Branched $SL(2,\mathbb{Z})$ Duality

Ziqi Yan

Nordita

31st Nordic Network Meeting on "Strings, Fields and Branes"

14 Nov 2022 NBI, Copenhagen

2208.13815 w/ Eric Bergshoeff, Kevin Grosvenor, Johannes Lahnsteiner, Utku Zorba

2112.09316 w/ Stephen Ebert, Hao-Yu Sun

4D Yang-Mills

$$S_{YM} = -\frac{1}{4g_{YM}^2}\int d^4x\, F_{\mu\nu}F^{\mu\nu} \qquad F = dA$$

S-duality: $S_{parent} = S_{YM} + \int \tilde{F}\wedge(F - dA)$

e.o.m. from varying A_μ: $\tilde{F} = d\tilde{A}$

$$\tilde{S}_{YM} = -\frac{g_{YM}^2}{4}\int d^4x\, \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$$

Strong-weak duality: $g_{YM} \longrightarrow \frac{1}{g_{YM}}$

[Tseytlin '96]

D3-Brane

$$S_{D3} = -\int d^4x \sqrt{-\det\left(\eta_{\mu\nu} + e^{-\frac{\Phi}{2}} F^B_{\mu\nu}\right)} \qquad g_{YM} \sim e^{\frac{\Phi}{2}}$$

$$F^B = dA^B$$

S-dual ↓

$$\tilde{S}_{D3} = -\int d^4x \sqrt{-\det\left(\eta_{\mu\nu} + e^{\frac{\Phi}{2}} F^C_{\mu\nu}\right)} \qquad \tilde{g}_{YM} \sim e^{-\frac{\Phi}{2}}$$

$$F^C = dA^C$$

[Bergshoeff, de Roo, Kerstan, Ortin, Riccioni '06]

- $SL(2,\mathbb{Z})$ duality $\quad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,b,c,d \in \mathbb{Z} \quad ad-bc=1$

$$S_{D3} = -T_3 \int d^4x \sqrt{-\det\left(\eta_{\mu\nu} + \frac{\Theta^T \mathcal{F}_{\mu\nu}}{\sqrt{\Theta^T \mathcal{M}^{-1} \Theta}}\right)}$$

$$\Theta = \begin{pmatrix} p \\ q \end{pmatrix} \leftarrow (p,q)\text{-string} \qquad \Theta \to \Lambda \Theta$$

$$\mathcal{F} = \begin{pmatrix} B + dA^B \\ C^{(2)} + dA^C \end{pmatrix} \qquad \mathcal{F} \to (\Lambda^{-1})^T \mathcal{F}$$

$$\mathcal{M} = e^{\Phi} \begin{pmatrix} |\tau|^2 & C^{(0)} \\ C^{(0)} & 1 \end{pmatrix} \qquad \mathcal{M} \to \Lambda \mathcal{M} \Lambda^T$$

$$\tau = C^{(0)} + i e^{-\Phi} \qquad \tau \to \frac{a\tau + b}{c\tau + d}$$

[Seiberg, Witten '99]

Open Strings in B-Field

- magnetic

$$B_{23} \sim \frac{\alpha'}{\theta} \longrightarrow \text{noncommutativity in } x^2 \text{ and } x^3$$

$$[x^i, x^j] \sim \alpha' \frac{1}{G+B} B \frac{1}{G-B} \qquad \text{Seiberg-Witten map}$$

particle limit: $\alpha' \to 0$

Non-Commutative Yang-Mills

$$[x^2, x^3] \sim \theta \qquad g^2_{NCYM} \sim \frac{g_s}{\theta}$$

[Gopakumar, Maldacena, Minwalla, Strominger '00]

S-dual: Non-Commutative Open String

- electric

$$\tilde{B}_{01} \sim 1 - \frac{\alpha'}{\alpha'_{eff}} e \longrightarrow \text{space/time noncommutativity}$$

$\alpha' \to 0$: still a string theory!

$$[X^0, X^1] \sim \frac{\alpha'_{eff}}{e} \qquad g^2_{NCOS} \sim g_s \sqrt{\frac{e}{\alpha'_{eff}}}$$

$SL(2,\mathbb{Z})$ generalization?

[Andringa, Bergshoeff, Gomis, de Roo '12]

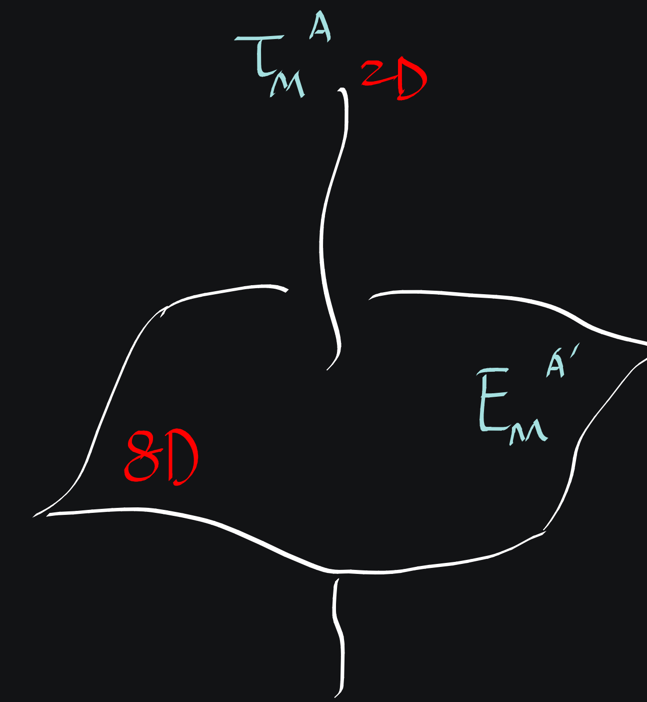
Generalized NCOS Limit $\omega \to \infty$

$$\hat{G}_{MN} = \omega^2 \underbrace{\tau_M{}^A \, \tau_N{}^B \, \eta_{AB}}_{\tau_{MN}} + \underbrace{E_M{}^{A'} E_N{}^{A'}}_{E_{MN}}$$

$$\hat{B} = -\omega^2 \, \ell + B$$

$$\hat{\Phi} = \Phi + \ln \omega$$

$$\hat{C}^{(q)} = \omega^2 \, \ell \wedge C^{(q-2)} + C^{(q)}$$ [Ebert, Sun, ZY '21]



$M, N = 0, 1, \cdots, 9$

longitudinal $A = 0, 1$

transverse $A' = 2, \cdots, 9$

$\rightsquigarrow$ nonrelativistic string theory [Review by Oling, ZY '22]

[Klebanov, Maldacena; Gomis, Ooguri; Danielsson, Guijosa, Kruczenski '00]

[Bergshoeff, Grosvenor, Lahnsteiner, ZY, Zorba '22]

Branched $SL(2,\mathbb{Z})$

- $w \to \infty$ limit of D3-brane
- D3-brane in nonrelativistic string theory

$$S_{D3} = -T_3 \int d^4x \sqrt{-\det \begin{pmatrix} 0 & \tau_\beta^{E} \\ \bar{\tau}_\alpha^{E} & E_{\alpha\beta}^{E} + \dfrac{\Theta^T \mathcal{F}_{\alpha\beta}}{\Theta^T \zeta} \end{pmatrix}}$$

$$\tau_\alpha = \tau_\alpha^0 + \tau_\alpha^1$$

$$\bar{\tau}_\alpha = \tau_\alpha^0 - \tau_\alpha^1$$

$$\Theta = \begin{pmatrix} p \\ q \end{pmatrix} \qquad \mathcal{F} = \Sigma + \begin{pmatrix} dA^B \\ dA^C \end{pmatrix}$$

$$\zeta = e^{\frac{\Phi}{2}} \begin{pmatrix} 1 \\ -C^{(0)} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} B \\ C^{(2)} \end{pmatrix} + \frac{q}{p - q\,C^{(0)}} \begin{pmatrix} 0 \\ \ell\, e^{-2\Phi}/2 \end{pmatrix}$$

$$\ell = \frac{1}{2} \epsilon_{AB}\, \tau^A \wedge \tau^B$$

- $\omega \to \infty$ limit of $SL(2,\mathbb{Z})$ transformations

finite $SL(2,\mathbb{Z})$ transformations only if $c\tau_0 + d > 0$!

$$\textcircled{\sqcup} \to \Lambda\, \textcircled{\sqcup}$$

$$\tau \to (\Lambda^{-1})^T \tau \qquad\qquad \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

quasi-doublet

$$\Sigma \to (\Lambda^{-1})^T \left\{ \Sigma + \alpha \begin{pmatrix} q \\ -p \end{pmatrix} \right\}$$

$\textcircled{\sqcup}^T \Sigma$ is invariant

(the vector $\begin{pmatrix} q \\ -p \end{pmatrix}$ is) orthogonal to $\textcircled{\sqcup}$

- branched $SL(2,\mathbb{Z})$ transformations

$$\Theta \to \Lambda \Theta$$

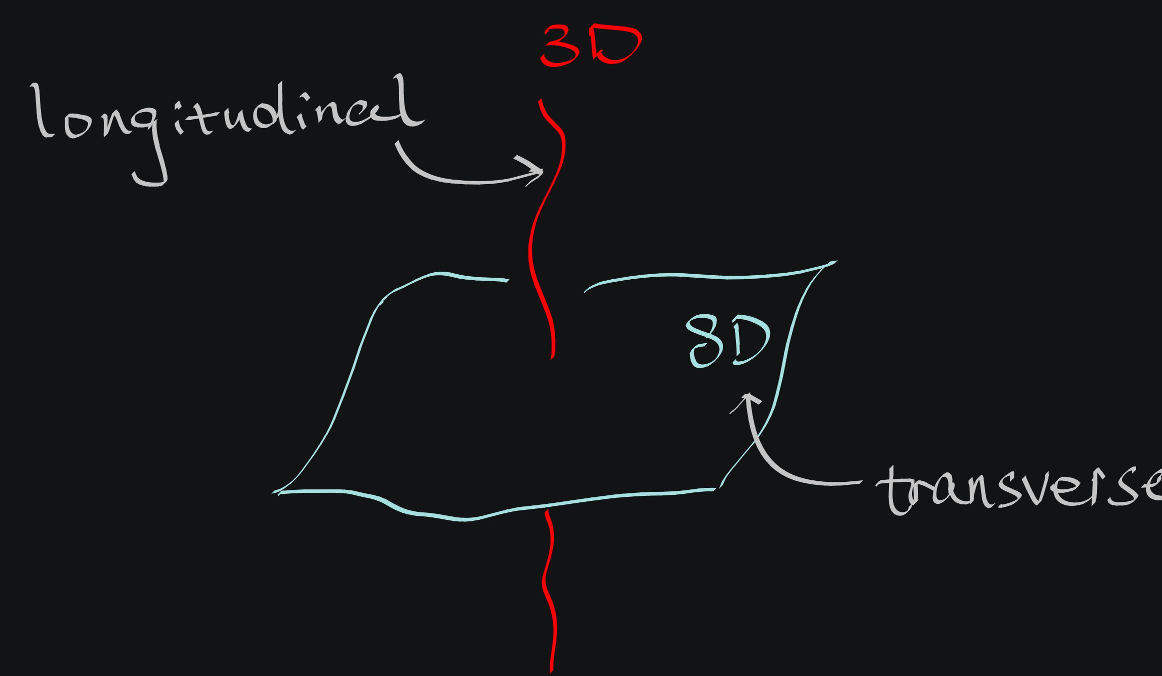
$$\mathcal{C} \to \mathrm{sgn}(cC_0+d)\,(\Lambda^{-1})^T \mathcal{C}$$

$$\mathcal{F} \to \mathrm{sgn}(cC_0+d)\,(\Lambda^{-1})^T \left\{ \mathcal{F} + \alpha \begin{pmatrix} q \\ -p \end{pmatrix} \right\}$$

[Berman '98]
[Ebert, ZY, in progress]

Geometrical Interpretation

nonrelativistic M5-brane on a split torus



$\Rightarrow$ branched $SL(2,\mathbb{Z})$ of D3-brane

Branched (p,q)-Strings

[Bergshoeff, Grosvenor, Lahnsteiner, ZY, Zorba '22]

- $\omega \to \infty$ limit of (p,q)-strings

finite action only if $p - q\,C^{(0)} > 0$!

- (p,q)-strings in nonrelativistic string theory

$$S_{+} = -\int d^2x \,(\Theta^T \zeta)\sqrt{-\tau^E}\,\tau_E^{\mu\nu} E^E_{\mu\nu} - \int \Theta^T \Sigma$$

$\uparrow$ $p - q\,C^{(0)} > 0$

$$p - q\,C^{(0)} \longrightarrow \frac{p - q\,C^{(0)}}{c\,C^{(0)} + d} \rightsquigarrow \text{sign change if } c\,C_0 + d < 0$$

$$p - q\, C^{(0)} > 0$$

$$S_{+} = -\left\{ \int d^2x \left(\Theta^T \mathcal{C}_\theta\right) \sqrt{-\tau^E}\, \tau_E^{\mu\nu} E^E_{\mu\nu} + \int \Theta^T \Sigma \right\} \quad \circlearrowleft \quad c\, C_0 + d > 0$$

$$\updownarrow \quad c\, C_0 + d < 0$$

$$S_{-} = +\left\{ \int d^2x \left(\Theta^T \mathcal{C}_\theta\right) \sqrt{-\tau^E}\, \tau_E^{\mu\nu} E^E_{\mu\nu} + \int \Theta^T \Sigma \right\} \quad \circlearrowleft \quad c\, C_0 + d > 0$$

$$p - q\, C^{(0)} < 0$$

To Be Continued :

Unbranched in nonrelativistic IIB supergravity?

[Bergshoeff, Grosvenor, Lahnsteiner, ZY, Zorba, to appear]

D-brane solutions?

Nonrelativistic holography? [Harmark, Hartong, Obers '17]

DLCQ supergravity and Matrix theory?

[Bergshoeff, Lahnsteiner, Romano, Rosseel '22]

Thank You!