanian, PC, Magan, Wu ’22]

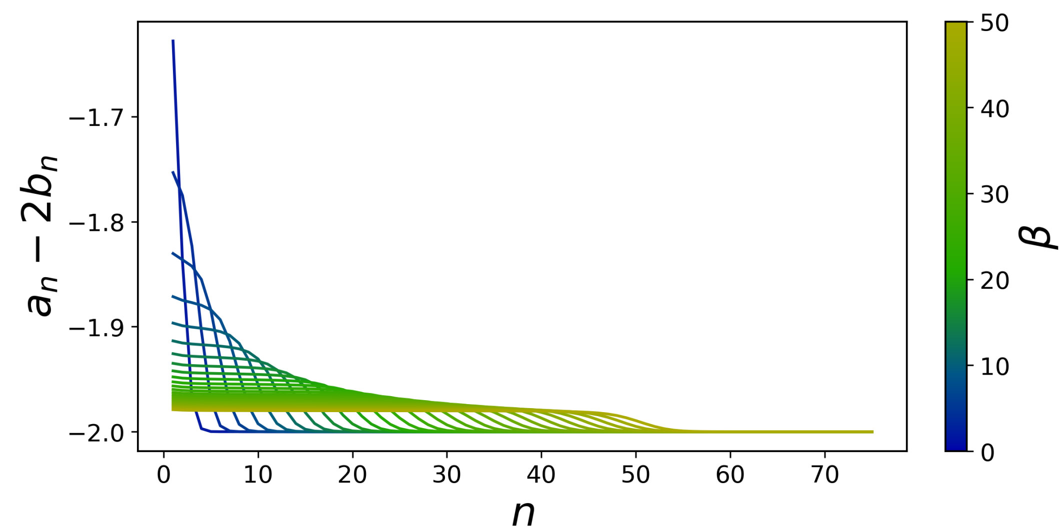
Examples Lanczos: GUE, N (up to 4096)





More detailed (early time)



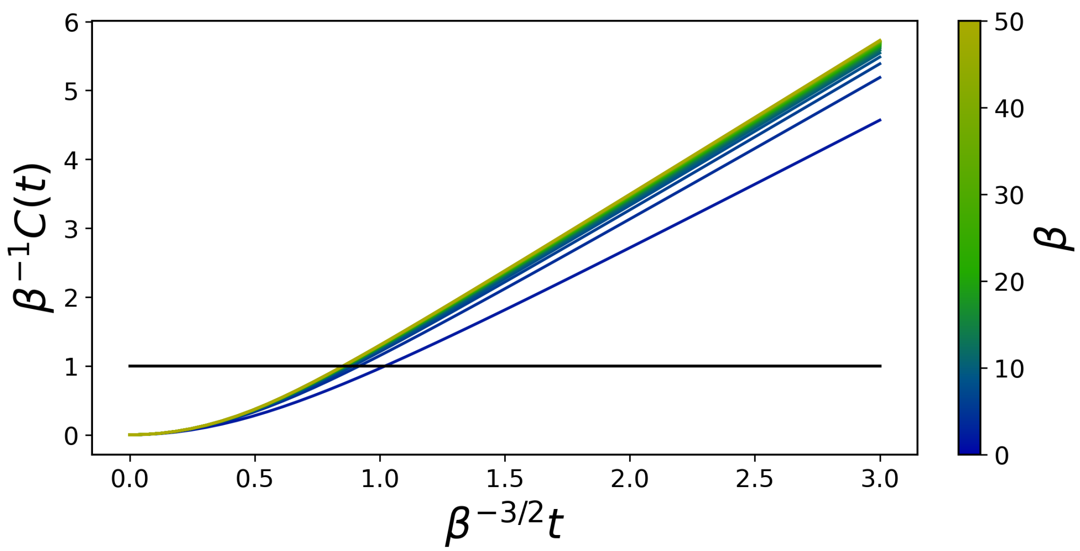


Ex.6 Evolution of the TFD for RMT

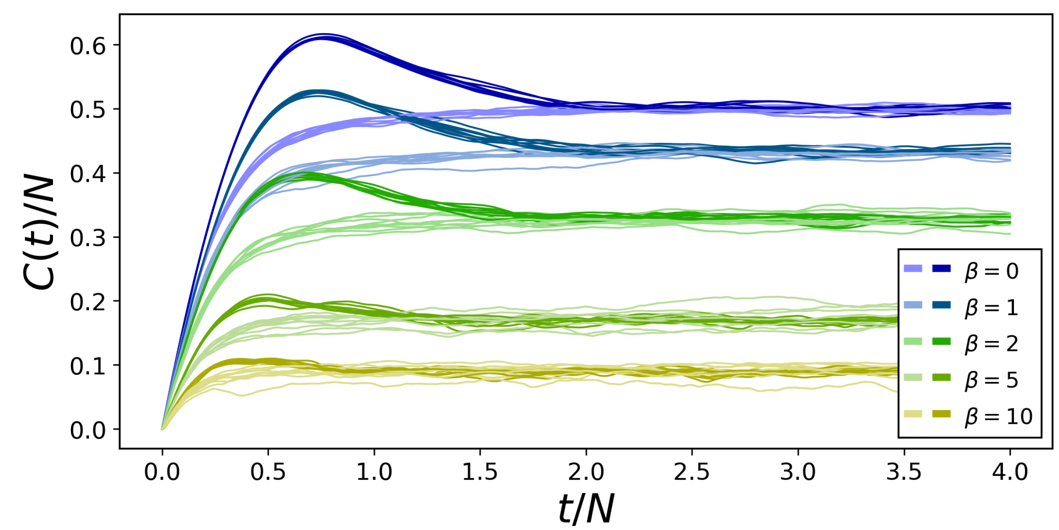
[Balasubramanian, PC, Magan, Wu ’22]

Complexity for TFD evolved with GUE Hamiltonian (Similar for GOE,GSE,SYK)

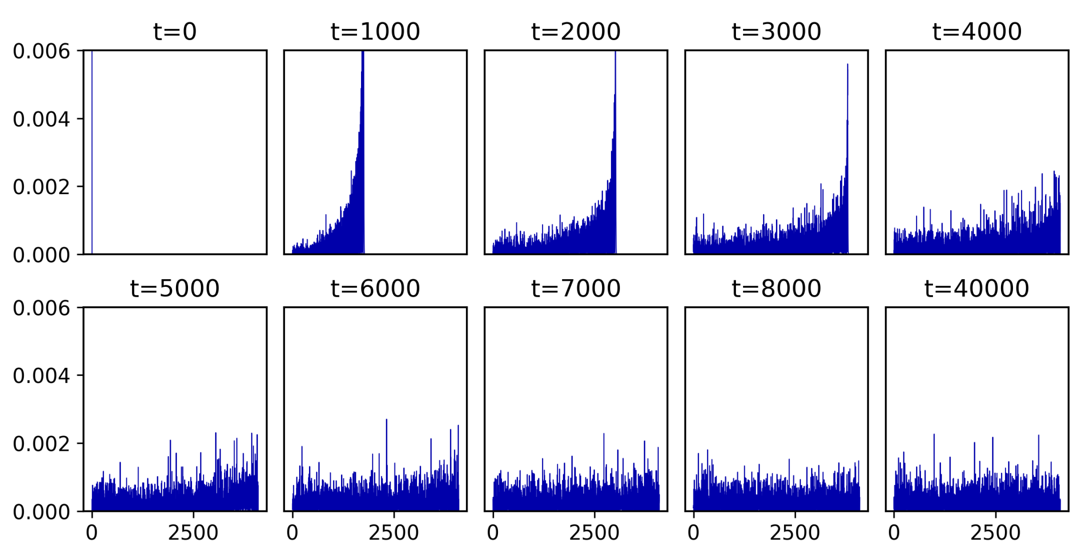
Early time



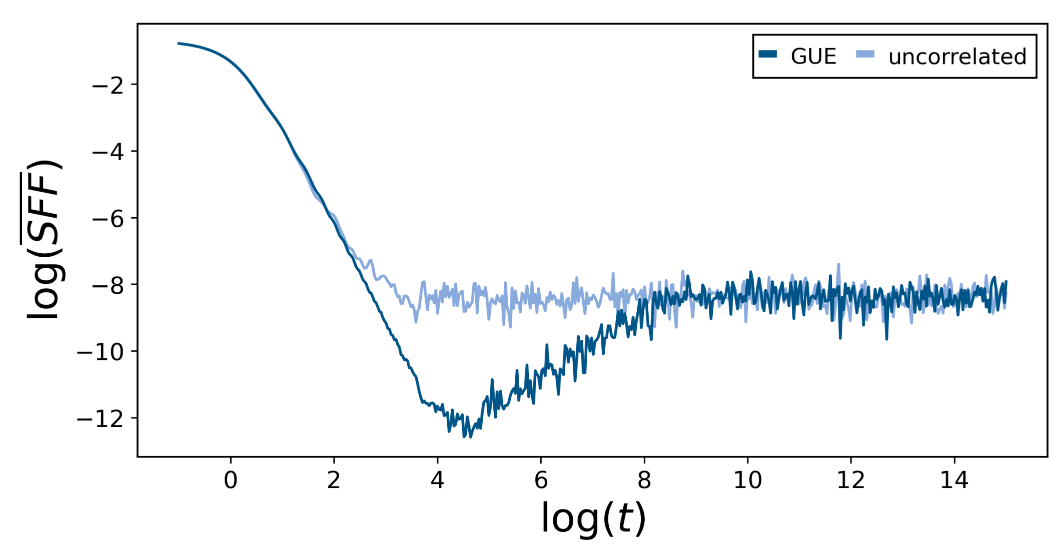
Ramp, Peak, Slope, Plateau



$N = \{1024, 1280, 1536, 1792, 2048, 2560, 3072, 3584, 4096\}$



Slope, Dip, Ramp, Plateau



$N = 4096$ and $\beta = 1$, averaged over 10 samples of the GUE

Conclusions

- New definition of Krylov/Spread Complexity for operators/states !
- Progress on a useful notion of “Complexity” in many-body systems
- Computable for operators and states; numerically for discrete models and QFTs
- Crucial ingredient: return amplitude (2- and higher-point function, SFF etc.)
- Evolution of TFD in RM: Ramp, Peak, Slope, Plateau
- Symmetry: new angle on Lanczos coefficients and growth in SYK, 2d CFT
- For SL(2,R) (semi-simple Lie alg.) we can “geometrize” it (coherent states) and interpret as phase space volume
- Straightforward to generalise to more interesting many-body setups (topological phases)

Many Open Problems

- Universal laws for Spread/Krylov complexity? Is it useful for QI or QC?
- Integrable vs Chaotic growth? Is it sensitive? At which time regime?
- Purely Integrable models? Can we study it using integrability (not just numerics)?
- Interesting states? More complicated objects (defects, boundaries)?
- Precise connection with Holography?
- Complexity and near-horizon geometry (AdS2)?
- More from Symmetry/Algebras? BMS, flat space, non-relativistic….?
- Late-time physics of AdS/CFT and Black-Holes?
- Why Krylov basis? Bulk understanding? (Chords in SYK? [Lin'22])

Thank You! Stay Tuned! Join the fun ;)

Backup slides

SU(2)

[PC, J.M.Magan, D.Patramanis '21]

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad J_\pm = J_1 \pm i J_2 \qquad [J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0$$

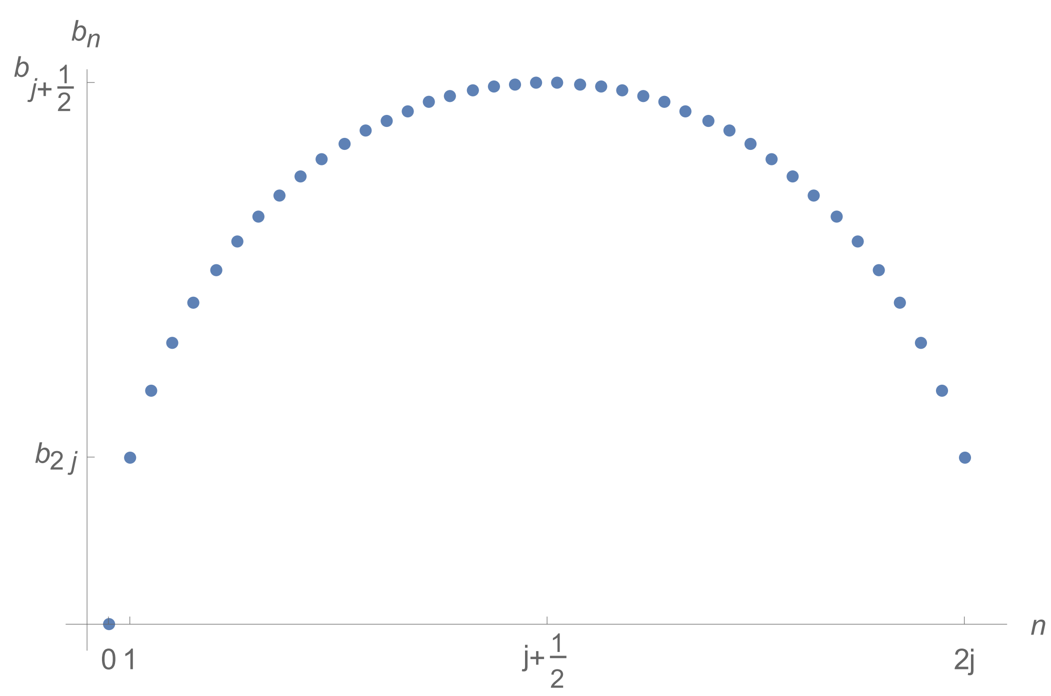
Liouvillian:

$$\mathcal{L} = \alpha(J_+ + J_-)$$

Representation:

$$\begin{aligned} J_0 \left|j, -j+n\right\rangle &= (-j+n) \left|j, -j+n\right\rangle, \\ J_+ \left|j, -j+n\right\rangle &= \sqrt{(n+1)(2j-n)} \left|j, -j+n+1\right\rangle \\ J_- \left|j, -j+n\right\rangle &= \sqrt{n(2j-n+1)} \left|j, -j+n-1\right\rangle. \end{aligned}$$

$$b_n = \alpha\sqrt{n(2j-n+1)}.$$



$$n = 0, .., 2j$$

SU(2)

[PC, J.M.Magan, D.Patramanis '21]

Spin coherent states:

$$|z,j\rangle = (1+z\bar{z})^{-j}\sum_{n=0}^{2j} z^n \sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}\,|j,-j+n\rangle \qquad z=\tan\left(\frac{\theta}{2}\right)e^{i\phi}$$

Trajectory: $\theta = 2\alpha t$ and $\phi = \pi/2$

$$\varphi_n(t) = \frac{\tan^n(\alpha t)}{\cos^{-2j}(\alpha t)}\sqrt{\frac{\Gamma(2j+1)}{n!\Gamma(2j-n+1)}}$$

Krylov complexity:

$$K_{\mathcal{O}} = \sum_{n=0}^{2j} n|\varphi_n(t)|^2 = 2j\sin^2(\alpha t)$$

Information Geometry

$$ds^2 = \frac{2j\,dz\,d\bar{z}}{(1+|z|^2)^2} = \frac{j}{2}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \qquad V_t = \int_0^{2\alpha t} d\theta \int_0^{2\pi} d\phi\sqrt{g} = 2\pi j\sin^2(\alpha t) = \pi K_{\mathcal{O}}$$

"Complexity Algebra"

[PC, J.M.Magan, D.Patramanis '21]

More generally lessons from the symmetry approach

$$\mathcal{L}|\mathcal{O}_n) = b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\mathcal{L} = \tilde{L}_+ + \tilde{L}_-$$

$$\mathcal{B}|\mathcal{O}_n) = -b_n|\mathcal{O}_{n-1}) + b_{n+1}|\mathcal{O}_{n+1})$$

$$\mathcal{B} = \tilde{L}_+ - \tilde{L}_-$$

Lets commute: From these definitions

$$\tilde{K} \equiv [\mathcal{L}, B]|\mathcal{O}_n) = 2(b_{n+1}^2 - b_n^2)|\mathcal{O}_n)$$

We can demand that the algebra closes at this first step. This gives

$$2(b_{n+1}^2 - b_n^2) = An + B$$

$$b_n = \sqrt{\frac{1}{4}An(n-1) + \frac{1}{2}Bn + C}$$

What if it doesn't? Number of steps to the closure? Classification?

"Complexity Algebra"

[PC, J.M.Magan, D.Patramanis '21]

For SL(2,R)

$$\mathcal{L} = \alpha(L_{-1} + L_1), \quad \mathcal{B} = \alpha(L_{-1} - L_1), \quad \tilde{K} = 4\alpha^2 L_0,$$

Geometrically, these are simply combinations of the isometry generators

$$ds^2 = \frac{h}{2}\left(d\rho^2 + \sinh^2(\rho) d\phi^2\right)$$

$$\begin{aligned} L_0 &= i\partial_\phi, \\ L_{-1} &= -ie^{-i\phi}\left[\coth(\rho)\partial_\phi + i\partial_\rho\right], \\ L_1 &= -ie^{i\phi}\left[\coth(\rho)\partial_\phi - i\partial_\rho\right]. \end{aligned}$$

In particular

$$\tilde{K} = 4\alpha^2(\hat{K}_{\mathcal{O}} + h) \ \sim \partial_\phi$$

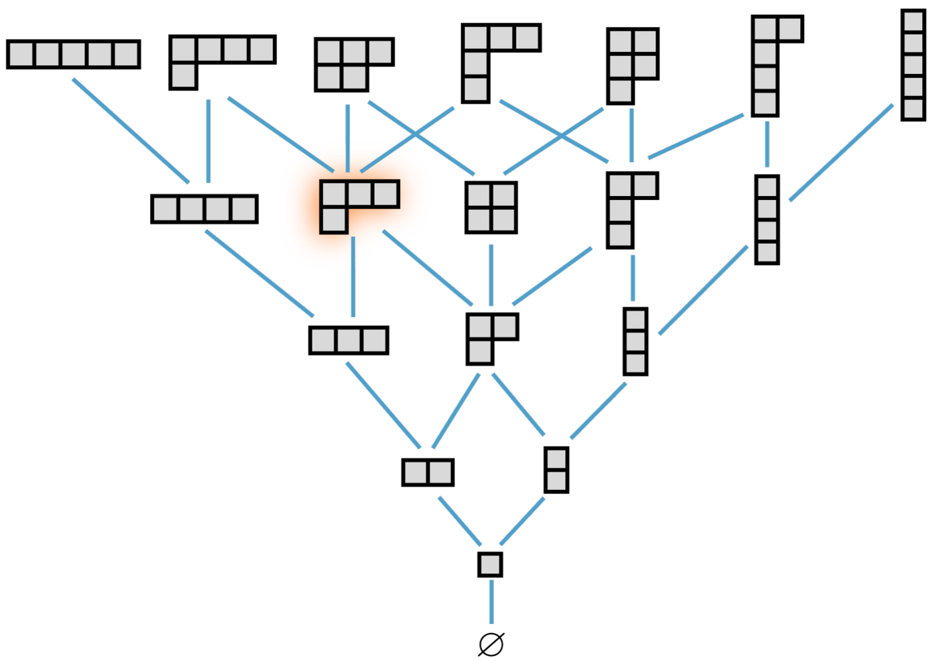
Relation between complexity and Isometries (Momentum/Boost)

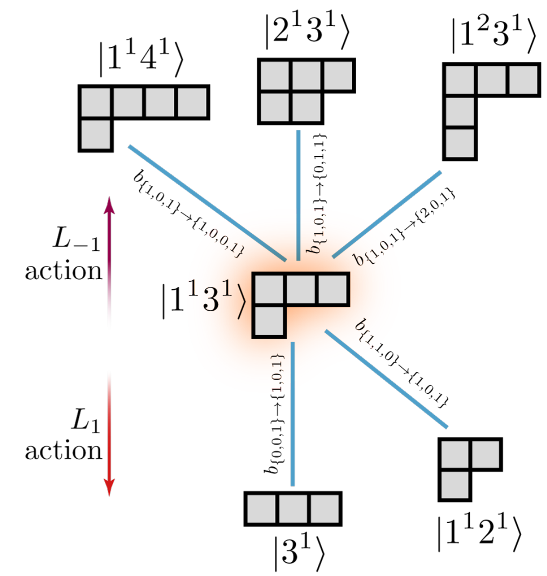
[Lin,Maldacena,Zhao'19]

Generalisations

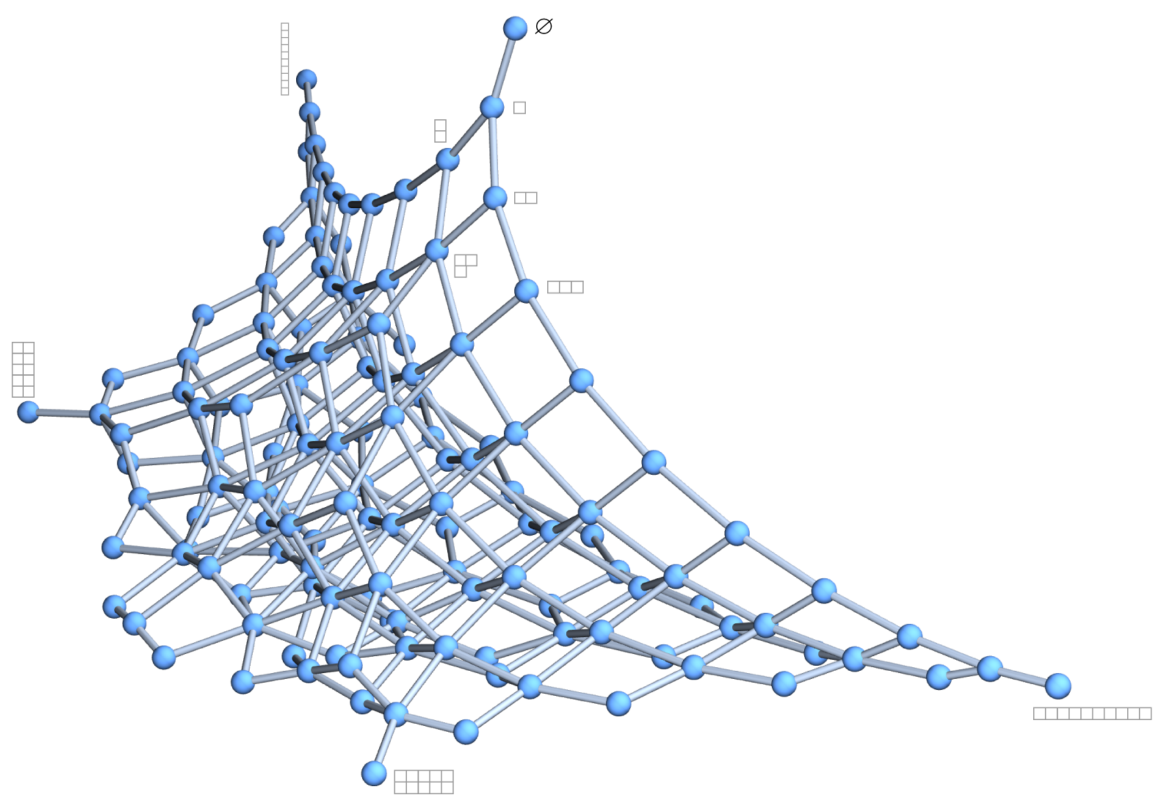
[PC, Datta '21]

Operator Growth in 2d CFTs: Primary flow into the bath of descendants.





Example:



Lanczos Coefficients

[PC, Datta '21]

Linear Growth of Lanczos coefficients corresponds to

$$\varnothing \to \square \to \square\square \to \square\square\square \to \square\square\square\square \to \square\square\square\square\square \to \cdots$$

Growth for "typical" states

$$b^{(1)}_{\text{typ}} \approx \frac{\sqrt{6N}}{\pi}$$

Slower than the initial linear growth (consistent with ETH)

Krylov complexity is the same as for the global (SL(2,R)) case (exponential growth) slowed down to polynomial for typical states and then is expected to saturate (for constant b, regime beyond CFT(?))